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STEREOGRAPHY,

OR, A

COMPLEAT BODY

OF

PERSPECTIVE,

In all its BRANCHES.

Teaching to deſcribe, by MATHEMATICAL RULES,

THE

Appearances of LINES, PLAIN FIGURES, and SOLID BODIES,

RECTILINEAR, CURVILINEAR, and MIXED, in all manner of Poſitions.

Together with their

PROJECTIONS or SHADOWS,

AND THEIR

REFLECTIONS by POLISHED PLANES.

The WHOLE performed by Uniform, Eaſy, and General METHODS,
For the moſt Part entirely New.

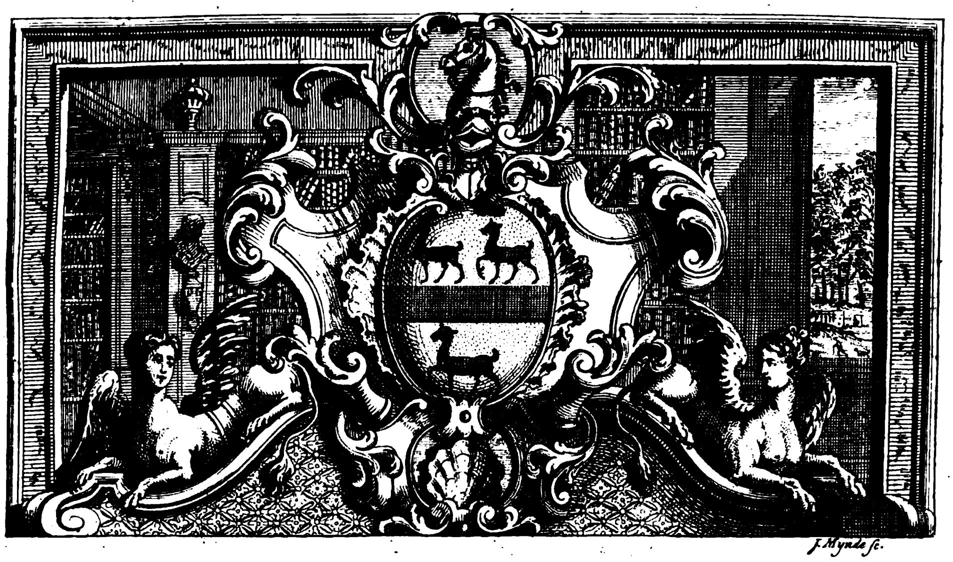
In SEVEN BOOKS.

By *J.* *HAMILTON*, Eſq; F. R. S.

In Two VOLUMES.

VOL. I.

LONDON,
Printed for the AUTHOR,
By W. BOWYER, in *White Fryars*, near the *Temple*, and Sold by S. AUSTEN, at the *Angel* and *Bible* in *St. Paul's Church-Yard*.
MDCCXXXVIII.



To the RIGHT HONOURABLE

Sir Joſeph Jekyll, *Knight*.

MASTER *of* the ROLLS,

ONE OF

His MAJESTY's Moſt Honourable PRIVY COUNCIL, *&c.*

May it pleaſe Your Honour,

I TAKE the Liberty humbly to offer to Your Honour's Protection, a Work over which You have a Parental Right of Guardianſhip, ſince it is owing to Your Honour's great Generoſity and unmerited Favour to its Author, that it is now in a Condition to ſee the Light.

IT was begun many Years ſince, and carried on at ſhort and diſtant Intervals of Receſs from a Hurry of Buſineſs, and might have ever remained unfiniſh'd, and in Obſcurity, had not Your Honour's Goodneſs, in placing me in a more eaſy Station of Life, allowed me Time to retrieve it from the Diſorder and Confuſion in which it lay, and to give it the Form in which it now appears before You.

THE Subject, indeed, is foreign to the Profeſſion in which I have been bred, and perhaps not of ſufficient Weight to claim Your Honour's Attention, always imploy'd in matters of the greateſt Moment, in the Diſtribution of impartial Juſtice and Equity, and an active Zeal for the Service and Good of Your Country; but as it hath Mathematical Reaſoning and Truth for its Foundation, and the Improvement of the Ornamental and Uſeful Arts of Painting, Sculpture, and Architecture for its End, I hope it will not appear altogether unworthy of Your Favour.

THE Honour I have had, SIR, of being ſometimes admitted to Your familiar Converſation, makes me too well acquainted with Your Merit in Private, as well as Publick Life, to follow the common Method of DEDICATIONS, and attempt ſo deſerving a Character, which I am perſwaded would be equally diſagreeable to You, and above my Talents to ſucceed in. Humanity, Integrity, Fortitude, and a ſincere Love of Your Country's true Intereſt, with all the Natural and Acquired Abilities, neceſſary for exerting thoſe Virtues in the moſt prudent and beneficial Manner, are ſuch ſhining Qualifications, as can receive no additional Luſtre from being enlarged upon in this Addreſs.

I SHALL not therefore longer interrupt Your Honour's more important Thoughts, than to beg Your kind Acceptance of this Product of my Studies, and to expreſs the Satisfaction I have, on this Occaſion, of teſtifying, in a publick Manner, the unfeigned Gratitude, Eſteem, and profound Reſpect, with which I am,

May it pleaſe Your Honour,

Your Honour's moſt obliged,

and moſt obedient humble Servant,

14th June, 1738.
Six Clerks Office.

JOHN HAMILTON.

PREFACE.

THE great Advantage of the Science of *Stereography* to all whoſe Profeſſion or Pleaſure leads them to the Practice of *Deſigning* or *Painting*, is now ſo generally known and allowed, that a Treatiſe on that Subject can want no Recommendation in that reſpect; but, as this Science hath, under the Name of *Perſpective*, been treated of by ſo many different Authors, for above two Centuries paſt, that it is natural to ſuppoſe the Subject muſt have been long ſince exhauſted, it ſeems incumbent on One, who adventures to offer any thing farther on that head to the Publick, ſo apt to be diſguſted with Repetitions, to make ſome Apology for treading in ſuch a beaten Path.

ARCHITECTURE, *Sculpture*, and *Painting*, have ever been the Delight of all Polite and Civilized Nations, and have improved in proportion to the Power and Grandeur of the States where they were cultivated, and by Turns have ſuffered the like Decays. The ancient *Greeks* and *Romans* have left us many Monuments of their great Skill in the two firſt, ~~and if~~ we believe ſome Paſſages of their Hiſtorians, they were as *little* deficient in the latter: However, if we may judge by ſuch ſmall Remains of their Paintings as are ſtill preſerved, we may thence reaſonably infer, that the Science of *Perſpective* was very little known to them; and this, their *Teſſelated* Pavements and *Bas-Reliefs* farther confirm, which, for want of *Perſpective*, are deſtitute of many Beauties which the Knowledge of that Art might have furniſhed them with. The Maſters of thoſe Times excelled in the Deſcription of ſingle Figures, or Groupes of Figures on the ſame Line, in giving them a beautiful Grace, a juſt Expreſſion, and proper Attitudes; but, with reſpect to the Diminution and Degradation of Objects in proportion to their Diſtances, that was a Secret they were not acquainted with, it depending greatly on the Science of *Opticks*, which in thoſe Times was very little underſtood; it being remarkable, that amongſt ſo many ancient Authors whoſe Writings have reached us, there is ſcarce any thing to be found on the Subject of *Opticks*, ſave a very ſhort and imperfect Piece aſcribed to *Euclid*, and not one Author who has wrote on *Perſpective*. And yet if any Treatiſe had been compoſed on a Subject ſo curious and entertaining, it is hardly probable it would have been ſuffered to periſh.

Perſpective may therefore be juſtly ranked amongſt the Inventions, or at leaſt the Improvements of latter Times. For the polite Arts having been involved in the Ruin of the *Roman* Empire, and ſucceeded by a long, dark, and ignorant Period; at length, towards the beginning of the 14th Century, thoſe Studies, and with them *Painting*, *Sculpture*, and ARCHITECTURE, began to revive; but theſe, from low Beginnings, had been gaining Ground a conſiderable Time before the Profeſſors diſcovered the Uſe of *Perſpective*, which was little regarded in their Works till about the beginning of the 15th Century, when *Paolo Uccello*, a *Florentine* Painter, made it his more immediate Study, and was therein imitated by ſucceeding Maſters, amongſt whom *Andrea Mantigne* of *Padua*, made the greateſt Proficiency, and far excelled the reſt in this particular.

But hitherto, *Perſpective* was only conſidered as a new and more correct Manner of drawing the Appearances of Objects with reſpect to their different Diſtances, for doing of which they had as yet no certain Rules, but barely the Judgment of the Eye: It was not until the following Century, ſo fruitful in great Maſters, ſuch as *Leonardo di Vinci*, *Michael Angelo*, *Raphael*, *Titian*, *Julio Romano*, and others, that this Art came to be conſidered as reducible to Mathematical Rules; 'twas then firſt, that divers of the Painters, Sculptors, and Architects of that Age applied themſelves to diſcover thoſe Rules, which gave Birth to ſeveral Eſſays, containing ſome of the moſt obvious Principles of the Art; but as the Subject was at that Time new, and the Writers not ſufficiently ſkill'd in Opticks and Mathematicks, the Advances they made amounted to little more than two or three Rules adapted to particular Caſes, and thoſe both laborious and inconvenient in the Practice.

Theſe firſt Attempts, and the uſefulneſs of the Subject, prompted others to purſue the Inquiry, which in a few Years ſpread itſelf, and became the Study of many Artiſts and Men of Learning in moſt of the polite Parts of *Europe*, who at different Times publiſhed their farther Diſcoveries, whereby the Books of *Perſpective* became at length greatly multiplied; and of late Years, no general Courſes of Mathematicks have been eſteemed compleat, without a particular Treatiſe on that Subject; inſomuch that had the Improvements made in this Science, been equal to what might have been expected from the Number and Abilities of thoſe who have treated it, all farther Writing on that Head muſt have been long ſince rendered ſuperfluous.

But as it is ſurpriſing, that this Branch of the Mathematicks ſhould have ſo long remained hid, before it was diſcovered and taken into Conſideration; it is no leſs ſo, that it ſhould have been left in ſo low a degree of Perfection, after paſſing through ſo many Authors Hands, who wanted neither Mathematical Knowledge, nor Skill in the Deſigning Part, to qualify them for the Work.

The Reaſon of this ſeems to be, that the firſt Writers having ſet out upon very narrow Principles, and preſcribed difficult and inconvenient Operations, thoſe who followed, rather applied themſelves to facilitate the Practice, than to enlarge the Foundation: This might induce the Mathematicians amongſt them to imagine the Subject incapable of any great Advancements in the Theory, and ſo not worthy of their cloſer Application; and the Artiſts, to ſubſtitute Drawing and

and Deſigning in its Stead, to ſupply the Imperfection and Deficiency of the Rules; it being obſervable, that the largeſt Works of this kind yet extant, are much more valuable for the curious Deſigns and Draughts with which they abound, and for executing of which they have given very little Instruction, than for any uſeful Rules whereby the practice of *Perſpective* may be made more extenſive.

It is not here my Intention to entertain the Reader with an Account of the many Authors who have already treated this Subject, or of the Excellencies or Defects of their Performances; all I ſhall ſay by way of Excuſe for adding to their Number is, that after a careful Peruſal of all the Writings of this Sort which have fallen into my Hands, there appeared to me to be ſufficient Room left for very great Improvements, both in the Theory and Practice: This was what alone induced me to purſue and finiſh the following Work, which was at firſt begun only by way of Amuſement at Times of Leiſure, without any View of making it publick, till in the Progreſs of it, I found Reaſon to think I could offer ſomething new and inſtructive on the Subject, worthy of being communicated.

This led me to re-examine the Principles of the Science as laid down by former Writers, to enlarge ſuch of them as appeared too narrow, and to ſupply what was wanting, in order to extend the Foundation, and make it capable of bearing a larger Superſtructure than heretofore. And I have had the Pleaſure to find that by general and uniform Rules of Practice built on that Foundation, a great Diverſity of Problems can be ſolved with Eaſe, which hitherto have been either left *wholly* untouched, or elſe, ſuch of them as have been *attempted*, have been required to be performed by ſuch tedious and entangled Methods as are very difficult to be underſtood, and more ſo to be put in Practice, and frequently falſe or inſufficient for the Purpoſe.

On examining the Nature of *Stereography* in general, it appears to have a much nearer Affinity to Conick Sections than has hitherto been obſerved; and as this is a Branch of the Mathematicks which is not ſo commonly learnt as the eaſier Parts of Geometry contained in *Euclid*, with which the Reader is ſuppoſed to be acquainted, I have thought it neceſſary to explain ſuch of the Principles of that Science, as are more immediately uſeful to our Subject, and particularly the Nature and Properties of the *Harmonical* Diviſion of Lines, the Application of which to *Stereography* is of great and extenſive Uſe, and affords a very conſiderable Improvement to it.

For although the *Marquis de l'Hopital*, in his Treatiſe of *Conick Sections*, has cenſured the Method of Demonſtration from the Properties of *Harmonical* Diviſion, uſed by M. *de la Hire* in his Book on the ſame Subject, as more difficult than the Analitical Way of Demonſtration which he himſelf hath choſen; yet, in regard that by the former Method, the Lines themſelves are determined, which is what is principally required in *Stereographical* Problems, it is much preferable, for that Purpoſe, to the Analitical way of Conſtruction, by which, not the Lines themſelves, but only their mutual

Proportions and Relations are diſcovered, which afterwards require difficult Geometrical Operations to reduce them into Lines.

In compoſing this Work, I have freely made uſe of all ſuch Materials as I could find any where fit for my Purpoſe, and in particular have taken all ſuch Aſſiſtance and Hints as were furniſhed me by Dr. *Brook Taylor*'s two ſmall Treatiſes on this Subject publiſhed ſome Years ſince, in which that learned Gentleman has, in a few Pages, made more real Advances towards perfecting the Science, than all the Writers who went before him: But as he has, in moſt of his Problems, given their Solution only in ſome of the eaſieſt Caſes, leaving it to his Reader to apply his Rules to the more difficult which might occur, which Application is frequently not very obvious, and ſometimes cannot be made, and the Solution of others of his Problems depending on Principles which he has no where explained; I have endeavoured to ſupply theſe Defects, by firſt laying down all ſuch Principles as are neceſſary to the Solution of the Problems under each particular Branch of the Inquiry, and have, for the moſt Part, conſidered each Problem firſt in its moſt difficult or complex Caſe, and given ſeveral different Methods of ſolving it, and thence in proceeding to the more eaſy, ſhewn how the ſeveral Methods before propoſed are applicable to them, and do by degrees unite with one another, and become the ſame, as the Caſes grow more ſimple.

The Objects I have choſen to treat of are but few, and thoſe for the greateſt Part very plain and familiar, having, as much as was conſiſtent with the Extent of my Deſign, avoided all complicated or laborious Examples, or filling up the Plates with Variety of Objects, foreign to the Rule immediately under Conſideration, which might indeed have diverted the Eye, but no ways informed the Judgment: But thoſe I have made choice of, *viz.* a *ſtraight Line*, a *Triangle*, and other *Regular Polygons*, the *Circle*, *Ellipſis*, and the reſt of the *Conick Sections*, the *five Regular Solids*, and the *Cone*, *Cylinder*, *Sphere*, and *Annulus*, are ſuch as enter into the Compoſition, and are, in a manner, the Elements of all other Objects; and I was the rather induced to confine my ſelf to theſe Particulars, as thinking it moſt for the Benefit of the young Artiſt, to teach him to deſcribe the component Parts of Objects in any required Poſitions, and to leave it to his Induſtry and Practice to combine them as he ſhould have Occaſion.

Theſe I have therefore ſhewn how to deſcribe, in all manner of Poſitions, either with regard to the Eye or the Picture; and alſo how to find their Projections or Shadows in all different Situations of the Light; as likewiſe to determine their reflected Images in poliſhed Planes: And under each of theſe Heads I have endeavoured to make the Rules as univerſal as poſſible, that they might ſerve not only for the Object in the Example, but for all others of the like kind; to which End, a Variety of Methods are every where propoſed, which may each have their particular Conveniency in different Circumſtances; which is what hath been generally wanting in moſt of the former Works of this ſort, where the Authors have contented themſelves with giving ſome few Rules for the Deſcription of ſome of the ſimpleſt Objects in the eaſieſt and moſt regular Situations, and thence immediately proceed to large Compoſitions, leaving the Learner at a Loſs how to deſcribe the eaſieſt Object, when its Poſition is a little out of the uſual and ordinary Way.

I am ſenſible this Work may, in ſome Places, be liable to juſt Exception, with regard to the Length of ſome of the Propoſitions, and the Number of Corollaries and Scholia belonging to them, ſeveral of which might perhaps more properly have made diſtinct Propoſitions of themſelves: But as, notwithſtanding all the Helps I could meet with, I have, in many Branches of this Inquiry, been obliged to travel without a Guide, ſo, in the Progreſs of the Work, new Lights ſometimes aroſe which did not at firſt offer themſelves, and which I thought better to add in the Shape of Corollaries, or *Scholia*, to the Propoſitions under which they moſt naturally fell, than to diſturb the Order of thoſe already written, and the References to them; being willing at the ſame Time, that the whole of what was to be ſaid on the ſame Head, might, as near as poſſible, appear together in one View.

The Care taken throughout, to inſert ſuch Obſervations and Remarks as appeared neceſſary or ſerviceable to the better underſtanding the Theory as well as the Practice of the Subject in hand, and the conſtant Endeavour to be every where plain and intelligible, have unavoidably drawn this Work into a greater Length than might be expected, and ~~doubtleſs~~ *given* occaſion to ſome Repetitions, as *well of the Matter as of the* Demonſtrations, eſpecially with regard to the more difficult Parts of the Science, which ſeemed to require a fuller Explication.

Theſe Superfluities I would gladly have retrenched, had I not been apprehenſive of falling into a contrary Error; it being too common with Writers, when fully poſſeſſed of the Ideas of the Subject they treat, to imagine they can convey them to their Readers by ſuch ſhort Hints, as, although intelligible enough to themſelves, may not be ſufficiently explicit for others; whereby in aiming at Conciſeneſs, they fall into Obſcurity, a Fault ~~much~~ more ~~tireſome~~ ~~and~~ diſcouraging to a Learner, eſpecially in Mathematical Studies, than a little Repetition, which may ſometimes ſerve to refreſh the Memory, or, by a Variation of the Expreſſion or Demonſtration, may help to the better underſtanding of what might appear dark in another Place: And therefore, if this Work ſhould in ſome Places appear too prolix or minutely circumſtantial to Readers of quick Apprehenſions, and ſuperior Skill in the Mathematical Sciences, yet, it being alſo intended for the Inſtruction of ſuch who may not have thoſe Qualifications in ſo great Perfection, the more Learned, it is hoped, will for the Sake of the others, paſs by all Faults of that kind; and if ſome particular Inquiries have been purſued farther than it may be thought was directly neceſſary to the main Subject, their Curioſity and Novelty was what induced me to it, and muſt plead my Excuſe.

PREFACE.

That the Reader may ſee a Plan of what he is to expect in the following Work, and have a View of the Order and Method obſerved in the Conduct of it, there is added a ſhort Abſtract of the principal Matters it contains; and it will, I hope, anſwer his Expectation, if upon Peruſal of the Work itſelf, he ſhall find the Science of *Stereography* therein reduced into a regular and uniform Syſtem, carried on from the firſt and eaſieſt Principles, to the more difficult and many hitherto untouched Problems, which may lead him to ſo perfect a Knowledge of that Science, as to enable him by his own Induſtry to make farther Advances.

This is what hath been endeavoured in the following Sheets, and if I have ſucceeded in the Attempt, and rendered the Practice of *Perſpective* more eaſy, general, and extenſive than hitherto it has appeared, which I flatter my ſelf to have done, I ſhall have attained the End I propoſed in writing.

It remains only to add a few Words touching the preſent Edition.

With regard to this, the Reader may perhaps be diſappointed to find the Figures referred to, leſs ornamental than thoſe which are to be met with in many Books of this ſort; but as the principal Intention of this Work is to inſtruct, Utility and Perſpecuity have been preferred to Ornament and Shew: Great Care hath been taken, both in the Text and in the Figures, that they ſhould be as correct as poſſible, and that the Letters with which the Figures are marked in the Plates ſhould be properly placed, and ſhould correſpond with thoſe in the Print, and alſo amongſt themſelves; the ſame Letters having, as near as could be done, been employed every where to denote the ſame Things; whereby they become, in a manner, conſtant Signs of the Points and Lines to which they are uſually annexed, and ſerve frequently inſtead of a longer Deſcription, and may ſometimes aſſiſt the Reader to underſtand the Text without an immediate Inſpection of the Figure.

The Notes in the Margent refer either to ſuch Places where the Demonſtration of what is advanced in the Text may be found, or to ſuch as ſhew how any required Operation may be performed, when not there immediately taught; or elſe direct to other Parts of the Work, where ſome farther Account of the Matter under Conſideration may be met with: All theſe have been carefully examined, that the Reader may not be miſguided, and ſent to a wrong Place.

In theſe References, the only Book to be reſorted to, beſides the preſent Work, is *Euclid's Elements*; in thoſe of this Sort, which are always marked *El.* the Figure which preceeds is the Number of the Propoſition, and that which follows marks the Book, as they ſtand in Doctor *Gregory's* Edition of that Author, printed in Folio at *Oxford* in 1703. And in the References to any Propoſition of this Work, the Number of the Book to which it belongs is diſtinguiſhed by Roman Capital Numerals; and where theſe are wanting, it ſhews that the Propoſition referred to, is in the ſame Book where the Reference ſtands.

PREFACE.

Moſt of the Lemma's, Definitions, and ſmaller Articles in the firſt ſix Books, lying diſperſed in ſeveral diſtant Places, it hath been thought convenient to add a Table for the more ready finding them, when occaſionally referr'd to, and to ſhew in what manner they are quoted.

There are alſo Directions to the Book-binder, where to inſert the Plates, if intermixed with the Print; and a Title Page to prefix to the ſecond Volume, for ſuch as ſhall not be diſpoſed to have the whole Work bound up in One; it being ſo printed as to be conveniently divided into two, the Second to begin with Book V. if the Plates be interſperſed; or the whole of the Text may be bound in one Volume, and the Plates in another.

Laſtly, there is added a Liſt of ſuch Perſons Names as have been pleaſed to incourage the Publication, by ſubſcribing to this Work; to whom I here return my particular Thanks for the Obligation.

AN ABSTRACT OF THE Principal Matters contained in the following WORK.

BOOK I.

Contains Five Sections.

SECT. I. *Treats of plain Viſion; wherein the Conſtruction of the Eye, the manner in which it is affected by viſible Objects, and the Rules by which we judge of their true Shapes and Diſtances, are conſidered; as alſo touching the Optick Angle, Erect Viſion, diſtinct and confuſed Sight, Light, Colours, and other Particulars relating to that Subject: all which are ſhortly diſcourſed of, by way of Introduction.* page 1.

SECT. II. *Shews the difference between* Drawing *and* Perſpective; *and what Aſſiſtance is particularly to be expected from either, towards compleating a finiſh'd piece of Painting.* p. 10.

SECT. III. *Gives an Account of the* Geometrical *and* Stereographical *Ways of deſcribing Objects on a Plane by Mathematical Rules. Of the firſt ſort are thoſe uſually called a* Plan, Ichnography, Orthography, Elevation, Profil, &c. *Of the other ſort are* Perſpective, Projection, *and* Tranſprojection: *The firſt of theſe is when the original Object is ſuppoſed to lye beyond the Picture, or the Plane on which the Deſcription is made; the ſecond is when the Object is between the Eye and the Picture; and the third is when the Eye is ſuppoſed to be between the Picture and the Object.* p. 13.

SECT. IV. *Deſcribes the* Preparatory *Planes, Lines, and Points uſed in* Stereography; *with their relations to each other, and to the Objects; together with the general Relations of the indefinite Images of Lines to their Originals.* p. 18.

SECT. V. *Treats of the Proportions of the Images of determinate Parts of Lines; and how theſe Proportions are affected by the alteration of the Height, Diſtance, or Place of the Eye.* p. 36.

BOOK II.

In this Book are taught the Rules of *Perſpective Geometry*, or the Method of deſcribing the Images of Points, Lines, and Figures, which lye in a given Original Plane; either by the help of the Objects themſelves drawn out in their proper Meaſures; or, without that Aſſiſtance, by having only the Image of ſome one Point or Line given, and knowing the Proportions of the Sides and Angles of the propoſed Object, with reſpect to that which is given; and is divided into three Sections.

SECT. I. *Teaches how to prepare the Original Plane and the Picture, for the Deſcription of the intended Objects; and to find the indefinite Images of any ſtraight Lines in the Original Plane, with their proper* Vaniſhing *Points; and the Reverſe.* p. 48.

SECT. II. *Shews how to find the Images of any propoſed Points, or Parts of Lines, in the Original Plane; and to divide any determinate Part of the Image of a Line, ſo as to repreſent any propoſed Diviſions of the Original. Alſo, by the given Image of any known Part of a Line, to draw the Image of any other propoſed Line, with any deſired Angle of Inclination, and in any Proportion to the firſt; and this without the Aſſiſtance of the Original Plane.* p. 55.

SECT. III. *Teaches how to deſcribe the Images of any Rectilinear Figures in the Original Plane; as* Triangles, Parallelograms, Pentagons, Hexagons, *&c. having any* *one Side given; alſo to deſcribe the Image of a Circle from any Diameter given; from whence general Methods are deduced, for drawing the Repreſentation of the Plan, or Ichnography of any Building, Fortification, Pavement, Garden, or any other Figures in the Original Plane, whether Regular or Irregular.* Page 71.

BOOK III.

Treats of the *Conick Sections*, ſo far as they may be conceived to be formed by the Image of a Circle ſeen in different Poſitions; and contains three Sections.

SECT. I. *Shews which of the* Conick Sections *is produced by the Image of a Circle, according to its Poſition with regard to the Eye and the Picture; and afterwards treats fully of the Nature and Properties of Lines* Harmonically *divided, and ſhews the Affinity of that kind of Proportion to* Stereography, *with its great and neceſſary Uſe in that Science.* p. 88.

SECT. II. *Gives a Deſcription of the ſeveral* Conick Sections, *and of ſuch of their Properties as are uſeful to the preſent Subject; alſo determines in a Circle in the Original Plane, the Lines whoſe Images ſhall become the* Axes, Diameters, *or* Ordinates *of the Section produced by the Image of that Circle. Likewiſe the* Axes, Conjugate Diameters, Ordinates, Centers, Foci, *&c. of the ſeveral produced Sections, are determined in the Picture, by the Help of the Image of any one Diameter of the forming Circle, without the Aſſiſtance of the Original Plane; with ſeveral other curious Matters relating to that Subject.* p. 102.

SECT. III. *Treats of the Tranſmutation of the* Conick Sections *into each other by the Rules of* Stereography; *and gives ſeveral eaſy Methods of deſcribing each of the Sections.* p. 132.

BOOK IV.

Treats of Points, Lines, and Plain Figures, not in a given Plane; and hath two Sections.

SECT. I. *Conſiders the Nature of* Vaniſhing *Points and Lines; their Generation and mutual relations; and ſhews how to find Vaniſhing Lines of all manner of Planes, having any Angles of Inclination to each other, or to the Picture, or the Reverſe; with the particular Limits of thoſe Problems.* p. 151.

SECT. II. *Gives great Variety of Methods for finding the Images of Points, Lines, Plain Figures, and Planes, whoſe relations to the Picture, or to any known Plane are given; and ſhews how to find the Seats of any given Points, Lines, or Plain Figures on any Planes propoſed, and alſo to determine their mutual Interſections.* p. 179.

BOOK V.

This Book is divided into three Sections.

SECT. I. *Shews how to find the* Projections *or* Shadows *of Points, Lines, and Plain Figures, on any one or more given Planes from a given* Luminous *Point, the Direct Image of the propoſed Object being given; the whole performed without the Aſſiſtance of the Original Objects, and in all poſſible Varieties of the Poſition of the Light, and of the Objects, with reſpect to the Planes on which the Projections are required to fall.* p. 209.

SECT. II. *Treats of the* Reflection *of Light from poliſhed Planes; wherein is ſhewn how to find the Appearance of the Light reflected on any Original Plane, from any determinate part of a Reflecting Plane, in all poſſible Situations of the Luminous Point and Reflecting Plane with reſpect to each other, or to the Picture, the Eye, or the Plane on which the Reflection is deſired.* p. 244.

SECT. III. *Treats largely of the Reflected Images of Objects in poliſhed Planes; and ſhews how to find the Reflected Images of Points, Lines, Planes, and Plain Figures in all various Situations of the Object, the Reflecting Plane, the Picture, and the Eye, with reſpect to each other.* p. 257.

ABSTRACT of the PRINCIPAL MATTERS

BOOK VI.

Treats of the Deſcription of ſolid Bodies; and is divided into five Sections.

SECT. I. *Treats particularly of each of the five* Regular Solids, *and of their* Ichnographies *and* Elevations *on Planes variouſly ſituated with reſpect to their Faces and Sides; and ſhews how to deſcribe their Images, either by the Help of their* Ichnographies *and* Elevations, *or, without their Aſſiſtance, by the Vaniſhing Lines and Points of their Faces and Sides. Alſo general Methods are propoſed for deſcribing their Shadows and Reflections by poliſhed Planes; and the Section cloſes by applying the Rules before given, to the Deſcription of any other ſolid Bodies, whoſe Surfaces are terminated by Planes.* Page 287.

SECT. II. *Treats of the* Cone, *and how to deſcribe its Image, and to determine its viſible Part, and likewiſe its Shadow on any given Plane; alſo to find the Boundary of the Light, which can enter its Concave Surface from any given Luminous Point; and laſtly to deſcribe the Images of its Sections by any propoſed Planes, and thence the Shadow of any ſtraight Line on its Surface: In all which Caſes, the Species of the ſeveral produced Curves, with their Diameters, Ordinates, and Tangents, are determined.* p. 313.

SECT. III. *Treats of the* Cylinder *after the ſame Method; likewiſe of the Sections of two Cylinders, or of a Cylinder with a Cone; and when theſe produce regular Curves, and when not, and how to deſcribe them; and hence Rules are given for the Deſcription of all kinds of Arches or Vaults, whether Circular,* Elliptical, *or Gothick, with their ſeveral Interſections, and* Miter Groyns, *whether the Arches be Right or* Rampant. p. 328.

SECT. IV. *Treats in the ſame manner of the Sphere or Globe, its viſible Part and Shadow on any given Plane; and likewiſe ſhews how from any given* Meridian *Circle of a Sphere, to find the Image of its* Equator, *or any Parallel of Latitude, or the contrary; and cloſes with a ſhort Account of the ſeveral Projections of the Sphere for Mathematical Purpoſes.* p. 345.

SECT. V. *Treats of the* Annulus, *and wherein it differs from the* Tore *of a Column; and ſhews how to determine the Viſible Part of its Exterior and Interior Surfaces; with an Application of the Methods there propoſed, to the finding the viſible Out Line of any Urn, Vaſe, or other Object, whoſe Sections by Planes parallel to its Baſe are Circles, let its Elevation be of what Figure it will.* p. 355.

BOOK VII.

This Book treats of ſeveral Matters relating to the general Practice of Painting; and contains eleven Sections.

SECT. I. *Of Fixed or immoveable Painting on flat Grounds, where the Picture is conſtantly to remain in the Place for which it was expreſly painted. Wherein is ſhewn a general Method of preparing and drawing a Picture to hide any Irregularity in a Room, either in Point of Height, Length, Breadth, or otherwiſe; ſo that the Picture, when placed in its proper Situation, ſhall tally with the other Part of the Building, and repreſent a Continuation of it in ſuch manner as may be deſired.* p. 367.

SECT. II. *Of* SCENOGRAPHY, *or the Conſtruction and Diſpoſition of* Scenes *in Theatres, with the Rules by which they ought to be painted; ſo that they may all correſpond with each other, and repreſent one intire View of the Deſign, without Breaks or Confuſion; which Subject is pretty largely handled.* p. 370.

SECT. III. *Of Painting on* Vaulted *Ceilings,* Domes, Cupola's, *or other Curvilinear Surfaces; giving a new and eaſy Method of* Reticulating *ſuch Surfaces, by which the propoſed Deſign may be the more juſtly deſcribed on them.* p. 381.

SECT. IV. *Of* Aereal Perſpective, Chiaro Oſcuro, *and* Keeping *in Pictures; ſhewing wherein they differ, and propoſing ſome Rules for the Painters Conduct therein: Alſo ſome Conſiderations touching the Difference between a painted Picture, and the Repreſentation of Objects in a plain Looking-Glaſs, and in the* Camera Obſcura. p. 383.

SECT. V. *Of the Poſition of the Picture; giving an Account of the various Situations uſually given to Pictures, and of the manner in which they ought to be painted, with the Objects proper for each Poſition.* p. 386.

contained in this WORK.

TABLE for the more ready finding ſuch *Definitions*, *Lemma's*, and other Articles, as lye diſperſed in this WORK.

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BOOK

BOOK VI.

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17	Running Title,	Of the different Methods, &c.	Objects by Mathematical Rules.
32	Line 15,	EF and *o*P	*o*P and EF
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Upon a ſecond Review of this Work, the following *Errata* have been diſcovered, beſides thoſe before ſet down.

Page	*for*	*read*
Preface, 2. *Line* 5. *in ſome Copies*	15	15th
Ibid. 6. *Line* 20.	Perſpecuity	Perſpicuity.
In the direction for inſerting the Plates 7, 8, 9, &c. *in ſome Copies*	pag. 2.	pag. 72.
Page 127. *Line* 11.	*qo*: *gl*:: *bo*: *ba*.	*qo*: *ql*:: *bo*: *ba*.
142. *Line* 3.	CL	*c*L.
175. *Line antepenult.*	placed at O	placed at I.
213. *Line* 7.	Side O	Side of O.
222. *Meth.* 7. *Lin.* 8.	*Tn*	*Tw.*
233. *Schol. Meth.* 4. *Lin.* 2.	FFGH	EFGH.
260. *Reference* f.	Caſe 1. prob. 43.	Caſe 1. prop. 43.
296. *Cor.* 2. *Lin.* 2.	*s a*	S *a*.
303. *Line* 3.	ya.	*y*a.
307. *Cor.* 1. *Lin.* 5.	*d*D, *d*C.	dD, dC.
Ibid. *Lin.* 6.	*d*D.	dD.
358. *Art.* 4. *Line* 7.	been	being.
360. *Cor.* 2. *Line* 12.	two thoſe	thoſe two.
In Plate 23. *in ſome Copies.*	Fig. 72. N° 2.	Fig. 71. N° 2.

STEREOGRAPHY,
OR A
COMPLEAT BODY
OF
PERSPECTIVE,
In all its BRANCHES.

BOOK I.

SECTION I.

Of plain Viſion.

THE chief affections or properties of Objects, perceivable by the Eye, are *Figure* and *Colour*, both which become viſible only by means of the Light which ſtrikes the Eye from the Object; Light being the Medium of Sight, without which that ſenſation cannot be excited.

Figure belongs more particularly to the Object itſelf, and ſeems inſeparable from the notion of Matter; it is that which terminates its Extenſion, and gives bounds to the Space it occupies; it is the ſame in the dark as in the light, and comes within the notice of the ſenſe of Feeling, as well as that of Sight; but *Colour* is only perceivable by the Eye, and is not inherent but accidental to Bodies, it ariſing from the nature of the Light which the Objects reflect to the Eye, and is therefore in ſtrictneſs a property of Light itſelf, and dependent upon it.

Light may be conſidered either as Uncoloured or Coloured.

By *Uncoloured* Light is meant the light which Bodies reflect to the Eye, according to the different figures and poſitions of their Surfaces, abſtracted from any *Teint* or *Colour*; and takes in all the degrees of Light and Shade, from the brighteſt White to the deepeſt Black. This ſort of Light is produced by a uniform reflection of all the rays of Light indifferently, whether in greater or ſmaller quantities, ſo as no one kind do predominate over another. Bodies, whoſe Surfaces are diſpoſed to reflect Light in this manner, are therefore uſually termed *Uncoloured*, they producing no ſenſation of Colour in the beholder, to whatever degree of Light they are expoſed; but this does not in the leaſt hinder their *Figure* from being perceived, of which the Eye is enabled to make a judgment by the various degrees or quantities of the Light, which the different parts of the Object reflect, and in many caſes with greater certainty, than when it is attended with variety of Colours.

Hence it is that Objects may be very juſtly repreſented, and a true Idea of them raiſed, ſo far as relates to their Figure and ſituation, without any reſpect had to their Colour. A drawing in black and white may give as perfect a reſemblance of the Features and Air of a Face, or of the parts and proportions of a Figure, a Building, or a Landſkape, as if it were done in colours. But the addition of Colouring gives life to the Picture, and takes in the only remaining circumſtance relating to viſible Objects, which *Nature* has thought fit to make the Eye capable of perceiving, if we only except *Motion*, which it is not in the power of Painting alone to repreſent.

Colour, as the great Sir *Iſaac Newton* has ſhewn, is the ſenſation produced by the impreſſion made on the Eye by certain kinds or ſorts of Rays of Light, ſeparated from others by means of their different *Refrangibility* and *Reflexibility*, whereby they are divided into ſeveral parcels, each endowed with its own diſtinct *Colour-making* Power: and Bodies, whoſe ſurfaces are diſpoſed to reflect one kind of thoſe Rays more abundantly than others, exhibit, and are ſaid to be of that colour which is peculiar to the Rays they moſt copiouſly reflect; and the infinite diverſity of the ſurfaces of Bodies, and the different mixtures and modifications of different colour-making Rays thereby occaſioned, muſt therefore produce that infinite variety of Colours which beautifies the face of Nature.

From this diſpoſition in coloured Bodies to reflect only part of the Rays of light which fall on them, and to ſuppreſs others, it follows, that they do not, when expoſed to equal degrees of light, appear ſo bright as thoſe which are uncoloured; nevertheleſs while the ſurface of the Object is every where diſpoſed to reflect the ſame kind of Rays, tho' in greater or ſmaller quantities, according as its ſeveral parts are expoſed to the light, the ſhape of the whole may be thence known, the proportion of the quantity of light reflected by every part being preſerved: but in Bodies, whoſe different parts reflect different Colours, their true ſhape is more difficult to be judged of.

For as a Body, whoſe ſurface is diſpoſed to reflect all ſorts of Rays indifferently, and is therefore properly of no colour, will, by being expoſed to different degrees of light, appear proportionally lighter or darker, according to the quantity of Light which falls on it; and its ſeveral parts will diſtinguiſh themſelves from each other in brightneſs, according to their ſituation with reſpect to the Light and the Eye, by which different appearance of Light, the Figure and poſition of the parts of the Object are judged of; ſo one diſpoſed to reflect only one ſpecies of Rays, ſuch as the *Red*, will put on all the different ſhades or *hues* of that colour, in proportion to the quantity of light it receives, and thereby exhibit a proportional diſtinction of its parts, by the different brightneſs or darkneſs of their colour, which however continues ſtill of the ſame ſpecies. The ſame is to be underſtood of any other ſpecifick uniform Colour, it being evident, that in every degree of uncoloured Light, a proportional quantity of Rays of each ſpecifick ſort muſt be contained; and while this proportion of reflected light from the ſeveral parts of an Object is preſerved, the judgment of its ſhape is not diſturbed, whatever its reigning or general Colour may be. But as ſome Colours carry with them a much greater proportion of Light than others, ſo that of two bodies expoſed to the ſame degree of light, that which reflects *(ex. gra.)* the *Yellow* Rays, ſhall appear much more enlightened than that which reflects the *Red*; ſo in Bodies whoſe ſeveral parts reflect different ſpecifick Colours, theſe by bringing with them different proportions of light in reſpect to the whole quantity, render the judgment of the ſhape of the ſurface, from whence they are reflected, more precarious; there not being that due proportional difference of light and ſhade, which the figure and poſition of the parts of the Object require: and hence it happens, that by an Artful diſpoſition of Light and Colours on a plain ſurface, it ſhall appear to have great variety of cavities and eminences, and be capable of producing that agreeable deception of the Eye, which renders the Art of Painting ſo admirable.

But the conſideration of *Colour* not being neceſſary for the explanation of what we intend to offer relating to *Viſion*, we ſhall here conſider Light abſtracted from that property, and as being uniformly reflexible; and on this ſuppoſition,

1. All Bodies, ſo far as they are the object of Sight, may be conceived as ſurfaces made up of an infinite number of ſmall points, each of which reflects the Light that falls on it, towards all ſides in ſtraight lines or *Rays*, ſo that the Eye, in whatever poſition it be with reſpect to any point of an Object, muſt receive ſome or other of the Rays proceeding from it, if no other Object lie in a direct line between the Eye and it to obſtruct their paſſage.

2. Now the Eye is of the nature of an Optick Glaſs, and by means of its conſtruction, all Rays of light which proceed from any point, and fall on its whole aperture, are by refraction collected again in its bottom, where the Optick Nerve ſpreads itſelf ſomewhat like net-work (whence that part is called the *Retina)* and there form the Image of the point from whence they came, after the ſame manner as a Convex Glaſs applied to a ſmall hole in a darkened Chamber, throws the Images of the Objects without, on a paper placed at a proper diſtance behind the glaſs to receive them; and by the impreſſion made by ſuch Image on the *Retina*, the ſenſation of Sight is produced.

Fig. 1. Let Y repreſent the Eye, HI the *Cryſtalline* humour of a Spherical ſhape, O the

Center of the Eye, DE its aperture, and FG part of the *Retina* of a concave ſpherical figure; ABC is an Object reflecting from its ſeveral points, A, B, and C, ſeveral rays of light ſpreading over the whole aperture of the Eye. Now, by the refracting powers of the humours of the Eye, all the Rays which proceed from any one of thoſe points, are again collected into a point in the *Retina*, and there form the Image of the point from whence they did proceed, which Image will be in that part of the *Retina*, where a ſtraight line from the Original point paſſing through the Center of the Eye falls [*]. Thus the rays AH, AO, AI, coming from A, and ſpreading over the aperture of the Eye, are again collected and form the Image of that point on the *Retina* at *a*, where the line AO which paſſes through the center O, meets it: in like manner the image of B is deſcribed in *b*, and that of the point C in *c*; ſo that the Image of the original line is inverted on the *Retina*. Therefore to determine the place of the Image of any point on the *Retina*, we need only conſider that ſingle Ray which proceeds from it, and paſſes through the Center of the Eye, which therefore may be called the *Optick Ray* of that point, and is the Axe of the two Cones of Rays AHI and HI*a*, of which A and *a* are the Vertices, and HI the common Baſe, which two Cones are by the Opticians called a *Pencil* of Rays.

3. The line BO which falls directly on the Eye, is the Eye's Axe, and all rays are ſaid to be more or leſs oblique, as they make a greater or leſs Angle with it. The point *b* where the line BO cuts the *Retina*, may be called its Center, and the nearer the Image of any Object falls to that point, the more clearly and diſtinctly it is ſeen.

This is by experience found true, and the reaſon of it may depend on that given in Dioptricks, why the Images of Objects which have an oblique ſituation with reſpect to an Optick Glaſs, are not ſo clear and well defined, as thoſe of Objects which are more direct, *viz.* becauſe the Rays which compoſe the Pencil, coming from an oblique point, are not all exactly refracted into the ſame point of the *Focus* or diſtinct Baſe, but ſome of them croſs before they arrive at it, and others paſs beyond it before their Union, which renders the Image both leſs defined and darker, by reaſon of the loſs of thoſe Rays of light which do not enter into its compoſition, as they would have done in a more direct ſituation of the Object.

4. Hence it is, that although the Eye, by means of its ſpherical ſhape, can take in at once a very large Area or extent of Objects, and have a confuſed view of ſuch as lie greatly remote from its Axe, yet it can ſee diſtinctly but a ſmall compaſs at a time, perhaps not exceeding an Angle of 1 or 2 degrees on either ſide; but the imperfect view we have of the Objects around, ſeems intended by Nature only to warn us of their neighbourhood, and to prompt the Eye to turn towards them for a clearer ſight, either for pleaſure or preſervation, which Motion is ſo quick and ſo little attended to, that the whole appears as if it were ſeen together.

5. The Optick Rays of an Object meeting and croſſing each other in the Center of the Eye, as already deſcribed, form two oppoſite and ſimilar Pyramids, of which the Center of the Eye is the common Vertex, and the Object and its Image on the *Retina* are the Baſes; and if the Eye's Axe be directed to the Center of the Object, as in order to diſtinct Viſion it ought to be, the Image form'd on the *Retina* will be ſimilar (but in an inverted poſition) to a ſection of the Optick Pyramid any where between the Object and the Eye, by a Plane perpendicular to the Eye's Axe: it appearing from the *Phænomena* of the *Camera Obſcura*, that the diſtinct Baſe of an Optick Glaſs, where the Rays of Objects lying in a Plane perpendicular to its Axe are united, is a Plane or nearly ſo, and parallel to the other.

6. It is uſually laid down as a Maxim, that Objects appear greater or leſs in proportion to the Angles under which they are ſeen; which would be ſtrictly true, were the *Retina* a portion of a Concave Sphere, having the Center of the Eye for its Center; but as the *Retina* muſt conform itſelf to the true *Foci* of the Rays which enter the Eye from different points, and which form a diſtinct Baſe not much differing from a Plane, it ſeems neceſſary that ſo much of the *Retina* as is fitted to receive a diſtinct Image at the ſame view, ſhould by the Muſcles of the Eye, or otherwiſe, be brought nearly to a Plane, or at leaſt to a portion of a Sphere of a much larger Radius, than the diſtance between the Center of the Eye and the *Retina*: ſeeing otherwiſe the

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[*] This not being deſigned as a Treatiſe of Opticks, the ſmall difference ariſing from the double refraction of the Ray which paſſes through the Center of the Eye, at its entring the cryſtalline humour, and its emerſion out of it, is neglected; ſeeing the incident and refracted rays are parallel, and the diſtance between them ſo inconſiderable, that they may be conceived to be one continued ſtraight line.

Rays could not be exactly united upon it, which would render the Image, and conſequently the ſenſation imperfect.

7. Now it being moſt reaſonable to believe, that all Objects appear bigger or leſs, in proportion to the ſpaces their Images occupy on the *Retina*, and theſe ſpaces not being exactly proportional to the Angles under which the Objects appear, as ſhall be ſhewn; it thence follows, that Objects do not appear ſtrictly in proportion to the Angles under which they are ſeen.

Fig. 2. N°. 1. To explain what has been ſaid: Let AE be an Object perpendicular to OC the Eye's Axe, and let O be the Center of the Eye, through which the Optick Rays AO, BO, CO, &c. paſs, and paint the Images of the points A, B, C, in the bottom of the Eye at a, b, c: if this Bottom were a portion of a Sphere having O for its Center, it is evident that the Images ab, bc of the parts AB, BC of the Object, would be the Arches of the Angles under which AB and BC are ſeen, and would therefore be proportional to thoſe Angles; but if the place of the union of the Rays proceeding from A, B, and C, be in a Plane perpendicular to OC, the ſtraight line ae will repreſent a ſection of that Plane, and then the Images ab, bc of the parts AB, BC, will be proportional to thoſe parts, but not to the Angles under which they appear, which is thus ſhewn.

Let the parts AB, BC of the Object be equal, it is evident from the ſimilitude of the Triangles OAB, Oab, and OBC, Obc, that the Images ab and bc are alſo equal; it muſt next be ſhewn that the Angle AOB is leſs than the Angle BOC.

In the Triangle AOC, the ſide AO is larger than OC, as ſubtending a greater Angle[a]. Divide AC in F, ſo that AF may be to FC as AO to OC, and draw OF; then AF will alſo be larger than FC, and conſequently larger than AB the half of AC; now by conſtruction, the Angles AOF and FOC are equal[b], but the Angle BOC is bigger than FOC, and the Angle AOB is leſs than AOF, the Angle AOB is therefore leſs than BOC, and conſequently the Images of the parts AB and BC appear equal, although ſeen under different Angles.

[a] 19 El. 1.

[b] 3 El. 6.

In the ſame manner it may be proved, that if AE were produced, and any other diviſion taken in it equal to AB, but farther diſtant from the Axe OC, it would be ſeen under a ſtill ſmaller Angle, and yet have an equal Image; this diſproportion nevertheleſs between the Image and the Optick Angle decreaſes, the nearer the points of diviſion approach to the Axe of Sight, and becomes inſenſible in ſuch ſmall Angles as the Eye is fitted to receive diſtinctly at the ſame view.

8. Likewiſe if we conſider the ſame Object, placed at different diſtances from the Eye, it will be found that its apparent ſize is not varied in proportion to the Angles under which it is ſeen, but in proportion to the Tangents of half thoſe Angles; the Axe of Sight being always ſuppoſed to be directed to the Center of the Object.

Fig. 2. N°. 2. Let O be the Center of the Eye, and OI its Axe; from the Center O deſcribe an arch lgm touching the *Retina* in its Center g, and let the ſtraight line ab repreſent a ſection of the *Retina*, ſo far as it can be extended to receive a diſtinct view of the Object.

The Object AB being placed at the diſtance OG from the Center of the Eye, appears under the Angle AOB, and its Image is painted at ab; now AB being ſuppoſed perpendicular to OG, and biſected by it in G, the Angle AOG is half the Angle AOB, and ga the half of the Image ab, is the Tangent of the Angle AOG, putting Og the diſtance between the Center of the Eye and the *Retina* as Radius; if the Object be removed to the poſition CD, it is then ſeen under the Angle COD, and gc the half of its Image, is the Tangent of COH half the Angle COD; and in like manner ge is the Tangent of EOI, half the Angle EOF under which the Object appears in the poſition EF: whence it follows, that the Images of AB, CD, and EF, are to each other as the Tangents of half the Angles under which they are reſpectively ſeen, and are not therefore in proportion to the Angles themſelves, ſeeing Angles are not proportional to their Tangents; the Tangent of a ſmaller Angle being leſs in proportion to the Arch which meaſures it, than the Tangent of a greater Angle is to its Arch: and the diſproportion of the Images of Objects to the Angles under which they are ſeen at different diſtances from the Eye, will appear to be ſtill greater, if it be conſidered, that as Objects recede from the Eye, the diſtance between the Center of the Eye and the diſtinct Baſe or *Focus* of the Optick Rays is leſſened, which of conſequence leſſening the Radius Og, the Tangents will be proportionally diminiſhed: but as what is here ſaid is not meant of any near Objects, but of ſuch as are at a conſiderable diſtance from the Eye, where the

Plate 1 Book 1 Sect. 1. 2. 3.

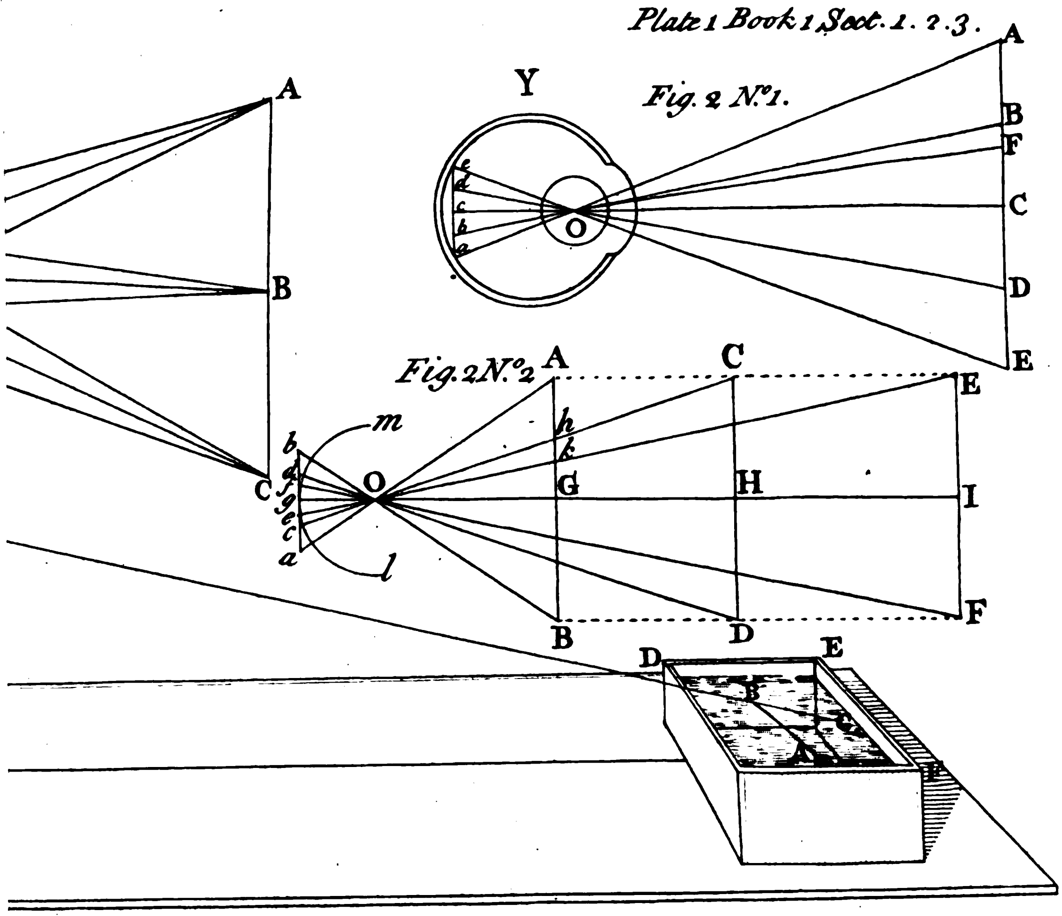
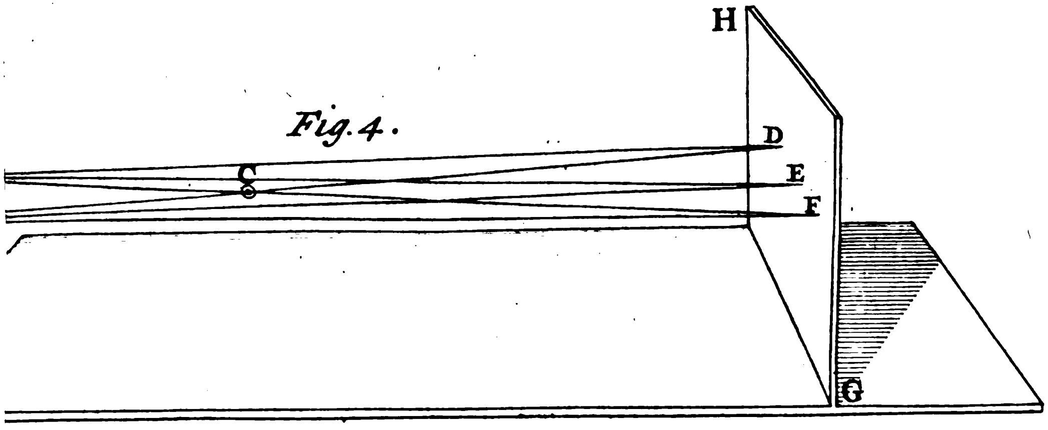


Fig. 4.



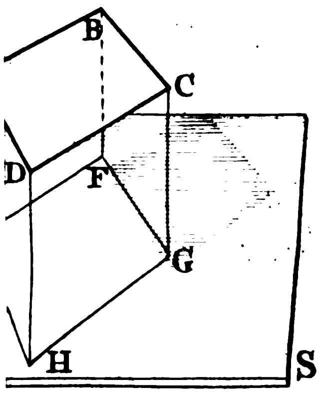


Fig. 6.

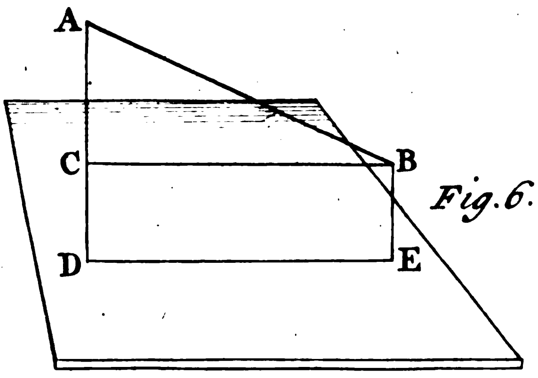
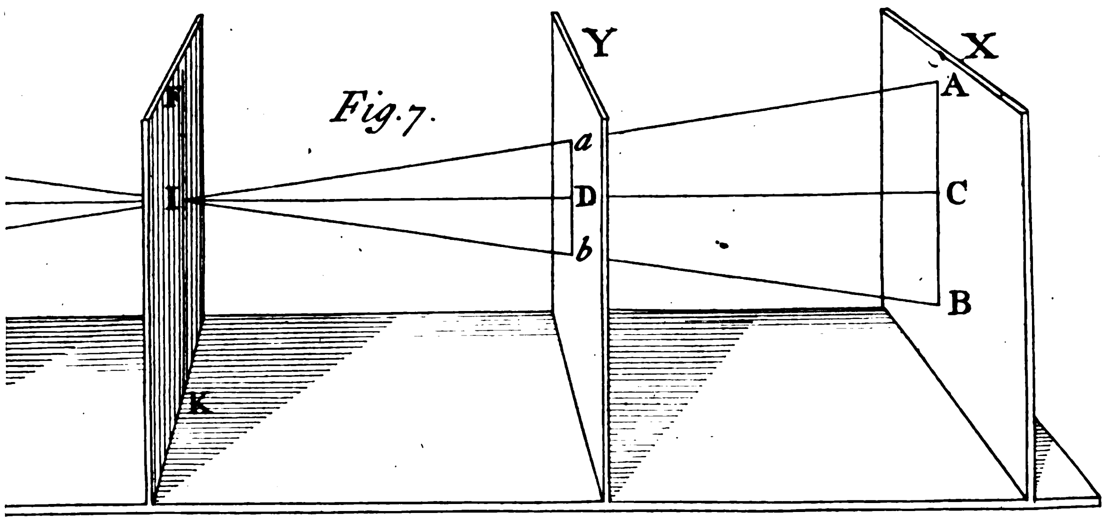


Fig. 7.



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the diverſity of Diſtances has but a ſmall effect on the Focal length, the above Poſition becomes more nearly true. And indeed on this is founded the General Rule laid down by Opticians, that the apparent Diameter of the ſame Object, at ſeveral Diſtances from the Eye, is reciprocally Proportional to thoſe Diſtances, which is thus demonſtrated:

In the Similar Triangles EIO, kGO $\quad EI = AG : kG :: OI : OG$
And in the Similar Triangles AGO, Oga $\quad AG : ag :: GO : Og$
And in the Similar Triangles kGO, Oge $\quad kG : eg :: GO : Og$
Conſequently $\quad AG : kG :: ag : eg :: OI : OG$

that is, *ag* the Image of the nearer Object A G, is to *eg* the Image of the more diſtant E I, reciprocally as O I the greater diſtance is to O G the leſs.

9. As Objects by their Diſtance appear ſmaller, ſo they alſo loſe in Proportion their Strength and Diſtinctneſs; for the Rays of Light which the Object reflects being *Diverging*, the farther they proceed before they enter the Eye, they ſpread the wider aſunder, and therefore fewer of them can be received by the Eye at a time, and conſequently the Image thoſe Rays make, will be the fainter; the denſity of Rays proceeding from a Radiant Point at ſeveral diſtances, being reciprocally as the Squares of thoſe diſtances: beſides, the Angles under which the Minute Parts of diſtant Objects appear, become ſo ſmall, that they do not ſenſibly affect the Eye, which makes thoſe Parts in a manner diſappear.

10. There is alſo a difference in the apparent Light of the ſeveral Parts of an Object, according as their Surfaces are more or leſs directly expoſed to the Eye, although the Light received by the whole Object be uniform and equal: it being evident, that the more directly any Surface is placed before the Eye, the more diſtinctly every point of that Surface is ſeen; and conſequently more of the Light which it reflects, will be received by the Eye in ſuch a Poſition, than when it is ſeen more ſlantingly, whereby its ſeveral Parts appear more crowded together, and in ſome meaſure to hide each other: and it is by the help of this different apparent Light of the Parts of an Object, that its true ſhape is the better judged of.

11. And as the Light, ſo the Colour of Objects receives a conſiderable diminution and alteration by their diſtance or obliquity; for the Colour of Objects proceeding from the diſpoſition of their Surfaces to reflect certain Rays of Light more copiouſly than others, as has been already obſerved, thoſe Rays in their progreſs from the Object becoming more rare, and being in their paſſage mixed with other Rays of Uncoloured Light, the true Colour of the Object is thereby much diluted, and conſequently affects the Eye more faintly: or if thoſe Rays in their way to the Eye happen to be mixed with other Rays of a different Colour, the original Colour of the Object will receive a tincture of the latter; it being plain from Experience, that the apparent Colour of all Bodies is in ſome meaſure affected by that of others which lie near them.

From a due conſideration of the various appearances of the Light and Colour of Objects, and of the diminution of the Teints of each particular Colour according to their different diſtances and ſituation, and of the effects of the mixture of Rays of ſeveral kinds, divers uſeful Rules might be had for Colouring in Pictures, ſo far as relates to what is called *Aerial Perſpective*; but that depending more on experience and obſervation of Nature, than on ſtrict Mathematical Rules, doth not fall ſo directly within our Subject: we ſhall therefore leave this Hint to be proſecuted by ſuch, whoſe Province it may more properly belong to.

12. It may ſeem difficult to conceive why Objects appear Erect, notwithſtanding their Images in the Eye are Inverted; but the reaſon of this may be, that all Rays received by the Eye, make an impreſſion on the *Retina* according to the direction with which they enter the Eye, and that impreſſion is felt as coming from that quarter, to which each particular Ray is directed; and therefore the Point, from which ſuch Ray proceeds, is judged to be ſomewhere in a ſtraight line with the Ray itſelf, and conſequently the whole Object is judged to be in its own natural ſituation.

Thus the Optick Ray which proceeds from the point A, paſſes in a ſtraight line thro' Fig. 2.
O the Center of the Eye, and paints its Image at *a* on the *Retina*, and this impreſſion N°. 1.
being made with the direction A O, it is therefore felt as proceeding from the point A, or at leaſt from ſome other point in the ſame ſtraight line; and in the ſame manner the impreſſion made on the *Retina* at *e*, is felt as coming from ſome Point in the line O E, and ſo of all the intermediate Points; ſo that although the Image of the line A E be inverted on the *Retina*, the upper part A being repreſented on *a* the lower part

of the *Retina*, and the lower part E repreſented at *e*, yet the whole line is judged to be in its proper Poſition from the Reaſon before given: and in like manner altho' the Images of thoſe parts of an Object which lie towards the right hand, are repreſented on the left ſide of the *Retina*, and thoſe that lie on the left are deſcribed on the right, yet the Judgment guided by the direction of the impreſſion gives the Object its true ſituation.

13. Hence it is that when any Rays happen, by paſſing out of one Medium into another, to be refracted, ſo as not to go on in a ſtraight line to the Eye, but to be bent at the common Surface of the two Mediums, the Eye judges the Object to be in a different place from where it really is, according to the direction of that part of the Ray which it receives.

Fig. 3. Let DEF repreſent a Veſſel, in the bottom of which let A repreſent a piece of Money, or any other fixed Object, and let I be the Eye, in ſuch a Poſition that a ſtraight line IA cannot paſs from the Object to the Eye, without being intercepted by the ſide of the Veſſel, ſo as to hide the Object; if afterwards a competent quantity of Water be poured into the Veſſel, the Eye and the Money remaining unmoved, the piece will become viſible: for the Ray AB being refracted on the Surface of the Water, will change its direction and paſs on from B to I, ſo that the Object will be viſible by the Ray IB, and according to that direction, it will be judged to lie ſomewhere in the line IBC.

14. From what hath been advanced touching the nature of *Viſion*, it may be inferred, that all Objects appear to the Eye with ſuch Proportions of Colour, Size, and Diſtinction, as their Images have on the *Retina*; and that the Eye does not ſee Objects as they are in themſelves, but only as they are repreſented there, according to the circumſtances of their ſituation and Diſtance. But the apparent Bigneſs and Shape of Objects ſeen by the Eye, is greatly different from that which the Judgment gives them; the firſt is governed by the Proportions of their Images on the *Retina*, the ſize of which is determined by certain Rules; but the Judgment acts on different principles, and judges of the real Bigneſs, Shape, and Diſtance of an Object, by comparing it with others of the ſame kind, or with ſuch whoſe uſual ſize is known, and by their different apparent Bulk, ſtrength of Colour, or diſtinction of Parts, or by ſeveral other methods of compariſon furniſhed by Experience, although by Cuſtom we make ſuch Judgments without attending to the means we uſe to form them.

15. Hence the impreſſion made by an Object on the Organ of Sight, the perception of which is properly the *Senſe of Seeing*, differs from the Judgment concerning the Object itſelf, formed in the Mind in conſequence of that impreſſion, which may be called the *Art of Seeing*, as a bare perception of the ſhapes of the Characters and Letters traced on a Paper, differs from the Art of reading, and underſtanding what is written. In order to the Senſe of Seeing, a proper diſpoſition of the Organ and a due Medium and Diſtance are only neceſſary; but for the Art of Seeing, frequent repeated Experience and Obſervation are requiſite, to enable the Mind to form a true Judgment of the Object, from the impreſſion it makes on the Eye: but the daily practice of this Art from one's Infancy renders it ſo natural and familiar, that by degrees the Idea of the Object ſeems (eſpecially in the uſual and common inſtances) to be immediately annexed to the Senſation, and the Judgment without any remarkable Act of Reflection, readily underſtands this ſilent language of the Eye.

And from this facility of judging of the real Shape or Figure of Objects ſeen by the Eye, ariſes a difficulty in drawing the Images of Objects at Sight: for in tracing out the ſeveral parts of the Image, we are apt to give them the meaſures ſuch as the Judgment upon ſight conceives the Originals to have, and not ſuch as they really appear to the Eye; which laſt nevertheleſs are the true meaſures the Images ought to have, in order to make them truly repreſent the Original: and it therefore requires ſome ſtudy and application to overcome that deception, and to be able truly to diſtinguiſh between the Shape of the Images of Objects in the Eye, and the Idea of thoſe Objects raiſed by their Images.

16. That the Judgment interpoſes in the raiſing of Ideas of Objects, different from the real Figure made by their Images in the Eye, will appear to be true, if it be conſidered, that when the uſual methods of judging become either impracticable, unfit, or uncertain, the Judgment made of the Size, Shape, or Diſtance of the Object, becomes either falſe or ambiguous; as in looking over a large Plane (either of Land or Water) at a diſtant Object, where there is no variety of Objects intervening, there, unleſs the uſual Size of the Object obſerved be known, its diſtance becomes altogether uncertain.

This kind of deception is no where greater than in looking up to the Heavens, where, for want of intermediate Objects to judge of the Diſtance by, the Sun, Moon, and Stars are imagined to be in a manner infinitely nearer both to us and to each other, and proportionally leſs, than what by other Rules and Methods contrived by Art they are found to be. And hence it is, that when the Sun, or Moon are near the Horizon, they appear or are judged to be conſiderably larger than when higher up in the Heavens; for we being accuſtomed to look along the ground, and to ſee the diſtant proſpect of Hills, with many other intermediate Objects which ſerve to ſet them off, acquire thereby a habit of judging the viſible face of the Earth to be a vaſt extended Plane on which thoſe Objects ſtand, and the Luminary, when near the Horizon, evidently appearing to be beyond the moſt diſtant viſible ground, is for that reaſon judged to be much farther from the Eye, than when it is ſeen higher up, where no Objects intervene by which the diſtance may be collected; and as the apparent Diameter of the Luminary is the ſame in either caſe, its greater imaginary diſtance makes it to be judged the larger. It is true, this appearance may in ſome ſort be aſſiſted by the Refraction occaſioned by the Vapours in the Air near the Horizon, but the principal cauſe of this Phenomenon is certainly owing to the reaſon abovementioned; which will appear ſtill more evident, if it be conſidered, that in looking at any Objects in the Sky, which have a conſiderable Elevation, we are apt to imagine them to hang perpendicularly over ſome part of the ground, the diſtance of which place from our own ſtation, is generally conceived to be the leſs, as the Angle of Elevation of the Object is greater. Upon the whole, in order to judge of the true Diſtance of an Object, its Size muſt be ſome way known, and to judge of its Size, its Diſtance ought to be aſcertained: when either of theſe is given, the other is more eaſily judged of; but when both are unknown, then the uſual methods of judging become uſeleſs, and recourſe muſt be had to other Rules whereby to form a true Judgment of the Object.

17. And as the Size and Diſtance, ſo likewiſe the true Shape or Figure of Objects, in ſome circumſtances, may be very uncertainly judged of; for as the whole compaſs of what can be ſeen at one view, is deſcribed on the *Retina* as a Plane, no one Image can be ſaid to be more diſtant than another; nor can the Cavities or Eminences of any particular Object be otherwiſe repreſented, than by the different degrees of Light *which its Parts receive*, according as they are more or leſs expoſed to it; ſo that in order to judge of the true Shape of an Object, it may often be neceſſary to know from what quarter the Light falls on it, ſeeing the ſame appearance of Light which, coming from one hand, would make the Object ſeem protuberant, if it proceeded from the other ſide, would make it to be judged hollow, and thoſe Parts which in the one caſe would ſeem to come nearer the Eye, in the other would appear to recede from it. But this ambiguity of Judgment happens more commonly when the Surfaces of the Objects are Curvilinear, there being a greater certainty in judging of the Figures of Bodies which are bounded by Planes. Thus if a Hemiſphere were placed with its flat ſide againſt a Wall, and expoſed to the Eye at ſome diſtance, it will be doubtful whether it be concave or convex, or even whether it be not a plane, if by other neighbouring Objects it cannot be diſcovered which way the Light falls on it.

18. And by reaſon that Diſtance makes the ſmall Parts of an Object in a manner diſappear, although the whole may be ſtill ſeen under a conſiderable Angle; it is uſual in Works of Art, ſuch as Pieces of Architecture, and Statues or other Ornaments relating to them, when they are to be ſeen at a conſiderable height or diſtance, to make the Features and other ſmall Parts which are deſigned to have an effect, much ſtronger in proportion to the whole, than if they were to be ſeen nearer at hand; otherwiſe the Work would appear flat, and its Parts not ſufficiently diſtinguiſhable: a due obſervance of which Rule adds a great beauty to the Deſign, and recommends the ſkill of the Artiſt. The ſame Rule has its place alſo in Painting, by the help of which the Eye may be agreeably deceived, and the ſeveral Objects be made to appear more or leſs diſtant, in proportion as they are deſcribed leſs and fainter, or bigger and ſtronger than the Life.

19. Nevertheleſs this Rule is to be uſed with diſcretion; for generally ſpeaking, in Pieces which are to be ſeen at a conſiderable diſtance, all ſmall and minute Ornaments ſhould be avoided; for if they were put there in their proper dimenſions, they could not be ſeen, and ſo would be an unneceſſary trouble: on the other hand, if thoſe Ornaments, which by their uſual ſize ought at the propoſed Diſtance to diſ-

appear, be made large and ſtrong enough to be diſtinctly ſeen, the Eye for that reaſon will judge them nearer than they really are, and bring along with them the other Parts of the Work, ſo as to leſſen the appearance of the whole.

This ſeems to be exemplified in the Figures ſet on the Weſt Front of St. *Paul*'s Cathedral, which by reaſon of their Gygantick ſize, are judged to be ſo much nearer the Eye, to reduce them to the common appearance of a human Body, that the grandeur of the Front on which they ſtand is thereby very much diminiſhed, as may appear at firſt ſight to a ſkilful Obſerver*.

20. But Bigneſs and Diſtance in themſelves being only comparative Quantities, no Object or Diſtance being otherwiſe to be called great or ſmall but by compariſon with ſome other; hence it proceeds, that in very large Buildings, when each part bears a due Proportion to the whole, nothing looks upon the general view to be monſtrous or too large: as on the other hand, in ſmall Deſigns, when the ſeveral Parts and Ornaments are proportionally leſſened, the beauty of the whole is ſtill preſerved. Thus two Pictures of the ſame thing, although the one be much larger than the other, may yet preſerve an equal Proportion, and be equally beautiful and agreeable to the Eye, and convey alike a true Idea of the Object repreſented.

21. Sight has hitherto been conſidered as performed by a ſingle Eye, we will now enquire what difference ariſes on looking with both Eyes at once: and in order thereto, we ſhall take notice of another property of the Eye not yet mentioned, which is its power of adapting itſelf for a diſtinct Viſion of Objects at ſeveral Diſtances.

Whether this be done by varying the Degree of Convexity of the Eye by the help of its Muſcles and Humours, or by the *Cryſtalline* Humours changing its Diſtance from the *Retina*, or by both, or any other means, it is not material here to enquire; but it is certain, were the Eye of a fixed form, as an Optick Glaſs, and the *Retina* always at the ſame Diſtance from the Center of the *Cryſtalline* Humour, it could then only ſee Objects diſtinctly at one determinate Diſtance; for the focal Diſtance of a Glaſs alters with the Diſtance of the Object, the greater Diſtance of the Object bringing the *Focus* nearer to the Glaſs, as the approach of the Object ſends the *Focus* farther off, as is demonſtrated in Dioptricks: ſo that if the *Cryſtalline* Humour were immoveable, and the Convexity of the Eye always the ſame, the different Diſtances of the Object would cauſe the place of Union of the Rays to fall ſometimes ſhort of, and ſometimes beyond the *Retina*, in either of which caſes there could be no diſtinct *Viſion*, which could only be at that particular Diſtance of the Object, from which the Rays proceeding to the Eye would be united on the *Retina:* and this is the reaſon why the Eye cannot, without the help of Glaſſes, ſee Objects that are brought very cloſe to it, for want of ſufficient power to change its form, ſo as that the *Retina* may be in the *Focus* of the Rays.

22. Hence it follows, that the Eye cannot ſee diſtinctly at the ſame time two Objects at different Diſtances, although they be nearly in the ſame line; but if the nearer Object be diſtinct, the other will be confuſed, and *vice verſa.* But this is to be underſtood of moderate ſmall Diſtances, for when Objects are a great way off, their different Diſtances from the Eye have but a very ſmall effect on the *Focus*, and make no perceiveable change at all therein when the Rays become ſenſibly parallel,

23. From this property of the Eye (were it ſingle) we might in ſome meaſure be able to judge of the different Diſtances of Objects, but by the help of two, the Judgment is much more certain, whether we look with both Eyes at once, or alternately with one after the other.

In looking with both Eyes at once at an Object, their Axes croſs each other at the Point obſerved, making an Angle, by the Bigneſs of which the Diſtance is judged of; and if the Object be ſuch as to permit the Eyes to ſee beyond it, each Eye will have a confuſed view of different Objects, according as they lie in the direction of either Axe. So that if the ſame direction of the Eyes be kept, and one of them be ſhut, the other will ſee ſuch Objects as lie in the ſame line with its own Axe, and the firſt being opened and the other ſhut, different Objects will appear in the ſame line with the Point obſerved, according to the direction of that Axe; by which means the principal Object will appear in two different places when looked at with each Eye alternately: and the more or leſs diſtant thoſe two apparent places of the Object are from each other, the nearer or farther off the Object is judged to be.

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* See M. *Perrault*'s *Vitruvius*, Second Edition, *Paris* 1684. Note 9. Chap. 5. Book IV. and alſo the Notes on Chap. 2. Book VI. where the ſubject of the four laſt Articles is judiciouſly treated of.

But

But this method of judging is only of ſervice, while the Diſtance of the Object bears a ſenſible Proportion to that between the Eyes; for when the Diſtance is great, the Angle made by the Axes of the Eyes becomes ſo ſmall that it is not perceivable, and the Axes being in a manner parallel, the apparent place of the Object is not ſenſibly varied whichever Eye looks on it.

Fig. 4. Let A and B repreſent the two Eyes, C an Object, and G H a Wall, or any other plane Surface beyond it; if both Eyes make C the principal Object of the Sight, then A F and B D are the Axes which croſs each other at C, making an Angle A C B. Here C is diſtinctly ſeen by both Eyes, but the Eye A ſees alſo the Point F by its Axe, and the Point D only by the oblique Ray AD; and the Eye B ſees in the ſame manner the Point D by its Axe, and the Point F only by the oblique Ray B F: wherefore to the Eye A, C appears as in F, and to the Eye B it appears as in D; but becauſe the Image of C is in the *Focus* of both Eyes, and ſeen by both Axes, it appears much plainer than either of the Points D or F, which are both of them out of the proper Diſtance, and are only ſeen by one Axe and an oblique Ray each.

If both Eyes look directly at D, then A D and B D become the Axes, ſo that D is diſtinctly ſeen by the Axes of both Eyes; but C, although it is out of the proper Diſtance, is ſeen as in D by the Eye B's Axe, and is only ſeen as in F by the oblique Ray A F, which makes C appear much ſtronger in D than in F; and if both Eyes be directed to F, C will for the ſame reaſon appear much ſtronger in F than in D.

If both Eyes be fixed on E, then A E and B E are the Axes, and E is diſtinctly ſeen by both Eyes, but C is only ſeen by an oblique Ray in D by the Eye B, and by another oblique Ray in F by the Eye A; ſo that C is ſeen double with an equal ſtrength in D and F, though in both places it is indiſtinct, being out of the proper Diſtance, and ſeen by neither Axe.

Laſtly, if both Eyes be directed to C, and (without altering their *Focus*) the Point E be conſidered, it will appear double; by reaſon that the Eye A ſeeing the Point C by its Axe, at the ſame time ſees E by an oblique Ray as on its outſide, and on that account it is thought to be on the left hand of C; and the Eye B ſeeing C by its Axe, *alſo ſees E by an oblique* Ray as on its outſide, and on that account it is thought to be on the right hand of C: thus E being ſeen both as on the left and on the right of C, it is judged to be in two places, and therefore double. Not unlike the deception ariſing from croſſing the fore Finger and middle Finger, and putting a Pea, or any ſmall round Body between the tips of thoſe Fingers, which will be then felt as double; the natural poſition of the Fibres in each Finger, which are then affected, being tranſpoſed, thoſe on the adverſe ſide of each Finger being joined together.

In trying the above Experiments it will be proper to mark the Points D, E, and F, with Chalk, or ſome other way to diſtinguiſh them.

24. Although in looking with both Eyes, there be a diſtinct Image of the Object repreſented on the *Retina* of each, yet while both the Axes are directed to the ſame Object, it is not ſeen double, but the Senſation is only the more lively.

The reaſon why this double Image does not occaſion a double Senſation, ſeems to be founded on what has been already obſerved, that every Point of an Object appears to lie ſomewhere in the direction of the optick Ray, by which it is ſeen; and the Axes of the Eyes being in this caſe both directed to the ſame Point of the Object, they are the ſame with the optick Rays, by which each Eye ſees that Point; which therefore muſt appear in the direction of each Axe, and conſequently in their common Interſection; and by that means makes but one diſtinct Senſation of the ſame part of the Image: whereas when the Axes of the Eyes are not directed to the Point obſerved, but croſs each other either before or behind it, neither Eye ſees it by its Axe, but by oblique Rays; and as theſe muſt have different inclinations to the Axe of each Eye, and the Object appearing to each Eye to be in the direction of that oblique Ray, by which it is ſeen, it muſt therefore be judged to be in two different places, and conſequently double, as was mentioned above.

25. Before the cloſe of this Section, it may not be amiſs to take notice of another property of the Eye, which, with thoſe before mentioned, ſhews the admirable conſtruction of that curious Organ; and that is, a Power it has of enlarging or leſſening its Aperture, ſo as to admit a greater or ſmaller quantity of Rays of Light, according to the different Brightneſs of the Objects looked at, or the quantity of Light, with which the Air is repleniſhed. For as too much Light cauſes ſo great a diſturbance in

the *Retina*, that the Sight is thereby diſordered; ſo when there is not a ſufficient quantity of it, the impreſſion on the *Retina* becomes ſo languid, that it is not eaſily perceived. Thus in coming out of a dark Room into a very light Place, the whole at firſt appears dazling and glaring, till the Eye by degrees hath contracted its Aperture, ſo as to admit no greater quantity of Light at a time, than is neceſſary for diſtinct *Viſion*; and on the contrary when we go out of a light place into the Dark, at firſt no Objects can be diſcerned, but after ſome ſtay there, when the Eye hath enlarged its Aperture, they will begin to appear, and that Place, which at firſt going into, ſeemed perfectly dark, will by little and little grow more lightſome, in proportion as the Eye becomes capable of receiving more Rays at a Time.

SECTION II.

Of the Difference between the Art of Drawing and Stereography.

A Picture painted in its utmoſt perfection, ought to be an exact Copy of the Image, which the Objects themſelves, in their true ſituation, would form in the bottom of the Eye, were they expoſed to it.

To execute this with Succeſs, amongſt many other requiſites, two things are abſolutely neceſſary; the Art of *Drawing* or *Deſigning*, and the Science of *Stereography*; which two are very diſtinct from each other, and have each of them their peculiar Province.

The *Art of Drawing* is an acquired Habit of repreſenting the appearances of Objects by Imitation or Copying, without the aſſiſtance of Mathematical Rules; and muſt be gained by long Practice and diligent Obſervation. This Art hath ſome reſemblance to that of Writing, where the Learner is firſt taught to imitate the ſhapes of the Letters, then to join them into Syllables and Words, and being poſſeſſed of theſe firſt Rudiments, attains by practice a Freedom and Neatneſs of Hand to tranſpoſe, combine, vary, adorn, and flouriſh them according to his Fancy. After the like manner a young Deſigner firſt learns to draw the reſemblance of the eaſieſt Objects, and thence proceeds to the more difficult: he begins with an Eye, a Noſe, or other ſingle Feature, then a Hand, a Foot, and other Limbs, which he afterwards puts together to compleat an entire Figure; and being Maſter of all the different Parts of the Body as of ſo many different Characters, he learns to combine them in ſeveral Poſtures, and thence by degrees to compoſe Groops of Figures in proper Attitudes: the ſame method he purſues with regard to other viſible Objects, Animate and Inanimate, learning firſt to deſcribe their ſingle parts, and thence to compoſe the whole; and having thus provided himſelf with a ſufficient ſtock of particulars, is enabled to introduce all the Variety he thinks proper for the execution of a more extenſive Deſign.

By a long practice of this, the Artiſt acquires a habit of readily drawing whatever Objects offer themſelves to his Imagination; and when this Art is poſſeſſed in a ſuperior Degree by a Man of a good Genius and Taſte, it renders him capable of Performances truly worthy of Admiration.

But whatever length an Artiſt may be able to attain to by the help of Drawing alone, it is impoſſible but he muſt be to ſeek in an infinite variety of Circumſtances. He may ſucceed very well when he copies after the Life, when his Work is an imitation of real Nature which he ſees before him, and where the different effects of Light and Shade, and other various appearances of the Objects according to their mutual poſitions and relations, offer themſelves to his View, and prompt his Deſcription; but when the Original of his Deſign exiſts only in his Imagination, he has no ſuch ſure guide to go by, and will be very liable to omit many neceſſary circumſtances, and to fall into great errors and inconſiſtencies, ſo as to make his performance diſagreeable to his Eye, without being able to diſcover particularly where the fault lies, or how to redreſs it.

Here the Science of *Stereography* comes to his aſſiſtance; it enables him readily to diſcover his Error, and points out a Remedy; it helps him to regulate his Deſign, and to ſupply its defects or retrench its ſuperfluities by ſuch ſure and certain Rules, that he cannot be deceived: This teaches him to give every Object its due apparent Size and

Place

Place by Mathematical Rules, without leaving him to ſearch for it by the Light of his own Fancy or Imagination; by this he is taught to draw a Piece, which ſhall appear agreeable and juſt from any given ſituation; nor doth he need to remove himſelf from place to place to examine the truth of his Picture by his Eye, but is able to know with certainty what effect it will have from ſuch a ſtation, though he never goes to view it from thence: it inſtructs him how to find, without the help of the Objects themſelves, what effects they would have on each other as to Lights and Shades, and other circumſtantials, were they really exiſting in the poſition his Imagination gives them, and which otherwiſe he could not know with any tolerable certainty, though aſſiſted with good Judgment and Experience, unleſs he were firſt to make a Model of his intended Work, there to view its effects in Miniature, and then copy them out into his Picture: and laſtly, it is of ſingular uſe to him when he works on any uneven Ground, or for an unuſual poſition of the Eye, when the Objects he repreſents muſt neceſſarily be protracted or foreſhortened, or otherwiſe diſtorted, in an uncommon manner, ſuited to the Ground, on which they are to be drawn, in order that they may preſerve their natural appearances, when ſeen from the intended ſtation. Upon the whole, without good ſkill in Drawing the Painter can do nothing, and without the knowledge of *Stereography* he can do nothing perfectly well.

STEREOGRAPHY conſiſts of two Parts, *Speculative* and *Practical.*

The Speculative Part, or Theory, makes a conſiderable Branch of direct Opticks; it regarding the appearances of all viſible Objects as they exhibit themſelves to the naked Eye, and reducing thoſe appearances to Mathematical Rules and Theorems.

The Practical Part is an application of theſe Rules to the actual deſcription of thoſe appearances, the doing of which in a moſt eaſy and Uniform manner for all different Caſes, is all that can be expected from it.

But as this Part is purely Mathematical, its Aſſiſtance towards Drawing is only what can be performed by Rule and Compaſs, and can therefore ſtrictly ſerve only for finding the Images of Points, ſtraight Lines, and plane right lined Figures, and of ſolid Bodies bounded by Planes: as to all Curvilinear Figures, they can be no otherways deſcribed according to theſe Rules, but by the Images of the Points, of which they are compoſed; and as theſe are infinite, it is endleſs to find them all by the ſtrict Rules; whence it becomes neceſſary, after a ſufficient number of them are found, to compleat the Image by the help of Drawing, to the better effecting of which theſe Points ſerve as a guide.

Thus when a Circle is to be deſcribed, the practical Rules ſerve to find a ſufficient number of Points in the Circumference, which being neatly joined by hand, will perfect the Image; ſo that in ſtrictneſs, nothing in this Image is found by Mathematical Rules, ſave the few particular Points: the reſt owes its being to the hand of the Drawer.

Thus alſo, if any complicated Figure be propoſed, it may not be eaſy to apply the practical Rules to the deſcription of every minute Part, but by incloſing that Figure in a Regular one, properly ſubdivided, and reduced into Perſpective, that will ſerve as a help, whereby a Perſon ſkilled in Drawing may with eaſe deſcribe the Object propoſed: upon the whole, where the boundaries of the propoſed Objects conſiſt of ſtraight Lines and plane Surfaces, they may be deſcribed directly by the Rules of *Stereography*; but when they are Curvilinear, either in their Sides or Surfaces, the practical Rules can only ſerve for the deſcription of ſuch right lined Caſes as may conveniently incloſe the Objects, and which will enable the Deſigner to draw them within thoſe known Bounds with a ſufficient degree of exactneſs.

It is therefore in vain to ſeek by the practical Rules of *Stereography*, to deſcribe all the little Hollows and Prominences of Objects, the different Light and Shade of their Parts, or their ſmaller Windings and Turnings; the infinite Variety of the Folds in Drapery; of the Boughs and Leaves of Trees, or the Features and Limbs of Men and Animals; much leſs to give them that Roundneſs and Softneſs, that Force and Spirit, that Eaſineſs and Freedom of Poſture, that Expreſſion and Grace which are requiſite to a good Picture: *Stereography* muſt content itſelf with its particular Province of exhibiting a kind of rough Draught to ſerve as a Ground-work, and to aſcertain the general Proportions and Places of the Objects according to their ſuppoſed Situations, leaving the reſt to be finiſhed, beautified, and ornamented by a hand ſkilful in Drawing.

'Tis true, *Stereography* is of moſt uſe where it is moſt wanted, and where a deviation from its Rules would be the moſt obſervable; as in deſcribing all regular Figures,

gures, Pieces of Architecture, and other Objects of that ſort, where the particular Tendency of the ſeveral Lines is moſt remarkable; the Rule and Compaſs in ſuch caſes being much more exact than any deſcription made by hand: but ſtill the Figure deſcribed by the Perſpective Rules, will need many helps from Drawing; the Capitals and other Ornaments of Pillars and their Entablatures, the Strength of Light and Shade, the apparent Roundneſs and Protuberance of the ſeveral Parts, muſt owe their Beauty and Finiſhing to the Deſigner's hand: but with regard to ſuch Objects as have no conſtant and certain determinate Shape or Size, ſuch as Hills, Clouds, Trees, Rivers, uneven Grounds, and the like, there is a much greater Latitude allowable, provided the general Bulk or uſual natural Shape of thoſe Objects be in ſome meaſure obſerved, ſo as not to make them appear unnatural or monſtrous.

But although the ſtrict practical Rules of *Stereography* are in a great meaſure confined to the deſcription of right lined Figures, yet the knowledge of the general Laws of that Science is of great and neceſſary uſe to inform the Judgment, after what manner the Images of any propoſed Lines ſhould run, which way they ſhould tend, and where terminate; and thereby the better enables it to determine what appearances any Objects ought to put on, according to their different Situations and Diſtances: it accuſtoms the Eye to judge with greater certainty of the relations between real Objects and their Stereographical Deſcriptions, and the Hand to draw the ſame accordingly; and directs the Judgment readily to diſcover any conſiderable error therein, which might otherwiſe eſcape notice. Beſides, that when the Ground or general Plan, and the principal Parts of a Picture are firſt laid down according to the Rules, every thing elſe will more naturally fall in with them, and every remarkable deviation from the juſt Rules will be the more readily perceived, and the eaſier avoided or rectified; ſo that although it may be infinitely tedious, or abſolutely impracticable to deſcribe every minute Part of a Picture by the ſtrict Mechanical Rules, yet the employing them where they can be moſt commodiouſly uſed, will give the Picture in general ſuch a look, as will guide the Artiſt in drawing the other Parts without any obvious inconſiſtency.

Without the knowledge of *Stereography* a Picture is drawn as it were by gueſs, without any certain determinate Points or Lines, or any other Rule than the Judgment of the Painter's Eye to guide him: here the Shape and Situation of his Objects are not previouſly determined, but left at large, to be modelled as they may happen in the progreſs of his Work to appear to ſtand beſt; this indeed is the too common way in which Painters work, and it allows them all kind of Latitude in their Deſigns, or rather permits them to Paint without any ſettled Deſign at all, but as it ſhall happen. If a Figure on examination appears too large for its Diſtance, it is by a ſtroke of a Pencil brought to ſtand on nearer Ground; Mountains are removed from place to place by raiſing or lowering their Foundations, till at laſt the Painter fixes them as ſuits beſt to that Bulk and Strength of Colour which he firſt gave them: as he has no fixed Deſign to work by, all that he can do, is to make his Eye the Judge, and to correct what on view appears to him amiſs; but often not knowing how to do it, he makes it worſe, and is obliged, after many repeated unſucceſsful trials, to hide that part under a Veil, or blot it quite out, and put ſomething elſe in its place that may look better: as he is not ſure of what he really intends, he is obliged to keep others as much in the dark as himſelf, by induſtriouſly avoiding all regular Figures and ſtraight Lines, and leaving the boundaries of his Objects as uncertain as may be; and thus at length the Piece is finiſhed, and the Painter almoſt as ignorant of the true Original or Model of his Performance as the greateſt Stranger, and if in this manner it can be compleated without any obvious and groſs faults in it, he is much more beholden to Chance and good Fortune, than to the Rules of the Art he profeſſes.

On the other hand, a Picture drawn by the Rules, may be eaſily reduced to its Model; nothing is ambiguous or uncertain in it, but what is ſo in Nature; the true Diſtance, Height, and Breadth of every Object may be meaſured by a Line, the Ground and Buildings may be reduced to their Original Plan, and from thence a new Picture may be drawn of the ſame things in any other View. A Painter working by theſe Rules knows what he is about, and lets the Spectator know it too; he is in no danger of falling into abſurdities, nor does he ſtand in need of blinds and ſhifts to cover Ihis gnorance; if any part of his Work hath not a good effect, he knows the fault lies in his Model or Deſign, and how and where to correct it; and has the pleaſure of working with certainty, without the ſlavery of being obliged to grope out

out every ſtep of his way, and not knowing in the end whether he be right or wrong.

If thoſe who have hitherto wrote on *Perſpective*, had more duly weighed the difference between that, and the Art of Drawing, and what part was the peculiar Buſineſs of each to contribute towards the making a Picture; they would not have embelliſhed their Works with ſuch a variety of Elaborate and coſtly Gravings, for the Executing of which they have not given Rules, or laid any ſufficient Foundation; nor by that means have raiſed in their Readers, fruitleſs expectations of becoming able to make ſuch beautiful Deſigns by the help of *Perſpective* alone: but by applying themſelves more particularly to ſearch and diſcover the Extent of what this Art was capable of doing, would have thereby advanced the real knowledge of it, and made their Writings on that Subject more generally uſeful and inſtructive.

This I ſhall endeavour to do in the following Work; wherein I ſhall confine myſelf to treat of the Deſcription of Objects, ſo far as it may be attained by Mathematical Rules; from whence I ſhall alſo take proper Opportunities to deduce ſuch Remarks, as may be uſeful to the Theory in general.

SECTION III.

Of the different Methods of deſcribing Objects by Mathematical Rules.

1. THE different ways of deſcribing Objects on a Plane by Mathematical Rules are two, *Geometrical* and *Stereographical*; the firſt of which is ſubſervient and neceſſary to the other. In both theſe, the original Objects are always ſuppoſed to be out of the Plane, on which they are to be deſcribed; which Plane may be called the *Plane of the Section*.

2. The *Geometrical* Deſcription of an Object is, when its Repreſentation or Image on the Plane of the Section, is formed by the Interſections of that Plane with parallel ſtraight Lines, falling either perpendicularly, or with any Angle of inclination on it, from the ſeveral Points of the Object; which Lines may be conſidered as the Rays which produce or Project the Images of thoſe Points on that Plane.

3. Now theſe *Projecting* Lines being ſuppoſed parallel to each other, it follows, that the Eye, conſidered as a Point, though removed ever ſo far off, can receive but one of them at a time; and therefore in this kind of Deſcription, the Diſtance of the Eye is not concerned, but the Eye is rather ſuppoſed to be at an Infinite Diſtance from the Plane of the Section; ſo that no part of the Figure is deſcribed with reſpect to its being either nearer to, or farther from the Eye.

4. In this manner, no more of an Object can be repreſented, than only ſuch Parts of it, from which Lines parallel to each other may be drawn to the Plane of the Section; and conſequently only two Dimenſions at a time can be expreſſed, ſuch as Height and Depth, without regard to Thickneſs; or Breadth and Length, without reſpect to Depth, *&c.*

5. Therefore in a Geometrical Deſcription, the Object may be conſidered as a Plane; and if there be any Inequalities in its Surface, they may be ſuppoſed to come forwards, or recede till the whole becomes even; and if the Plane of the Figure thus made, be parallel to the Plane of the Section, the Image will be equal and ſimilar to its Original.

Let A B C D repreſent an Object, in a Plane parallel to R S, the Plane of the Section; and from the Points A, B, C, and D, draw parallel Lines, either perpendicular or inclining in any given Angle on the Plane R S, and meeting it in the Points E, F, G, and H: the Figure E F G H will then be the Geometrical Deſcription of its Original A B C D. Now becauſe the Lines A E and B F are parallel and equal, therefore A B will be alſo parallel and equal to E F[a]; and for the ſame reaſon, the ſides B C, C D, and D A will be alſo reſpectively parallel and equal to the ſides F G, G H, and H E; and all the Angles of the Original Figure being equal to the correſponding Angles of its Image[b], that Image will therefore be equal and ſimilar to its Original. Fig. 5.

[a] 33 El. 1.

[b] 10 El. 11.

6. If the Plane of the Figure be inclined to the Plane of the Section, then the Image and its Original will be neither equal nor ſimilar; but, when the Projecting Lines are perpendicular to the Plane of the Section, each Line of the Original Figure wlll be to its Image, as *Radius* to the Coſine of the Angle of Inclination of that Line to the Plane of the Section.

Fig. 6. Let A B be a Line inclining to D E the Plane of the Section, the Perpendiculars A D and B E will determine D E, the Image of the original Line; draw B C parallel to D E, which will therefore be equal to it; but in the Rectangular Triangle A B C, if AB be ſuppoſed the *Radius*, B C will be the Coſine of the Angle A B C, which is equal to the Angle of Inclination of the original Line to the Plane of the Section; therefore AB will be to B C, or its equal D E, as *Radius* to the Coſine of the Angle of Inclination.

7. And if the Plane of the Figure be perpendicular to the Plane of the Section, the whole Figure will be projected into a ſtraight Line; for then all Lines drawn from any Point of the Plane of the Figure, perpendicular to the Plane of the Section, will fall in the common Interſection of thoſe Planes [a], which is a ſtraight Line [b].

8. This kind of Deſcription is uſed for ſeveral purpoſes, and takes different Names according to the Nature of what it repreſents; and the Projecting Lines are generally taken to be perpendicular to the Plane of the Section.

9. On which ſuppoſition, it is called a *Plan* or *Ichnography*, when it deſcribes the perpendicular Seat or Place that any Objects have on the Ground, without reſpect to their Height above it. Thus the Ichnography or Plan of a Town, a Church, or any Building, is a Geometrical Deſcription of the perpendicular Seat or Place of all their Parts on the Ground, conſidered as a Plane.

10. It is termed the *Orthography* or *Elevation*, when it repreſents the Face of any Objects, as to their Breadth and Height above the Ground, without regard to the Space they occupy on it. Thus the Elevation or Orthography of a Building, is a Geometrical Deſcription of ſome one Front or Side, as to its Breadth and Height, without reſpect to its Depth: and if the Building were ſuppoſed to be cut through by a Plane, when the ſeveral Parts of it which are cut by that Plane, are deſcribed in their proper Meaſures, it is then called a *Section*.

11. And when it is uſed in Fortification, to deſcribe the ſeveral different Heights of the Works, as if they were cut a-croſs by a Plane perpendicular to the Ground, it takes name of the *Profile*; as it does that of a *Chart* or *Map*, when it deſcribes the true Situation and Shape of the Countrey, and Places there repreſented, in their proper Dimenſions or Proportions; eſpecially while the Space deſcribed is not ſo large, but that it may be conſidered as a Plane, without regard to the Convexity of the Body of the Earth.

12. In all theſe Caſes, the Plane of the Section is ſuppoſed to be parallel to the Plane of the thing deſcribed; and therefore the Image, according to the Rule before laid down, would be equal and ſimilar to its Original. But theſe Deſcriptions being moſt commonly uſed with intent to reduce great things into a ſmaller Compaſs, they notwithſtanding retain the ſame Names, ſo long as they remain Similar to what is intended to be repreſented, although they be proportionally diminiſhed in any Degree.

13. Geometrical Deſcription is alſo uſed in ſome kinds of Projections of the Sphere, where the ſeveral Circles in it are conſidered as lying in Planes, ſome parallel, ſome inclining, and others perpendicular to the Plane of the Section, and are accordingly projected into Circles, Ellipſes, or ſtraight Lines.

14. By this Account of Geometrical Projection it appears, that whether the Plane of the Section be before or behind the Plane of the Object, or farther from, or nearer to it, the Deſcription will not be varied, whilſt the ſeveral Situations of the Plane of the Section remain parallel: for it is evident, that whether the Plane R S be nearer to, or farther from the Plane of the Object A B C D, or whether it be above or below it; yet ſo long as the new ſuppoſed Situation of the Plane R S, remains parallel to that which it had before, the Projection or Image E F G H will be the ſame. Fig. 5.

15. Although it is before laid down, that in Geometrical Deſcriptions no regard is had to the Diſtance of the Eye, ſeeing that whether it be ſuppoſed nearer or farther off, the Image will ſtill be the ſame; yet uſually in Projections of the ſeveral ſorts that have been mentioned (except ſuch as are uſed for Mathematical Purpoſes) when the Object repreſented hath ſome of its viſible Parts more diſtant than others, the more

[a] 38 El. 11.

[b] 3 El. 11.

diſtant Parts are expreſſed by fainter Lines, and thoſe more near by ſtronger, in order to give a better Idea of the thing deſcribed; and even Shadowing is often uſed to help the Appearance: but theſe Methods are not ſtrictly proper to a Geometrical Deſcription, but are borrowed from the Stereographical.

16. A *Stereographical Deſcription* is when the Lines, which by their Interſection with the Plane of the Section form the Image of the Object, are not parallel, but are all ſuppoſed to meet in ſome one Point: this Point is taken as the Place of the Eye, and the Lines which produce the Image are conſidered as the Optick Rays which compoſe the Image on the *Retina*, as was explained in the firſt Section of this Book: ſo that this kind of Deſcription regards the Appearance that Objects have, when ſeen from one certain Point, and is therefore capable of repreſenting all the three Dimenſions at a Time, as Length, Breadth, and Thickneſs, or as it were the Solidity of Objects, whence it takes the Name of *Stereography*.

Stereographical Deſcription is of three ſorts, which take their Denominations from the ſeveral Situations of the Object, and the Plane of the Section, with reſpect to the Point from whence the Object is ſuppoſed to be ſeen, or the Place of the Eye.

17. Firſt, when the Plane of the Section is between the Eye and the Object, it takes the name of *Perſpective:* and here the Rays proceeding from the Object to the Eye, are ſuppoſed to be cut by the Plane of the Section, and by their Interſection with it, to form the Image of the Object: this kind of Stereographical Deſcription is therefore termed *Perſpective*, the Object being as it were ſeen through a Plane placed between the Eye and it, as if that Plane were tranſparent.

18. The Image thus formed muſt always be ſmaller than the Object, in proportion to the diſtances between the Eye and the Plane of the Section, and between the Eye and the Object: for the nearer the Plane of the Section is brought to the Eye, the Diſtance between the Eye and the Object remaining the ſame, the Image becomes the ſmaller, till at laſt, if the Plane of the Section were ſuppoſed to touch the Eye, the whole Image would be reduced to a Point; ſeeing all the Rays from the Object to the Eye, would cut that Plane only in its Point of Contact with the Eye.

19. Secondly, When the Object is between the Eye and the Plane of the Section, it is then called *Projection:* and here the Rays proceeding from the Object to the Eye, are ſuppoſed to be continued on beyond the Object, till they cut the Plane; it is therefore called Projection, the Image of the Object being in a manner projected or thrown forward upon a Plane beyond it.

20. The Image thus formed muſt always be larger than the Object, in proportion to the Diſtances between the Eye and the Plane of the Section: for the nearer the Object is brought to the Eye, the Diſtance between the Eye and the Plane of the Section remaining the ſame, the Image becomes the larger, till at laſt, if the Object were ſuppoſed to touch the Eye, or to be in the ſame Plane with it, and parallel to the Plane of the Section, the Image would become Infinite, or rather it would have no Image at all; becauſe the Rays from the Object to the Eye would then be parallel to the Plane of the Section, and ſo could never meet it to form the Image.

21. Laſtly, when the Eye is ſuppoſed to be between the Object and the Plane of the Section: here the Eye muſt be conſidered only as a Point, through which all the projecting Rays paſs, and are continued on till they cut the Plane of the Section on the oppoſite Side. This kind of Deſcription may be therefore called *Tranſprojection*, the Image of every Point of the Object being in a manner projected through the common Point upon the Plane of the Section; and hence it ariſes, that the Image thus formed is Inverted, and bears the ſame Similitude to its Original, as the Image formed in the *Retina* of the Eye doth to the Object ſeen by it.

'Tis true this kind of Projection is only Imaginary; for if the Point, through which the Viſual Rays are here ſuppoſed to paſs, be conſidered as the Eye looking on the Plane of the Section, then the Object will be behind it, and therefore muſt be out of Sight; Nevertheleſs the Projections of Points in that Situation with reſpect to the Eye, being in many Caſes neceſſary to be found, we have therefore here given a Place to this laſt kind of *Stereography*.

22. The Image of an Object formed in this laſt way, may be either bigger or leſs than its Original, or of an equal Size with it, in proportion as the Point, through which the Viſual Rays are ſuppoſed to paſs, is taken nearer to the Object, or to the Plane of the Section, or at an equal Diſtance from each.

To illuſtrate what has been here ſaid by an Example: Let I repreſent the Place of the Fig. 7.
Eye

Eye in a Plane FK, A B an Object in a Plane X, and Y the Plane of the Section, all thoſe Planes being parallel. The Rays AI and BI from the Points A and B meeting in I, cut the Plane Y in a and b, the Images of the Points A and B, and thereby determine ab the Image of A B, and the Plane of the Section Y being between the Eye and the Object A B, the Image ab is the Perſpective of A B[a].

It is evident from the Figure, that ab muſt always be leſs than A B; and that the Diſtance between I and the Plane X remaining the ſame, if the Plane Y be moved nearer to the Plane X (they continuing parallel) the Image ab will become larger; if the Plane Y be moved nearer to I, that Image will become leſs; till at laſt, if the Plane Y be moved ſo as to coincide with the Eye and the Plane F K, the whole Image of A B would be only a Point, the ſame with the Point I[b].

If ab be the original Object in the Plane Y, and X be the Plane of the Section, then Ia and Ib produced to the Plane X, cut it in A and B, and thereby determine A B the Image of ab, and the Image AB is then the Projection of the Object ab[c]; and here it is alſo apparent, that the Image A B muſt always be larger than its Original ab; if ab be brought nearer to the Plane X, the Image A B will be leſſened, the Lines Ia and Ib then making a ſmaller Angle; but if ab be brought nearer to I, the Image A B will be increaſed, till at laſt, if the Plane Y coincide with the Plane F K, the Point D coinciding with I, the Lines Da and Db will coincide with I F and I K, and ſo become parallel to the Plane X, and therefore can never meet it to determine the Length of the Image[d].

Laſtly, if $\alpha\beta$ be the Object in the Plane Z, and Y be the Plane of the Section, then the Lines αI and βI drawn through I, and produced till they cut the Plane Y in a and b, will determine ab the Transprojection of the Object $\alpha\beta$[e]; and in this caſe it is evident, that if the Diſtances I E and I D be equal, the Image ab will be equal to its Original $\alpha\beta$; if I be brought nearer to E, the Image will be larger, if nearer to D, it will be leſs; and if I were brought to D, the Image would be but a Point, as it would become Infinite if I were removed to E[f].

23. So that in all Kinds of Stereographical Deſcription, there muſt always be ſuppoſed ſome Diſtance between the Eye, the Plane of the Section, and the Object; it having been ſhewn, that if either the Object, or the Plane of the Section touch the Eye, no Image can be formed; and if the Object coincide with the Plane of the Section, then the Object and its Image become the ſame thing.

24. Stereographical Projection and Perſpective (as well as Geometrical Projection) are uſed in the Projections of the Sphere, ſometimes ſingly, and ſometimes both together, and are called by the general Name of *Projection*, without diſtinction. As if the Eye were ſuppoſed to be in the Pole of the Sphere, and the Plane of the Section were imagined to paſs through the other Pole perpendicular to the Axe; the whole Deſcription thus made is properly *Projection*, the Sphere being in this caſe all between the Eye and the Plane of the Section. But if the Plane of the Section were ſuppoſed to paſs through the Equator, or any of the Parallels of Latitude, the Deſcription will then be partly *Projection* and partly *Perſpective*; thoſe Parts which are between the Eye and the Plane of the Section being Projected on it, and thoſe which are beyond that Plane being deſcribed by their Perſpectives. Now the Eye being here ſuppoſed to be in the Pole of the Sphere, it is evident that this Pole cannot be repreſented, and that the Parts of the Sphere near to it will be projected fartheſt out, till they become almoſt infinitely diſtant from the Center of the Projection, in proportion as they are nearer to the place of the Eye: it is therefore uſual in this caſe, to confine the Projection to ſome Circle in the Sphere, ſuch as the *Polar Circle*, or any other at pleaſure, and then no part of the Sphere is deſcribed, that lies between that Circle and the Eye; and if the Plane of the Section be ſuppoſed to paſs by this Circle, then the Projection becomes all *Perſpective*, the Plane of the Section lying between the Eye and the whole Object deſcribed: but in all theſe Caſes the Projections are ſtill ſimilar, and differ in nothing but in Size, the Image becoming larger as the Plane of the Section is ſuppoſed farther diſtant from the Eye.

But the Projection of the Sphere being only an Application of the general Rules of *Stereography* to a particular Subject, we ſhall not here purſue it any farther, but refer to what has been wrote of it by thoſe, whoſe proper Theme it was; Intending to treat of the ſeveral kinds of *Stereography* ſo far only, as they relate or are ſubſervient to the Deſcription of the Appearances of Objects, when ſeen from any determinate Point, and more particularly of that kind of it which is called *Perſpective*.

25. We ſhall therefore Define *Stereography* to be the Art of finding by certain

[a] Art. 17.

[b] Art. 18.

[c] Art. 19.

[d] Art. 20.

[e] Art. 21.

[f] Art. 22.

Rules, the Interſections of the Optick Rays proceeding from any Object to the Eye, with a Plane through which thoſe Rays paſs, or to which they may be produced. To which, if we add the Art of deſcribing that Paſſage upon the Plane, according to the Degree of Strength and Light of each Ray, in ſuch manner, that the Rays of Light proceeding from the Deſcription on the Plane, may enter the Eye in the ſame Order, and with the ſame apparent Strength, as they would from the Object itſelf, in its true and real Situation, we ſhall have all that is requiſite for the exact Deſcription of the Object propoſed.

A Picture drawn by theſe Rules, and expoſed to the Eye at the proper Diſtance and Place, muſt therefore convey to it the like Appearance, as would be produced by the Objects themſelves, were the Picture tranſparent, or conſidered as a Window, without and beyond which, the Objects were ſuppoſed to be in the ſame Situation, wherein they are intended to be repreſented.

26. *Stereography*, in this View, has been generally called by the Name of *Perſpective*, and hath been by divers Authors divided into ſeveral Sorts; ſome have diſtinguiſhed it, with regard to the Poſition of the Picture in reſpect to the Ground, into *Direct*, *Inclining*, and *Horizontal*; that is, when the Picture is ſuppoſed to be either perpendicular, inclining, or parallel to the Ground: others have placed the Difference in the various Situations of the Picture with reſpect to the Eye, either directly before it, or above, below, or on one Side of its Axe: and others, with regard to the ſeveral Diſtances of the Eye from the Picture. But all theſe kinds, as to the Practice, are in effect the ſame, and have no eſſential Difference, the ſame Rules ſerving alike for all of them, let the Situation of the Objects with regard to the Picture, or of the Picture with reſpect to the Eye, be what it will.

There is therefore no real Diſtinction in Stereographical Deſcriptions, but what ariſes from the different Shape of the Surface, on which the Objects are repreſented, or which is ſuppoſed to cut the Rays from the Object to the Eye. And upon this Foot it may be divided into *Plain* and *Uneven*.

27. *Plain Stereography* is what hath already been deſcribed, and that, on which the other depends, and to which it muſt ſome way or other be reduced, in order to be put in Practice; and it is therefore that, which we intend chiefly here to treat of, as being the Foundation.

28. *Uneven Stereography* is when the Surface, on which the Objects are deſcribed, is not a Plane, but of any other Shape, as Concave, Convex, or otherwiſe uneven or Irregular. Of this kind are Paintings on Cupalo's, Vaulted Ceilings, or uneven Walls. And alſo the Painting on Scenes for Theatres, which is a kind of Stereographical Deſcription made upon ſeveral different Planes, at ſeveral Diſtances, and variouſly ſituated with reſpect to the Eye, and is uſually called *Scenography*. Of all which we ſhall ſay ſomething in their proper place, and ſhew in what manner the Rules of *Plain Stereography* may be applied to them.

29. There is alſo a ſort of *Stereography* by Reflection; which is, when Objects are ſo repreſented on a Picture, that by putting a Reflecting Body, either of Glaſs or Metal, of a determinate Shape, in a certain Place, the Picture ſhall from thence be reflected in its due Proportions to the Eye in a proper Situation; although, if ſeen without ſuch Reflection, it would appear a confuſed Heap of Colours or Lines, without any intelligible Shape.

30. There may be many other kinds of *Stereography*, compoſed or complicated of thoſe already mentioned; any Deſcription of Objects being in ſome Senſe to be called *Stereographical*, which at laſt brings the Image truly to the Eye, whether the Rays which compoſe the Image, have ſuffered Reflection, Refraction, or any other Diſtortion in their way to it. But theſe being more of Curioſity than Uſe, we ſhall not here take any farther notice of them.

31. Beſides the Geometrical and Stereographical Ways of deſcribing Objects here mentioned, ſome Authors have added a third kind, under the Name of *Military Perſpective* or *Geometrical Elevation*; by which all the three Dimenſions of Bodies are attempted to be made appear at one View, by raiſing their Sides on their Geometrical Plan or Ichnography, keeping to the true Meaſures of thoſe Sides, without regard to their ſeveral Diſtances from the Eye, but varying the Angles of Elevation Stereographically; and by this means producing an inconſiſtent medley Appearance, partly Geometrical, and partly Stereographical, unnatural and diſagreeable to the Eye, and ſerving no purpoſe, but what may be much better attained by true Perſpective; for which reaſon, it deſerving no place here, we ſhall content ourſelves with barely

mentioning it, and refer ſuch, whoſe Curioſity ſhall lead them to inform themſelves farther of its Nature, to the latter end of the firſt Volume of the Second Edition of the *Jeſuits Practical Perſpective*, printed at *Paris* in 1679. where a more particular Account of it may be found; it being time to proceed to our Subject.

SECTION IV.

Of the ſeveral preparatory Planes, Lines, and Points uſed in Stereographical Deſcriptions; their Definitions and Relations to each other, and to the Objects intended to be repreſented.

IN order to the Stereographical Deſcription of any Objects, ſeveral preparatory Planes, Lines, and Points are imagined, by the help of which the Images of the Objects are found: what theſe are, and how related to each other, and to the Objects and their Images, ſhall be ſhewn in the following Definitions and Theorems, and their Corollaries.

DEF. 1.

Fig. 8. The Plane of the Section, or that Plane, by which the Rays from the Object to the Eye are ſuppoſed to be cut, is called the *Plane of the Picture*; and is taken as a Plane indefinitely extended at ſome Diſtance from the Eye, and ſo ſituated, with reſpect to the Objects intended to be repreſented, as to be cut by all Rays which can enter the Eye from them, the Eye remaining fixed in one certain Place.

This Plane differs from the *Table* or *Picture* whereon the Objects are deſcribed, only as the whole from a part; for although this Plane be ſuppoſed Indefinite, yet the Picture muſt always have ſome Limits, and can be only ſome part or other of that Plane, taken and bounded at Diſcretion.; this Plane however is generally called ſimply the *Picture*, and is repreſented by the Plane EFGH.

DEF. 2.

That Point where the Rays are ſuppoſed to meet, is ſometimes called the *Point of Sight*, and is the Vertex of the Cone, or Pyramid of Rays, ſuppoſed to proceed from the Object to the Eye, and to be cut by the Plane of the Picture: this Point is therefore the Place where the Spectator's Eye ought to be, in order to ſee the Picture with Exactneſs; and for that reaſon it is moſt commonly called the *Place of the Eye*, or ſimply the *Eye*, and is marked with the Letter I.

DEF. 3.

By *Original Object* is meant, the real Object intended to be repreſented, placed in the true Situation it is ſuppoſed to have, with reſpect to the Picture and the Eye.

DEF. 4.

By *Original Plane* is meant, a Plane, wherein is ſituated any original Point, Line, or plain Figure, or that whereon any Original Objects are Geometrically deſcribed, or to which they may ſome way or other be referred.

Thus the Plane LMGH repreſents an Original Plane, in which the Line QB is an Original Line.

DEF. 5.

The Stereographical Deſcription of any Original Object, as of a Point, Line, or Figure, whether Perſpective, Projective, or Tranſprojective, is called in general the *Image of that Object*; and is the Section of the Plane of the Picture by the ſingle Ray, or by the Plane, Cone, or Pyramid of Rays, which proceed from the Object to the Eye, according as the Original Object is either a Point, Line, or Figure.

DEF. 6.

If from I, a Line IO be drawn perpendicular to the Plane of the Picture, cutting it in O, the Point O is called the *Center of the Picture*, and the Line IO is called the

Axe

Plate 2 Book 1 Sect. 4.

Fig. 8.

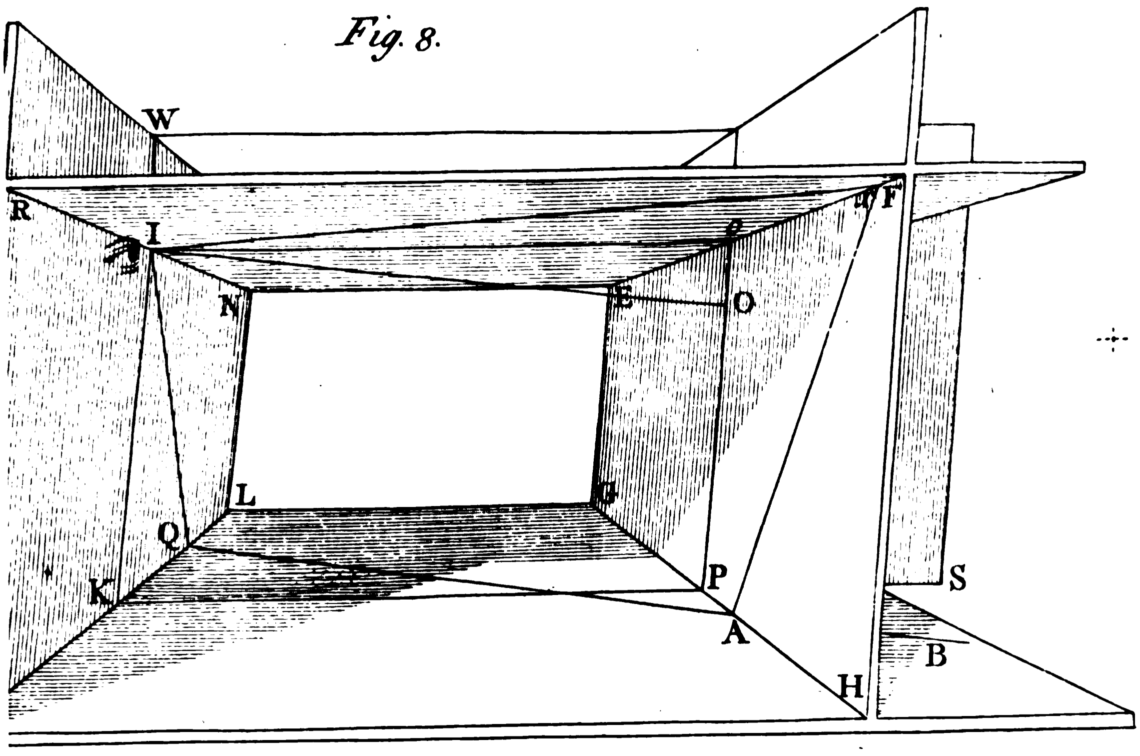


Fig. 9.

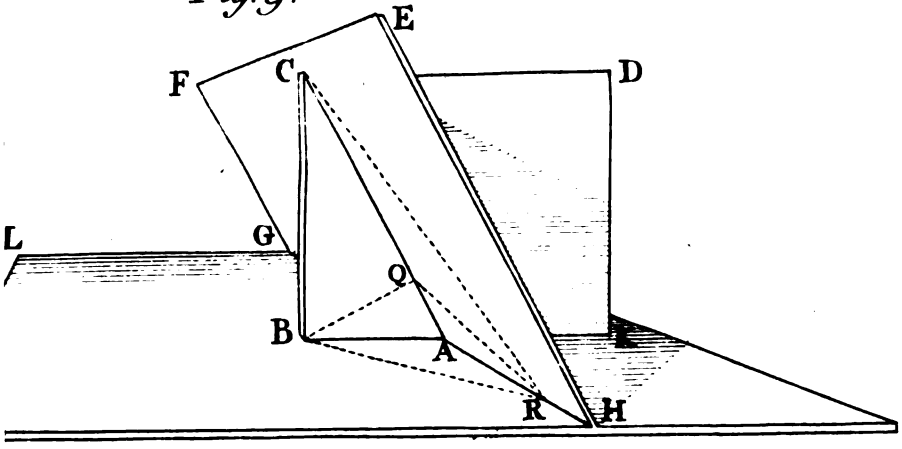
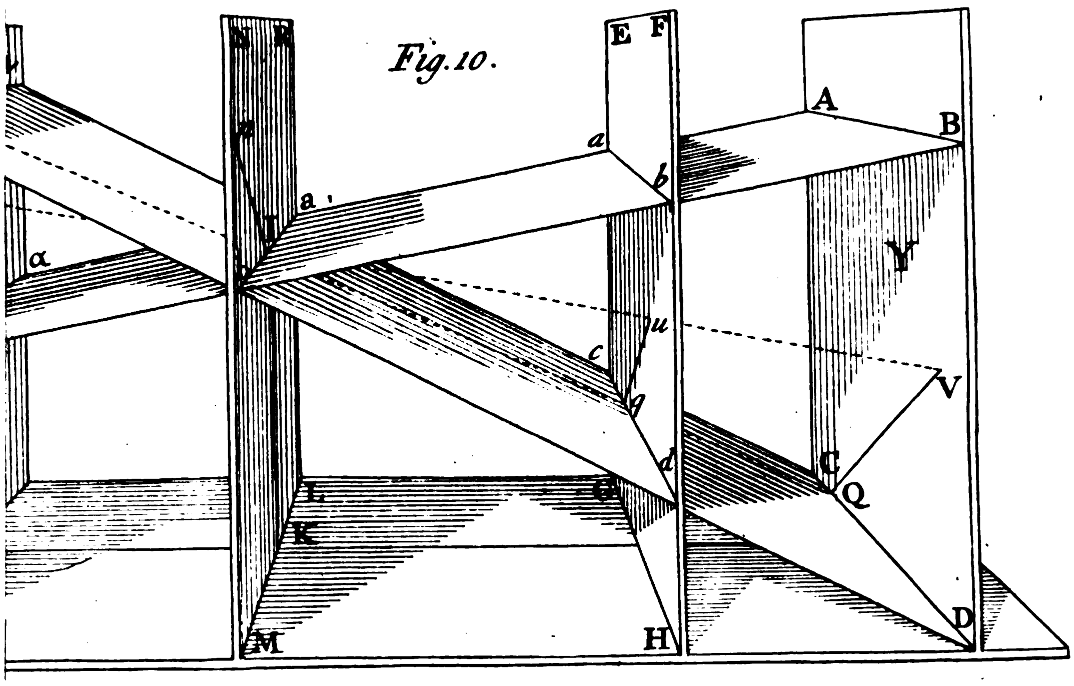


Fig. 10.



J. Mynde sc.

Axe of the Eye; and it being the meaſure of the Diſtance of the Eye from the Picture, it is, when conſidered only with regard to its Length, called ſimply the *Diſtance of the Picture*, and ſometimes the *Diſtance of the Eye.*

DEF. 7.

The Diſtance of the Picture being ſettled, there is a Plane imagined to paſs through the Point of Sight I, parallel to the Picture, and this is called the *Directing Plane.*

This Plane is repreſented by the Plane LMNR, in which the Point I is ſuppoſed to be, and is taken as a Plane indefinitely extended, paſſing through the Eye parallel to the Plane of the Picture.

COR. 1.

Whilſt the Situation of the Eye and the Picture remain fixed, the Directing Plane continues the ſame: for there cannot two different Planes paſs through the Eye parallel to the Picture.

COR. 2.

The Axe of the Eye IO is perpendicular to the Directing Plane; it being perpendicular to the Picture, to which the directing Plane is parallel.

DEF. 8.

An Original Plane may have any Situation with reſpect to the Picture, either Parallel, Perpendicular, or anywiſe Inclining to it; and with reſpect to the Eye, it may be either above, or below, or on either ſide of it.

If an Original Plane be not parallel to the Picture, it will, if produced, cut both the Picture and the Directing Plane.

The Interſection of the Original Plane with the Picture is called the *Interſecting Line* of that Plane, and its Interſection with the Directing Plane is called its *Directing Line.*

Thus if the Original Plane LMGH be not parallel to the Picture, the Line GH, where it cuts the Picture, is the Interſecting Line of that Plane; and the Line LM, where it cuts the Directing Plane, is its Directing Line.

DEF. 9.

If an Original Plane, not parallel to the Picture, be propoſed, another Plane is ſuppoſed to paſs through the Eye parallel to the Original Plane, and cutting the Picture, which Plane is called the *Vaniſhing Plane of the Original Plane.*

Thus the Plane NREF which paſſes through I, parallel to the Original Plane LMGH, is the Vaniſhing Plane of this Original Plane.

DEF. 10.

The Interſection EF of the Vaniſhing Plane with the Picture, is called the *Vaniſhing Line of the Original Plane*; and NR, the Interſection of the ſame Vaniſhing Plane with the Directing Plane, is called the *Parallel of the Eye.*

COR. 1.

The Vaniſhing Line EF, the Interſecting Line GH, the Parallel of the Eye NR, and the Directing Line LM, which relate to the ſame Original Plane, are parallel to each other.

Becauſe they are the Interſections of parallel Planes by parallel Planes[a], the Directing Plane and the Picture being parallel[b], as are alſo the Original and Vaniſhing Planes[c].

COR. 2.

Hence if any one of theſe Lines, and one Point in any other of them, be given, the whole of that other Line is given.

Becauſe a Line drawn through the given Point, parallel to the given Line, is the other Line required.

DEF. 11.

If through the Axe of the Eye IO, a Plane IKoP be ſuppoſed to paſs, perpendicular to the Original Plane LMGH, the Plane IKoP indefinitely produced, is called the *Vertical Plane of the Original Plane.*

This Plane in the Figure is for conveniency ſuppoſed to be Tranſparent.

COR.

[a] 16 El. 11.

[b] Def. 7.

[c] Def. 9.

C O R.

The Vertical Plane IKoP is perpendicular to the Picture, the Directing Plane, and the Vaniſhing Plane, as well as to the Original Plane. And it is alſo perpendicular to the Vaniſhing, Interſecting, and Directing Lines, and to the Eye's Parallel of the Original Plane.

Becauſe the Vertical Plane paſſes through IO the Axe of the Eye, which is perpendicular to the Picture[a], that Plane is therefore perpendicular to the Picture[b], and alſo to the directing Plane, which is parallel to the Picture; and it being perpendicular to the Original Plane, it is alſo perpendicular to the Vaniſhing Plane, to which the Original Plane is parallel; and being perpendicular to all theſe four Planes, it is alſo perpendicular to their common Interſections EF, GH, LM, and NR[c].

D E F. 12.

The Interſection IK, of the Vertical Plane IKoP, with the directing Plane, is called the *Director of the Eye*, and is taken as the meaſure of the Height of the Eye above the Original Plane; and therefore when conſidered only as to its Length, it is called the *Height of the Eye*; and the Point K, where the Eye's Director cuts the Directing Line LM, is called the *Foot of the Eye's Director*, or the *Point of Station.*

D E F. 13.

The Interſection oP of the Vertical Plane IKoP, with the Picture, is called the *Vertical Line of the Original Plane*; the Point o, where this Line cuts the Vaniſhing Line EF, is called the *Center of that Vaniſhing Line*; and the Point P, where it cuts the Interſecting Line GH, is called the *Foot of the Vertical Line.*

C O R.

Hence the Vertical Line of every Original Plane, paſſes through the Center of the Vaniſhing Line of that Plane, as well as through the Center of the Picture.

D E F. 14.

The Interſection Io of the Vertical Plane, with the Vaniſhing Plane, is called the *Radial of the Original Plane*; and the Diſtance between the Eye and the Center o, or the Length of the Line Io, is called ſimply the *Diſtance of the Vaniſhing Line.*

D E F. 15.

The Interſection KP of the Vertical Plane, with the Original Plane, is called the *Line of Station of the Original Plane.*

C O R. 1.

The Eye's Director IK is perpendicular to the Directing Line LM, and to the Eye's Parallel NR; the Vertical Line oP is perpendicular to EF and GH the Vaniſhing and Interſecting Lines; the Radial Io is perpendicular to NR and EF; and the Line of Station KP is perpendicular to LM and GH.

For EF, GH, LM, and NR being all perpendicular to the Vertical Plane IKoP[d], they are therefore perpendicular to all Lines in that Plane which touch them[e]; but IK, oP, Io, and KP are all Lines in the Vertical Plane, therefore all theſe Lines are perpendicular to ſuch of the Lines EF, GH, LM, and NR, as they ſeverally touch.

C O R. 2.

Hence a Line IK drawn from I perpendicular on the Directing Line LM, is the Eye's Director, and cuts the Directing Line in K, the Point of Station[f]: a Line drawn through O, the Center of the Picture, perpendicular to EF, or GH, the Vaniſhing or Interſecting Line of any Original Plane, is the Vertical Line of that Plane, and cuts EF in o its Center, and GH in P, the Foot of the Vertical Line[g]: a Line drawn from I, perpendicular on the Vaniſhing Line, is the Radial of that Vaniſhing Line, and cuts it in o its Center[h]: and a Line drawn through K or P in an Original Plane, perpendicular to the Directing or Interſecting Line of that Plane, is the Line of Station of the Original Plane[i].

C O R. 3.

The Radial Io of any Original Plane, is parallel and equal to KP, ſo much of the

[a] Def. 6.

[b] 18 El. 11.

[c] 19 El. 11.

[d] Cor. Def. 11.

[e] Def. 3 El. 11.

[f] Def. 12.

[g] Def. 13.

[h] Def. 14.

[i] Def. 15.

the Line of Station of that Plane as lies between K and P; and the Eye's Director I K is parallel and equal to *o* P, ſo much of the Vertical Line as lies between *o* and P.

They are reſpectively parallel, becauſe they are the Interſections of parallel Planes with the Vertical Plane [a]; and the Figure I K *o* P being therefore a Parallelogram, its oppoſite Sides are equal [b].

[a] 16 El. 11.
[b] 34 El. 1.

D E F. 16.

If an Original Line be not parallel to the Picture, it will, if produced, cut both the Picture and the Directing Plane; the Point where it cuts the Picture, is called the *Interſecting Point*; and that where it cuts the Directing Plane, is called the *Directing Point of the Original Line.*

Thus if the Original Line Q B, being produced, cut the Picture and the Directing Plane in the Points A and Q; the firſt is the Interſecting Point, and the other is the Directing Point of the Original Line.

D E F. 17.

If an Original Line Q B, not parallel to the Picture, be propoſed, another Line I *x*, is ſuppoſed to be drawn from the Eye to the Picture, parallel to the Original Line, and cutting the Picture in ſome Point *x*; the Point *x* is called the *Vaniſhing Point*, and the Line I *x* is called the *Radial of the Original Line*; and I *x* being the meaſure of the Diſtance of the Point *x* from the Eye, it is, when conſidered only with regard to its Length, called ſimply the *Diſtance of the Vaniſhing Point x.*

C O R.

The Radial I *o* of the Original Plane L M G H, is alſo the Radial of the Line of Station K P; and the Center *o* of the Vaniſhing Line E F, is the Vaniſhing Point of the Line K P; I *o* and K P being parallel [c].

[c] 16 El. 11.

D E F. 18.

If a Plane I *x* Q A be imagined to paſs through the Original Line Q B and its Radial I *x*, that Plane is called the *Radial Plane of the Original Line*; the Interſection I Q of this Plane with the Directing Plane, is called the *Director of the Original Line*; and the Interſection *x* A, of the ſame Plane with the Picture, indefinitely produced both ways beyond *x* and A, is called the *Indefinite Image of the Original Line*; and that part of it, which lies between *x* and A, is called the *whole Perſpective of the Original Line.*

C O R. 1.

The Radial I *x* of an Original Line Q B, is parallel and equal to Q A, ſo much of the Original Line as lies between Q and A its Directing and Interſecting Points; and the Director I Q of the Original Line, is parallel and equal to *x* A, the whole Perſpective of that Line, or ſo much of its indefinite Image as lies between *x* and A, its Vaniſhing and Interſecting Points.

Becauſe I Q and *x* A are the Interſections of the Picture and Directing Plane, with the Radial Plane I *x* Q A, thoſe Lines are therefore parallel [d]; and the Radial I *x* being parallel to the Original Line [e], the Figure I *x* Q A is therefore a Parallelogram, and conſequently its oppoſite ſides are equal [f].

[d] 16 El. 11.
[e] Def. 17.
[f] 34 El. 1.

C O R. 2.

The Eye's Director I K, is alſo the Director of the Line of Station K P; and the Vertical Line O P, produced at Pleaſure, is the Indefinite Image of that Line; and the part *o* P is its whole Perſpective; for the Vertical Plane I K *o* P is the Radial Plane of the Line K P.

C O R. 3.

Hence the Diſtance between the Eye's Parallel and Directing Line of any Plane, is equal to the Diſtance between its Vaniſhing and Interſecting Lines, which Diſtance may be called the *Depth of the Original Plane.*

D E F. 19.

If two Planes which Interſect, be not perpendicular, they are ſaid to Incline to each other; and the *Angle of Inclination* of thoſe two Planes, is the Acute Angle comprehended between two Lines drawn, one in each of the Planes, from the ſame Point of their common Interſection, perpendicular to it. Or if a Plane be ſuppoſed to cut both the Inclining Planes perpendicular to their common Interſection, it is the Acute Angle made by the Interſections of this perpendicular Plane with the other two Planes.

Fig. 9. Thus if two Planes LMGH and EFGH interſect in GH, and if from any Point A in the Line GH, a Line AC be drawn in the Plane EFGH, perpendicular to GH, and from the ſame Point A another Line AB be drawn in the Plane LMGH, alſo perpendicular to GH, then the Acute Angle CAB is the Angle of Inclination of the Planes LMGH and EFGH to each other.

It is evident alſo, that if a Plane CBDK were imagined to paſs through the Lines AC and AB, that Plane would be perpendicular to GH[a], and conſequently perpendicular to both the Planes LMGH and EFGH[b], and CA and AB would be the Interſections of this perpendicular Plane with thoſe two Planes.

[a] 4 El. 11.

[b] 18 El. 11.

COR. 1.

The Angle of Inclination CAB of the Planes LMGH and EFGH, is the largeſt that can be made by the Interſections of thoſe Planes with any other Plane, perpendicular to either of them.

From any Point C, in the Line AC, draw CB perpendicular on AB, which will be therefore perpendicular to the Plane LMGH; and all Planes paſſing though CB, will alſo be perpendicular to that Plane[c].

[c] 18 El. 11.

Imagine then any other Plane paſſing through CB, different from the Plane CAB, and if it be not parallel to GH, the Interſection of that Plane with the Plane LMGH will cut GH in ſome Point R. Let therefore BR be the Interſection of the ſuppoſed Plane with the Plane LMGH, and conſequently CR its Interſection with the Plane EFGH.

Then becauſe CA and BA are by ſuppoſition perpendicular to GH, they are the ſhorteſt Lines that can be drawn from the Points C and B to the Line GH, and therefore ſhorter than CR and BR: therefore in the Rectangular Triangles CBA, CBR, the Sides AC and AB being each ſhorter than the correſponding Sides RC and RB, and the Side BC being common to both Triangles, and the Angle at B in both of them Right (CB being perpendicular to the Plane LMGH) the Angle CAB muſt be larger than the Angle CRB[d], which laſt is the Angle made by the Interſections of the Planes EFGH and LMGH with a Plane CBR, perpendicular to LMGH, and different from the Plane CBDK.

[d] 21 El. 1.

It may in the ſame manner be ſhewn, that the Angle CAB is larger than the Angle made by the Interſections of the two given Planes, with any other Plane perpendicular to the Plane EFGH, and different from the Plane CBDK.

For if from B a Line BQ be drawn perpendicular on AC, it will be perpendicular to the Plane EFGH, to which all Planes paſſing through BQ will alſo be perpendicular; and QA and BA, being both perpendicular to GH, are the ſhorteſt Lines that can be drawn to that Line from the Points Q and B, and therefore contain a larger Angle than any two other Lines QR and BR do, which are drawn from Q and B to any other Point R in the Line GH.

COR. 2.

The nearer the Point R is taken to the Point A, the Angle CRB becomes the larger; and the farther the Point R is taken from A, that Angle is the leſs.

DEF. 20.

The Angle of Inclination of a Line CA to a Plane LMGH, is the Acute Angle CAB, made by the given Line CA with AB, the Interſection of the Plane LMGH with a Plane CDBA, paſſing through CA perpendicular to the given Plane.

Note, By a Line is always meant a ſtraight or right Line, if not expreſſed to be otherwiſe.

LEM. I.

Fig. 8. If two Planes RNML, EFGH be parallel, and any Line ML in the one Plane, be parallel to a Line GH in the other; then if any two other Lines IQ and FA be drawn, one in each of theſe Planes, inclining the ſame way on LM and GH, and making equal Angles with them, theſe laſt Lines IQ and FA will likewiſe be parallel.

Dem. Draw QA. Then if a Plane be imagined to paſs by IQ and QA, this Plane muſt cut the Plane EFGH in ſome Line Ax, which will be parallel to IQ[e]; wherefore IQ and QM in the Plane RNLM, being reſpectively parallel to Ax and AH

[e] 16 El. 11.

A H in the Plane E F G H, the Angles IQM and *x*AH will be equal [a]; but by the Suppoſition, the Angles I Q M and F A H are equal, therefore the Angle F A H is equal to the Angle *x* A H; but from the Point A, in the Line A H, there cannot be two different Lines drawn in the Plane E F G H, making the ſame Angle with, and inclining the ſame way on the Line A H; therefore A F muſt coincide with A *x*, and is conſequently parallel to I Q. *Q. E. D.*

GENERAL COROLLARY.

The Director's Radials, and whole Perſpectives of all Lines in an Original Plane, will continue the ſame, although the Angle of Inclination of the Picture to that Plane, be anywiſe changed, while the ſame Interſecting and Directing Lines are retained, and the Eye continues in the ſame Point of the Directing Plane.

Fig. 8.

Dem. Becauſe the Director I Q, of the Original Line Q B, is parallel to A *x*, the whole Perſpective of that Line [b]; and the Directing Line L M is parallel to the Interſecting Line G H [c]; therefore the Angles I Q M and *x* A H are equal [d]. Now if the Directing Plane R N L M be anywiſe turned on the Line L M, the Point I continuing in the ſame Point of that Plane, and moving along with it, the Line I Q and the Angle I Q M will not be thereby varied; neither will the Line *x*A, or the Angle *x* A H in the Picture E F G H, be changed by turning the Picture on the Line G H. But in all Inclinations of the Picture to the Original Plane, the Picture and Directing Plane are conſtantly ſuppoſed parallel; therefore if the Picture and Directing Plane be anywiſe equally turned on the Lines G H and L M, ſo as to continue parallel to each other, the Lines G H and L M continuing parallel, and the Angle *x* A H in the Picture remaining equal to the Angle I Q M in the Directing Plane; the Lines *x* A and I Q muſt continue parallel [e]; and I Q being ſtill the Director of the Original Line Q B, the Line A *x* in the Picture, which paſſes through the Interſecting Point A of the Original Line, and is parallel to its Director I Q, is therefore the whole Perſpective [f]; and as I Q and A *x* continue equal as they were before, the Figure I *x* Q A will ſtill be a Parallelogram; and conſequently I *x* being parallel and equal to Q A [g], it will continue to be the Radial of that Line [h].

The ſame may be ſhewn of the Director's Radials and whole Perſpectives of any other Lines in the Original Plane. *Q. E. D.*

Of the General Relations of Objects to the preparatory Planes, Lines, and Points uſed in Stereography.

THEOR. I.

An Original Line parallel to the Picture, hath no Vaniſhing, Interſecting, or Directing Points; or thoſe Points may be imagined to be at an Infinite Diſtance.

Fig. 10.

Let E F G H be the Picture, R N L M the Directing Plane parallel to it, and I the Place of the Eye in the Directing Plane, and let A B be an Original Line parallel to the Picture.

Dem. If a Plane A B a b, be conceived to paſs through the given Line A B and the Eye at I, cutting the Picture and the Directing Plane in *a b* and a b, theſe two Lines will be parallel to each other [i] and alſo to the Line A B, ſeeing a Plane Y may paſs through A B parallel to the Picture: now the Vaniſhing Point of A B being that Point where a Line drawn from I parallel to A B, cuts the Picture [k], Ia is therefore the Line which ought to produce that Vaniſhing Point; but I a being parallel to *a b*, a Line in the Plane of the Picture, it can never cut the Picture to determine that Point; wherefore AB hath no Vaniſhing Point, or its Vaniſhing Point may be conceived to be Infinitely diſtant; and the Original Line A B being itſelf parallel to the Picture and Directing Plane, it can never cut either of thoſe Planes to determine its Interſecting or Directing Points [l]. *Q. E. D.*

THEOR. II.

The Indefinite Image of a Line parallel to the Picture, is parallel to its Original.

Fig. 10.

Dem. The Indefinite Image of the Original Line AB, is the Interſection of the

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[a] 10 El. 11.

[b] Cor. 1. Def. 18.

[c] Cor. 1. Def. 10.

[d] 10 El. 11.

[e] Lem. I.

[f] Cor. 1. Def. 18.

[g] 34 El. 1.

[h] Cor. 1. Def. 18.

[i] 16 El. 11.

[k] Def. 17.

[l] Def. 16.

Picture with a Plane paſſing through the Eye and that Line [a]; but the Plane A B a b paſſes through A B and the Point I, therefore *a b* the Interſection of that Plane with the Picture is the Indefinite Image of A B, which two Lines are parallel [b].

If $\alpha\beta$ were an Original Line behind the Eye, parallel to the Picture, the Plane $\alpha\beta$ab paſſing through I, will cut the Picture in *a b*, the Indefinite Tranſprojected Image of $\alpha\beta$; and $\alpha\beta$, a b, and *ab* will be parallel [c]. *Q. E. D.*

COR. 1.

The Line a b, which paſſes through I, may be taken either as the Radial, or as the Director of the Original Line A B or $\alpha\beta$.

It may be taken as the Radial of the Original Line, as being parallel to it [d]; and it may be taken as the Director of that Line, as being parallel to its Image [e].

COR. 2.

All parallel Original Lines, as A B and C D, which are parallel to the Picture, have parallel Images *a b* and *c d*, and have the ſame Imaginary Radial or Director a b.

They have parallel Images, becauſe theſe are parallel to their Originals [f]; and they have the ſame Radial or Director, becauſe there can be but one Line a b drawn through I, parallel to the Original Lines and their Images.

COR. 3.

If two Original Lines C D and Q V, parallel to the Picture, make with each other any given Angle D Q V, their Images *c d* and *q u*, will make together the like Angle *d q u*; and ſo will their Imaginary Radials or Directors a b and I *n*.

For the Images are reſpectively parallel to their Originals, and alſo to their Imaginary Radials or Directors [g].

THEOR. III.

An Original Plane, parallel to the Picture, hath no Vaniſhing, Interſecting, or Directing Lines; or thoſe Lines may be imagined to be at an Infinite Diſtance.

Fig. 10. *Dem.* Becauſe a Plane paſſing through the Eye, parallel to the Original Plane Y, and which, by its Interſection with the Picture, ought to determine the Vaniſhing Line [h], is here parallel to the Picture, and the ſame with the Directing Plane, and therefore can never cut the Picture to determine the Vaniſhing Line; and the Original Plane being parallel to the Picture and Directing Plane, can never cut either of theſe Planes to determine its Interſecting or Directing Lines [i]. *Q. E. D.*

COR.

If an Original Plane Y, parallel to the Picture, cut any other Plane whatſoever, as L M C D; their common Interſection C D will be parallel to the Vaniſhing, Interſecting, and Directing Lines of this laſt Plane.

For the Plane Y being parallel to the Picture and Directing Plane, their Interſections C D, G H, L M, with the Plane L M C D, are parallel [k]; but G H and L M are the Interſecting and Directing Lines of the Plane L M C D, to which its Vaniſhing Line is alſo parallel [l], wherefore C D is alſo parallel to that Vaniſhing Line [m].

GENERAL COROLLARY.

All Lines in an Original Plane which is parallel to the Picture, are alſo parallel to the Picture, and therefore come within the Rules of the firſt and Second Theorems and their Corollaries.

THEOR. IV.

If an Original Line, not parallel to the Picture, be produced indefinitely on each Side of the Directing Plane, its Indefinite Image will be a Line drawn in the Picture through the Vaniſhing and Interſecting Points of the Original Line, and indefinitely produced on both Sides of the Vaniſhing Point.

Fig. 11. Let E F G H be the Picture, N R L M the Directing Plane, and I the Place of the Eye

[a] Def. 5.

[b] Theor. I.

[c] 16 El. 11.

[d] Def. 7.

[e] Cor. Def. 18.

[f] 9 El. 11.

[g] 10 El. 11.

[h] Def. 10.

[i] Def. 8.

[k] 16 El. 11.

[l] Cor. 1. Def 10.

[m] 9 El. 11.

Eye in that Plane; and let T B be an Original Line cutting the Picture and Directing Plane in P and K its Interſecting and Directing Points; and let I x be the Radial, and x the Vaniſhing Point of the given Line, and I x K P its Radial Plane: it muſt be proved, that the Indefinite Image of T B is the Line ds, drawn in the Picture through P and x, and indefinitely produced beyond d and s.

Dem. Becauſe Ix and T B are parallel [a], all Lines drawn from I to any Point in T B, will be in the ſame Plane with them [b], *viz.* the Radial Plane I x K P; and becauſe x and P are Points in this Plane, and alſo in the Picture, the Line ds drawn through P and x, is the Interſection of the Radial Plane with the Picture; wherefore all Lines drawn from I to any Point in T B (except the Point K) muſt, if produced, cut the Picture ſomewhere in the Line ds produced; but the Image of every Point in the Line T B is where a Line drawn from the Eye to that Point, cuts the Picture [c]; therefore the Image of every Point in the Line T B (except only of the the Point K) and conſequently of the Line itſelf, muſt be in the Line ds indefinitely produced. *Q. E. D.*

C O R. 1.

The Directing Point K of the Original Line hath no Image, or its Image may be imagined to be at an infinite Diſtance.

For the Line I K, which ought by its Interſection with the Picture to produce that Image, lies in the Directing Plane, and is therefore parallel to the Picture, and ſo can never cut it.

C O R. 2.

The Image of any Point in the part P B of the Original Line, indefinitely produced beyond B, will fall ſomewhere in Px, between P and x its Interſecting and Vaniſhing Points, and the Image of the moſt diſtant Point in the Original Line beyond B can never reach to x.

The Point P, being the Interſection of the Original Line with the Picture, is its own Image; and the Images of A and B are at a and b, where I A and I B cut Px between P and x. Now if the Point B, in the Original Line, were taken at ever ſo great a Diſtance from P, the Line I B will ſtill make an Angle with P B, to which the Angle xI B will be equal [d]; and therefore the Points x and b can never coincide, unleſs the Point B were ſuppoſed infinitely diſtant, in which Caſe the Angle I B P would vaniſh, and I B would become parallel to P B, and coincide with I x.

D E F. 21.

The Point x is therefore called the *Vaniſhing Point of the Original Line* [e]; and as the Images a and b of the Points A and B, and of all other Points in P B, indefinitely produced beyond B, are Perſpective; Px, in which all thoſe Images lie, is therefore called the *whole Perſpective of the Original Line* [f]; and the Indefinite Part P B of the Original Line, is called its *Perſpective Part.*

C O R. 3.

The Image of any Point in the part P K of the Original Line, which lies between P and K its Interſecting and Directing Points, cannot fall nearer to the Vaniſhing Point x than P, but may be any where in Pd, indefinitely produced beyond d.

The Point P is its own Image; now if any Point C be taken between P and K, its Image will be at c, where I C cuts Pd, which Image muſt neceſſarily fall farther from x than P does; and if the Point D be taken nearer to K, its Image d will fall ſtill farther from x in the Line Pd; and the nearer the Point D is taken to K, its Image will fall the farther from x beyond P, until if D and K coincide, its Image will be at an Infinite Diſtance in the Line Pd [g].

D E F. 22.

As the Images of all Points in P K are Projected on the Line Pd, the part Pd of the Indefinite Image is called the *Projective Part of that Image*; and the part P K is called the *Projective Part of the Original Line.*

C O R. 4.

The Image of any Point in K T, that part of the Original Line which lies behind K, muſt fall ſomewhere in xs, that part of the Indefinite Image which lies on the contrary Side of x from P, indefinitely produced beyond s; and the Image of

[a] Def. 17.

[b] 7 El. 11.

[c] Def. 5.

[d] 29 El. 1.

[e] Def. 17.

[f] Def. 18.

[g] Cor. 1.

the moſt diſtant Point in KT produced beyond T, can never reach to the Vaniſhing Point x.

The Image of any Point S in the Line KT, is at s, where SI cuts xs; and if S were taken nearer to K, its Image would be ſtill farther diſtant from x in the Line xs, until if S and K ſhould coincide, its Image would be infinitely diſtant in the Line xs; ſeeing KI being parallel to sd, it may be conceived to meet it at an infinite Diſtance at either Extremity: again, if any Point T be taken in KT beyond S, its Image will fall at t between s and x; and the farther the Point T is taken from K, its Image will fall the nearer to x, but can never reach to that Point, until the Original Point T becomes infinitely diſtant; in which Caſe TI may be conceived to become parallel to TK, and ſo to coincide with Ix.

DEF. 23.

As the Images of all Points in KT are Tranſprojected on the Line xs, the part xs of the Indefinite Image, indefinitely produced beyond s, is called the *Tranſprojective Part of that Image*; and the part KT of the Original Line, indefinitely produced beyond T, is called its *Tranſprojective Part*.

SCHOL.

It may be here obſerved, that as the Images of the moſt diſtant Extremities of the Original Line TB, indefinitely produced both ways, are at the Vaniſhing Point x; ſo the Originals of the moſt diſtant Extremities of the Indefinite Image ds, produced in like manner, are at the Directing Point K.

Hence it follows, that the Image of any determinate part TC of the Original Line which paſſes through K, is not one continued Line in the Picture, but two diſtinct and indefinite Lines; the Image of the part CK of the Original Line, being cd indefinitely produced beyond d, and the Image of the part TK, being ts indefinitely produced beyond s.

After the ſame manner, the Original of any determinate part as, of the Indefinite Image which paſſes through the Vaniſhing Point x, is not one continued Line, but two diſtinct and Indefinite Original Lines; ax repreſenting the part AB of the Original Line, indefinitely produced beyond B, as sx repreſents the part ST of the ſame Line, indefinitely produced behind T.

DEF. 24.

The Line tc which joins the Images of T and C, the Extremities of an Original Line which paſſes through the Directing Line, is called the *Complement of the Image of* TC; and the Line TC which joins the Originals of t and c, the Extremities of a Line tc which paſſes through the Vaniſhing Line, is called the *Complement of the Original of* tc.

THEOR. V.

All parallel Original Lines, not parallel to the Picture, have the ſame Radial and Vaniſhing Point, and their Images all meet in that Vaniſhing Point.

Dem. Becauſe a Line drawn from the Eye, parallel to any one of the Original Lines, is parallel to all the reſt[a], which Line is therefore their common Radial, and the Point where that Radial cuts the Picture, is their common Vaniſhing Point[b]; but the Indefinite Image of every Line paſſes through its Vaniſhing Point[c]; and this Point being common to all the propoſed Parallels, their Images therefore all meet in that Point. *Q. E. D.*

COR. 1.

No two Original Lines, which are not parallel, can have the ſame Radial and Vaniſhing Point.

Becauſe the ſame Line drawn from the Eye cannot be parallel to them both.

COR. 2.

All Original Lines, perpendicular to the Picture, have the Center of the Picture for their Vaniſhing Point, and the Axe of the Eye for their Radial.

Becauſe the Axe of the Eye is parallel to all ſuch Lines[d].

[a] 9 El. 11.

[b] Def. 17.

[c] Theor. 4.

[d] Def. 6, and 8 El. 11.

THEOR. VI.

All Original Lines, which have their Directing Points any where in the ſame Director, have parallel Images.

Dem. Becauſe thoſe Images being all parallel to the ſame Director[a], they are parallel amongſt themſelves[b]. Q. E. D.

COR.

The Directors of any two Original Lines make at the Eye an Angle equal to that made by their Images.

Becauſe the Directors are reſpectively parallel to thoſe Images[c], and all Directors meet at the Eye.

THEOR. VII.

If two Original Lines meet or croſs each other, their Indefinite Images will alſo meet or croſs in the Image of the Interſection of the Original Lines.

Dem. Becauſe the Interſection of the Original Lines is a Point common to both thoſe Lines, the Image of that Interſection is therefore a Point common to both their Images; which Images muſt therefore meet or croſs in that Point. Q. E. D.

COR. 1.

If the Original Lines meet, or croſs in the ſame Point of the Directing Plane, their Images will be parallel.

For the Interſection of the propoſed Lines being a Directing Point, its Image is at an infinite Diſtance[d]; the Images therefore of the propoſed Lines meet at an infinite Diſtance, that is to ſay, they are parallel.

COR. 2.

If the Original Lines be parallel, their Images will meet in the ſame Vaniſhing Point.

For the Original Lines being parallel, their Interſection may be conceived to be at an infinite Diſtance in each of thoſe Lines; and their Common Vaniſhing Point being the Image of thoſe infinitely diſtant Points[e], the Images therefore meet there.

Theſe two Corollaries are the ſame in effect with the two preceding Theorems, but are deduced from a different Conſideration.

COR. 3.

The Images of all parallel Lines whatſoever, are either parallel or meet in ſome one Point.

If the Original Lines are parallel to the Picture, their Images will be parallel[f]; and if the Original Lines are not parallel to the Picture, their Images meet in their common Vaniſhing Point[g].

THEOR. VIII.

If an Original Line being produced, paſs through the Eye, its Vaniſhing and Interſecting Points will coincide, and its Directing Point will be the ſame with the Place of the Eye.

Dem. Becauſe the Original Line and its Radial muſt coincide, and the Interſection of that Line with the Directing Plane is in the Eye. Q. E. D.

COR.

The Indefinite Image of ſuch a Line is only a Point, which Point is the Image of every poſſible Point in the Original Line.

Becauſe a Line from the Eye to any Point in the Original Line, coincides with that Line, which can cut the Picture but in one Point.

THEOR. IX.

If an Original Plane be not parallel to the Picture, the Eye's Director

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[a] Cor. 1. Def. 18.

[b] 9 El. 11.

[c] 10 El. 11.

[d] Cor. 1. Theor. 4.

[e] Cor. 2 and 4. Theor. 5.

[f] Cor. 2. Theor. 2.

[g] Theor. 5.

rector and Vertical Line of that Plane, will make with the Line of Station and Radial, Angles equal to the Angle of Inclination of the Original Plane to the Picture.

Fig. 8.

Dem. Becauſe the Vertical Plane IK*o*P is perpendicular to the Picture, and to the Vaniſhing and Directing Planes, as well as to the Original Plane[a], the Interſections of the Vertical Plane with thoſe Planes, meaſure the Angle of Inclination of thoſe Planes to each other[b]; conſequently the Angle *o*PS, made by the Vertical Line *o*P with the Line of Station KPS, is the Angle of Inclination of the Original Plane to the Picture, to which the Angles IKP, WI*o*, and I*o*P are equal, IK*o*P being a Parallelogram[c]. *Q. E. D.*

[a] Cor. Def. 11.

[b] Def. 19.

[c] Cor. 3. Def. 15. and 29 El. 1.

COR. 1.

If the Original Plane be perpendicular to the Picture, its Vaniſhing Line will paſs through the Center of the Picture; the Axe of the Eye will be its Radial, and the Eye's Director and Vertical Line will be perpendicular to the Line of Station, and conſequently to the Original Plane.

Becauſe, if the Original Plane be perpendicular to the Picture, the Vaniſhing Plane will alſo be perpendicular to the Picture, and muſt therefore paſs through IO the Axe of the Eye[d].

[d] 18 El. 11.

COR. 2.

If the Original Plane be not perpendicular to the Picture, its Vaniſhing Line cannot paſs through the Center of the Picture.

Becauſe all Planes, which paſs through the Eye's Axe, are perpendicular to the Picture[e]; but a Plane perpendicular to the Picture, cannot be parallel to one which is not; therefore the Vaniſhing Plane of an Original Plane, not perpendicular to the Picture, cannot paſs through the Axe of the Eye, and conſequently the Vaniſhing Line cannot paſs through the Center of the Picture.

[e] 18 El. 11.

THEOR. X.

All Lines in an Original Plane have their Vaniſhing, Interſecting, and Directing Points, in the Vaniſhing, Interſecting, and Directing Lines of that Plane.

Fig. 8.

Let QB be a Line in the Original Plane LMGH, cutting the Picture and Directing Plane in A and Q, its Interſecting and Directing Points[f].

Dem. The Line QB being in the Original Plane, it can cut the Picture and Directing Plane only in their Interſections with the Original Plane, which Interſections are GH and LM[g]; wherefore A, the Interſecting Point of QB, is in GH the Interſecting Line; and Q, the Directing Point of QB, is in LM the Directing Line of the Original Plane: in the next place, I*x* drawn parallel to the Original Line QB, by its Interſection with the Picture, determines *x* the Vaniſhing Point of that Line[h]; but I*x* being parallel to QB, it is alſo parallel to the Original Plane, wherefore a Plane may be drawn through I*x* parallel to that Plane: and as no two different Planes can be drawn through I*x* parallel to the Original Plane, this Plane muſt be the Vaniſhing Plane[i], the Interſection of which with the Picture being the Vaniſhing Line EF[k], the Vaniſhing Point *x* of the Line QB is therefore in that Line. *Q. E. D.*

[f] Def. 16.

[g] Def. 8.

[h] Def. 17.

[i] Def. 9.

[k] Def. 10.

COR. 1.

The Image of any Point, Line, or Figure, in that part of an Original Plane which lies beyond its Interſecting Line, muſt fall between the Interſecting and Vaniſhing Lines of that Plane; the Image of any Point, Line, or Figure, in that part of the Original Plane which lies between its Interſecting and Directing Lines, cannot fall nearer to the Vaniſhing Line of that Plane than its Interſecting Line; and the Image of any Point Line, or Figure, in that part of the Original Plane which lies behind its Directing Line, muſt fall on the oppoſite Side of the Vaniſhing Line to the Interſecting Line; and laſtly, the Image of the moſt diſtant Point in the Original Plane, either before or behind the Directing Plane, can never reach the Vaniſhing Line of that Plane.

All this evidently follows from the ſecond, third, and fourth Corollaries of Theor. IV.

DEF.

DEF. 25.

Hence the Vaniſhing Line is ſo called; and it being evident from the preceding Corollary, that when the Original Plane repreſents the Ground, the Vaniſhing Line of that Plane will be the utmoſt bounds, to which the Image of any part of the Ground, ſuppoſing it level, can paſs; this Line therefore determines the Horizon, and is in this View called the ***Horizontal Line***; as on the ſame Suppoſition the Interſecting Line is called the *Ground Line*, it then repreſenting the Interſection of the Picture with the Ground.

DEF. 26.

As at the twenty firſt, twenty ſecond, and twenty third Definitions, an Original Line and its Image are diſtinguiſhed into their Perſpective, Projective, and Tranſprojective Parts; ſo the correſponding Parts of an Original Plane, and of the Images of any Lines or Points in thoſe Parts, may have the ſame Names given them: and as the Images of all Points, Lines, and Figures in the Perſpective Part of the Original Plane, even of the moſt diſtant, are confined between the Interſecting and Vaniſhing Lines of that Plane; that Space is therefore called the *Depth of that Plane*[a].

COR. 2.

A Line drawn through any two Vaniſhing Points of Lines in an Original Plane, is the Vaniſhing Line of that Plane; a Line drawn through two Interſecting Points, is the Interſecting Line; and a Line drawn through two Directing Points, is the Directing Line of the Plane, in which the Original Lines lie.

For any two Points in a ſtraight Line being given, the whole Line is given.

COR. 3.

All Original Lines, parallel to an Original Plane, have their Vaniſhing Points in the Vaniſhing Line of that Plane.

Becauſe Lines may be drawn in the Original Plane parallel to the propoſed Lines; and all parallel Lines having the ſame Vaniſhing Points[b], the Vaniſhing Points of the propoſed Lines are therefore in the Vaniſhing Line of the Original Plane, to which they are parallel[c].

THEOR. XI.

The *Radial* of a Line in an Original Plane makes the ſame Angle with the Eye's Parallel and the Vaniſhing Line of that Plane, as the Original Line makes with the Directing and Interſecting Lines, or any other Line in that Plane parallel to them.

Dem. Becauſe the Radial is parallel to the Original Line[d], and the Eye's Parallel and Vaniſhing Line are parallel to the Directing and Interſecting Lines[e]; therefore the Radial makes the ſame Angle with the Eye's Parallel and Vaniſhing Line, as the Original Line makes with the Directing and Interſecting Lines[f]. *Q. E. D.*

COR. 1.

The Radial of a Line in an Original Plane makes the ſame Angle with the Radial of that Plane, as the Original Line makes with the Line of Station.

Becauſe they are reſpectively parallel.

COR. 2.

If the Original Line be parallel to the Line of Station, its Radial will coincide with the Radial of the Original Plane[g].

COR. 3.

The Radials of any two Lines in an Original Plane, make together an Angle equal to that made by the Original Lines.

THEOR. XII.

The Director of a Line in an Original Plane makes the ſame Angle with the Eye's Parallel and Directing Line of that Plane, as the Image of the given Line doth with the Vaniſhing and Interſecting Lines of that Plane.

Dem. Becauſe the Director and the Image of the Original Line are parallel[h]. *Q. E. D.*

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[a] Cor. 3. Def. 18.

[b] Theor. 5.

[c] Theor. 10.

[d] Def. 17.

[e] Cor. 1. Def. 10.

[f] 10 El. 11.

[g] Cor. Def. 17. and Theor. 5.

[h] Cor. 1. Def. 18.

COR. 1.

The Director of a Line in an Original Plane makes the ſame Angle with the Eye's Director of that Plane, as the Image of the Original Line doth with the Vertical Line of that Plane.

Becauſe the Eye's Director and Vertical Line are parallel [a], and ſo is the Director of the Original Line to its Image [b].

COR. 2.

The Indefinite Images of all Lines whatſoever, whoſe Directing Points fall any where in the Eye's Parallel, relating to an Original Plane, are parallel to the Vaniſhing Line of that Plane.

For the Images are parallel to their Director, which, in this caſe, being the Parallel of the Eye, is therefore parallel to the Vaniſhing Line of the Original Plane [c].

COR. 3.

The Indefinite Images of all Lines whatſoever, whoſe Directing Points fall any where in the Eye's Director, relating to an Original Plane (whether the propoſed Lines be in or out of that Plane) are parallel to its Vertical Line.

For the Eye's Director, which is the Director of the propoſed Lines, is parallel to the Vertical Line [d].

COR. 4.

If any two Lines in an Original Plane cut the Directing Line of that Plane in the ſame Point, their Images will be parallel.

Becauſe the Original Lines muſt then have the ſame Director [e].

COR. 5.

If the Images of any two Original Lines in the ſame Plane be parallel, the Original Lines (if they be not parallel to the Picture) muſt have the ſame Directing Point.

For there can be but one Director drawn parallel to both the given Images, and this Director can cut the Directing Line of the Original Plane only in one Point, which Point is therefore the Common Directing Point of the propoſed Original Lines.

THEOR. XIII.

All parallel Original Planes have the ſame Vaniſhing Line, Center, and Radial; and their Interſecting and Directing Lines are parallel.

Fig. 12. Let EFGH be the Picture, NRLM the Directing Plane, and I the Place of the Eye, and let LMAB and *lm*ab be two parallel Original Planes.

Dem. Becauſe if through I, a Vaniſhing Plane NREF be drawn parallel to either of the Original Planes, it will alſo be parallel to the other; and the ſame Vaniſhing Plane can form but one Vaniſhing Line EF, of which IO is the Radial, and O the Center. Laſtly gh and GH, the Interſecting Lines, and *lm* and LM, the Directing Lines of the Original Planes, muſt be parallel, they being all parallel to the ſame Vaniſhing Line EF [f]. *Q. E. D.*

COR. 1.

No two Planes which are not parallel, can have the ſame Vaniſhing Line.

Becauſe the ſame Vaniſhing Plane NREF cannot be parallel to them both.

COR. 2.

All parallel Original Planes have the ſame Vertical Line, Vertical Plane, and Parallel of the Eye.

Becauſe there can be but one Line O*p* drawn through the Center of the Picture O, perpendicular to the ſame Vaniſhing Line EF [g], and but one Line NR drawn through the Eye parallel to that Vaniſhing Line [h]: and no two different Vertical Planes can paſs through the Eye and the ſame Vertical Line.

THEOR. XIV.

If two Original Planes cut each other in a Line parallel to the Picture, their Vaniſhing Lines will be parallel, and their Interſecting and Directing Lines will alſo be parallel, if none of theſe laſt coincide.

Fig. 12. The ſame things being ſuppoſed as before, let LMAB and *lm*AB be two Original Planes cutting each other in AB, a Line parallel to the Picture.

Dem.

[a] Cor. 3. Def. 15.

[b] Cor. 1. Def. 18.

[c] Cor. 1. Def. 10.

[d] Cor. 3. Def. 15.

[e] Cor. 1. Def. 18.

[f] Cor. 1. Def. 10. and 9 El. 11.

[g] Cor. 2. Def. 15.

[h] Cor. 1. Def. 10.

Dem. If a Plane Y be imagined to paſs by the Line AB parallel to the Picture and Directing Plane, then the Interſections AB, GH, and LM of the Plane LMAB with thoſe three Planes will be parallel [a]; and for the ſame reaſon AB, gh, and lm the Interſections of the ſame three Planes with the Plane lm AB, will be parallel; conſequently GH and gh the Interſecting Lines of the two Original Planes, and LM and lm their Directing Lines will be parallel [b]. But the Vaniſhing Lines of theſe Planes are parallel to their Interſecting Lines [c], and the Original Planes themſelves not being parallel, they cannot have the ſame Vaniſhing Line [d]; therefore the Vaniſhing Lines EF and ef are diſtinct and parallel. *Q. E. D.*

COR. 1.

If the Interſection AB of the given Planes were in the Picture, that would be their common Interſecting Line; and if the Interſection of the given Planes were in the Directing Plane, that would be their common Directing Line; but in either Caſe their other Lines would be parallel.

COR. 2.

All Planes, whoſe Vaniſhing Lines are parallel, have the ſame Vertical Line, Vertical Plane, and Parallel of the Eye.

For oP which is perpendicular to EF, is alſo perpendicular to ef; and NR parallel to EF, is alſo parallel to ef.

COR. 3.

If the Vaniſhing Lines of two Original Planes be parallel, their Radials will make together an Angle equal to the Angle of Inclination of the Original Planes.

For the Vertical Plane Ikop being perpendicular to both the Original Planes, the Interſections KS and kS of that Vertical Plane with the Original Planes, which are their Lines of Station, meaſure their Angle of Inclination [e]; and the Lines of Station KS and kS being reſpectively parallel to the Radials IO and Io, they make together equal Angles [f].

THEOR. XV.

If the Vaniſhing Lines of two Original Planes be parallel, their Common Interſection will be parallel to the Picture.

Dem. For the Original Planes having the ſame Vertical Plane Ikop [g], which is perpendicular to them both [h], it is therefore perpendicular to their Common Interſection AB [i]; but the Vertical Plane is perpendicular to the Vaniſhing Lines of the Original Planes [k], therefore AB, the common Interſection of thoſe Planes, is parallel to their Vaniſhing Lines EF and ef, and conſequently parallel to the Picture. *Q. E. D.* Fig. 12.

COR. 1.

If an Original Line AB be parallel to the Picture, it will be parallel to the Vaniſhing, Interſecting, and Directing Lines of all Original Planes which can paſs through it.

For all Planes whatſoever which paſs through AB, cut the Plane Y in that Line; conſequently the Vaniſhing, Interſecting, and Directing Lines of all ſuch Planes are parallel to AB [l].

COR. 2.

If the Image of any Line in an Original Plane be parallel to the Vaniſhing Line of that Plane, the Original Line itſelf muſt be parallel to the Picture.

For a Line in the Original Plane parallel to the Picture, being parallel to the Vaniſhing Line of that Plane [m], its Image muſt be parallel to the ſame Line [n]; therefore a Line AB, parallel to the Picture, may be found in the Original Plane, which will produce the given Image; but no two different Lines in the ſame Plane can produce the ſame Image, therefore if the given Image be the Image of a Line in the Original Plane, it muſt be the Image of AB, a Line in that Plane parallel to the Picture.

THEOR. XVI.

If two Original Planes cut each other in a Line not parallel to the Picture, their Vaniſhing, Interſecting, and Directing Lines will alſo cut each other. And the Interſection of the Vaniſhing Lines will be the Vaniſhing Point, the Interſection of the Interſecting Lines will be the Interſecting Point, and the Interſection of the

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[a] 16 El. 11.

[b] 9 El. 11.

[c] Cor. 1. Def. 10.

[d] Cor. 1. Theor. 13.

[e] Def. 19.

[f] 10 El. 11.

[g] Cor. 2. Theor. 14.

[h] Cor. Def. 11.

[i] 19 El. 11.

[k] Cor. Def. 11.

[l] 16 El. 11.

[m] Cor. 1.

[n] Theor. 2.

Directing Lines will be the Directing Point of the Common Interſection of thoſe Planes.

Fig. 13. Let EFGH and NRLM be the Picture and Directing Plane, and I the Place of the Eye, and let $ABMm$ and $LMgb$ be two Original Planes cutting each other in Mg.

Dem. Becauſe Mg is not parallel to the Picture, it muſt have a Vaniſhing, Interſecting, and Directing Point: now Mg being a Line common to both the Original Planes, its Vaniſhing Point o muſt be in both their Vaniſhing Lines[a], and therefore ef, the Vaniſhing Line of the Plane $LMgb$, muſt croſs oP, the Vaniſhing Line of the Plane $ABmM$ in the Point o, ſeeing ef and op cannot coincide, the propoſed Planes not being parallel[b]; and for the ſame reaſon, EG and gb, the Interſecting Lines, and mM and ML, the Directing Lines of the given Planes, muſt croſs in g and M, the Interſecting and Directing Points of Mg, thoſe Lines being parallel to their reſpective Vaniſhing Lines. *Q. E. D.*

COR. 1.

If two Original Planes $ABMm$ and GHLM be both perpendicular to the Picture, their Vaniſhing Lines EF and oP will make together an Angle equal to the Angle of Inclination of the Original Planes; and their Common Interſection will be in O the Center of the Picture, which will be the Common Center of thoſe Vaniſhing Lines.

For the Picture EFGH being perpendicular to both the Original Planes, the Interſections EG and GH of thoſe Planes with the Picture (which are their Interſecting Lines) determine their Angle of Inclination[c]; and if the Interſecting Lines determine that Angle, the Vaniſhing Lines EF and oP muſt do ſo too[d]. And the Common Interſection MG of the Original Planes being perpendicular to the Picture[e], its Vaniſhing Point is in O the Center of the Picture[f], which is alſo the Common Center of the Vaniſhing Lines of the propoſed Planes[g].

COR. 2.

If two Original Planes $ABMm$ and GHLM be perpendicular to each other as well as to the Picture, their Vaniſhing Lines will be perpendicular, and the Vaniſhing Line of either Plane will be the Vertical Line of the other.

The Vaniſhing Lines EF and oP will be perpendicular, becauſe they make together the ſame Angle as the propoſed Planes do[h], and the Vertical Line of every Plane paſſing through the Center of the Picture perpendicular to the Vaniſhing Line of that Plane[i], the two Vaniſhing Lines EF and oP which paſs through O, muſt be each the Vertical Line of the other.

COR. 3.

If two Original Planes $ABMm$ and $gbLM$ be perpendicular to each other, and only the Plane $ABMm$ be perpendicular to the Picture, their Vaniſhing Lines oP and ef will ſtill be perpendicular; and the Vaniſhing Line oP of the Plane $ABMm$, which is perpendicular to the Picture, will be the Vertical Line of the Plane $gbLM$; but the Vaniſhing Line ef of this laſt Plane will not be the Vertical Line of the firſt, but only parallel to it.

Becauſe the Plane $ABMm$ is perpendicular to the Picture, all Lines perpendicular to that Plane are Parallel to EF its Vertical Line, and alſo to the Picture[k]; and becauſe the Planes $gbML$ and $ABMm$ are perpendicular, a Line CD may be drawn in the Plane $gbML$ perpendicular to the Plane $ABMm$[l], and conſequently parallel to EF and to the Picture; and CD being therefore parallel to the Vaniſhing Lines of all Planes which paſs through it[m], it muſt be parallel to ef, the Vaniſhing Line of the Plane $gbML$; which Vaniſhing Line is therefore parallel to EF, the Vertical Line of the Plane $ABMm$, and conſequently perpendicular to oP, its Vaniſhing Line. Laſtly oP, which paſſes through O, and is perpendicular to ef, is the Vertical Line of the Plane $gbLM$; but in regard this Plane is not perpendicular to the Picture, its Vaniſhing Line ef cannot paſs through O, and conſequently it cannot be the Vertical Line of the Plane $ABMm$, but only parallel to it.

THEOR. XVII.

If an Original Plane, being produced, paſs through the Eye, its Vaniſhing and Interſecting Lines will coincide, and its Directing Line will be the ſame with the Parallel of the Eye.

Fig. 10. *Dem.* If the Original Plane $ABab$ paſs through the Eye, and cut the Directing Plane in a b, no other Plane parallel to the Original Plane, can paſs through a b; ſo that

[a] Theor. 10.

[b] Cor. 1. Theor. 13.

[c] Def. 19.

[d] 10 El. 11.

[e] 19 El. 11.

[f] Cor. 2. Theor. 5.

[g] Cor. 1. Theor. 9.

[h] Cor. 1.

[i] Cor. 2. Def. 15.

[k] Cor. 1. Theor. 9. and 6 El. 11.

[l] 38 El. 11.

[m] Cor. 1. Theor. 15.

the Original Plane and its Vaniſhing Plane coincide, conſequently ab is both the Directing Line, and the Parallel of the Eye of the Original Plane, and *ab*, the Interſection of the Picture with the Original Plane, is both its Interſecting and Vaniſhing Line. *Q. E. D.*

C O R. 1.

The Indefinite Image of an Original Plane which paſſes through the Eye, is only a ſtraight Line, in which Line the Images of all poſſible Points, Lines, and Figures in the Original Plane muſt lie; and therefore the Original Plane in this Caſe has no Depth[a].

For a Line drawn from the Eye to any Point in the Plane AB*ab*, muſt cut the Picture ſomewhere in the Line *ab*, in which Line therefore the Image of that Point muſt be found.

[a] Def. 26.

C O R. 2.

All Original Lines in the Plane AB*ab* have the ſame Line ab for their Director.

For no Line in the Original Plane can cut the Directing Plane but only in the Line ab.

THEOR. XVIII.

Any Point in the Picture, may be the Image of any Point in an Original Line paſſing through the Eye and the given Point in the Picture; or it may be taken as the Vaniſhing Point of that Line, or as the Image of its Interſection with all Planes or Lines whatſoever which it cuts.

This evidently follows from *Cor. Theor.* VIII.

C O R.

If a Point be given in the Picture, its Original cannot be determined, unleſs the Situation of the Original Point, with reſpect to ſome other known Point, Line, or Plane, be given.

THEOR. XIX.

Any Line drawn in the Picture, may be the Indefinite Image of any Original Line in a Plane paſſing through the Eye and the given Line in the Picture; or it may be taken as the Vaniſhing Line of that Plane, or as the Image of its Interſection with all other Planes whatſoever which it cuts.

This evidently follows from *Cor.* 1. *Theor.* XVII.

C O R.

If a Line in the Picture be given, its Original cannot be determined, unleſs ſome two Points in that Line be known, one of which at leaſt muſt be an Original Point, the other may be the Vaniſhing Point of the Original Line.

For if the Vaniſhing Point alone be given, the Direction of the Original Line is determined, ſeeing it muſt be parallel to its Radial[b]; but the Original Line itſelf may be any Line parallel to that Radial[c], and lying in a Plane paſſing through the Eye and the given Line in the Picture: but when the Direction of the Original Line, and an Original Point in that Line are known, the Original Line itſelf is then determined, ſeeing there cannot two different Lines be drawn through a given Point, parallel to the ſame Radial.

[b] Cor. 1. Def. 18.

[c] Theor. 5.

THEOR. XX.

The Original of any Figure in the Picture, may be any Object which is bounded by the ſame Pyramid of Rays, indefinitely produced.

Dem. If EFGH be the Picture, and I the Place of the Eye in the Directing Plane, the Image of the Original Figure ABCD is *abcd*[d]. Now *ab* may be the Image of any Original Line in the Plane IAB, bounded by the Lines IA and IB, indefinitely produced; and *ad* may be the Image of any Original Line in the Plane IAD, bounded by IA and ID[e], and ſo of the other Sides of that Figure. Wherefore *abcd* may be the Image of any Figure whatſoever, bounded by the Pyramid of Rays IABCD, indefinitely produced. *Q. E. D.*

Fig. 14.

[d] Def. 5.

[e] Theor. 19.

COR.

If the Image of any Plane Figure be given in the Picture, its Original cannot be determined, unleſs ſome three Points in that Figure be known, one of which at leaſt muſt be an Original Point, the other two may be Vaniſhing Points.

For the two given Vaniſhing Points will determine the Vaniſhing Line of the Plane, in which the Original Figure lies[a], and conſequently the Direction of that Plane; ſeeing it muſt be parallel to the Vaniſhing Plane which produces that Vaniſhing Line[b]: but ſtill the Original Plane may be any Plane parallel to that Vaniſhing Plane[c]. Wherefore ſome one Original Point in the Original Plane is neceſſary to be known, by which that Plane may be aſcertained; which Point being given, the Original Plane is thereby determined, ſeeing there cannot two different Planes paſs through the ſame Point parallel to the ſame Vaniſhing Plane; and the Original Plane, in which the Original Figure lies, being thus found, the Original of the given Image in the Picture is aſcertained.

THEOR. XXI.

Any Line in the Picture parallel to the Vaniſhing Line of an Original Plane, if it be the Image of an Original Line, muſt be either the Image of a Line parallel to the Picture, or of one whoſe Directing Point is ſomewhere in the Eye's Parallel of that Plane.

Dem. If the Original Line be parallel to its Image, it muſt alſo be parallel to the Picture; but if the Original Line be not parallel to its Image, that Image muſt then be parallel to the Director of the Original Line[d], which Director being therefore parallel to the propoſed Vaniſhing Line[e], muſt be the Parallel of the Eye relating to that Vaniſhing Line[f]; ſeeing there cannot be drawn two different Lines through the Eye, parallel to the ſame Vaniſhing Line. *Q. E. D.*

COR. 1.

Any Line in the Picture parallel to the Vertical Line of an Original Plane, if it be the Image of an Original Line, muſt be either the Image of a Line parallel to the Picture, or of one, whoſe Directing Point is ſomewhere in the Eye's Director of that Plane.

For if the Original Line be not parallel to the Picture, its Image muſt be parallel to its Director; which Director being therefore parallel to the propoſed Vertical Line, muſt be the Director of the Eye relating to the Original Plane[g].

COR. 2.

Any two parallel Lines in the Picture, if they be the Images of any Original Lines, muſt be either the Images of Lines parallel to each other and to the Picture, or of ſuch Original Lines as have the ſame Director.

If the Original Lines be parallel to their Images, they muſt alſo be parallel to the Picture; but if the Originals be not parallel to their Images, they muſt have the ſame Director, ſeeing there can be but one Director drawn parallel to the given Images.

THEOR. XXII.

If an Original Line, produced indefinitely on both Sides of its Directing Point, be divided by any Number of Points, and through each of thoſe Points Lines be drawn parallel to the Indefinite Image of the Original Line; and if other Lines be drawn through the ſeveral Images of thoſe Points parallel to the Original Line, until they meet reſpectively with the Lines drawn through their reſpective Originals; Then a Curve Line paſſing through the Interſections of the Parallels drawn through the Perſpectives and Projections, with the Parallels drawn through their reſpective Originals, will be a Portion of an *Hyperbola*; and another Curve Line paſſing through the Interſections of the Parallels which proceed from the Tranſprojections, with the Parallels drawn through their reſpective Originals, will be a Portion of the oppoſite *Hyperbola*.

[a] Cor. 2. Theor. 10.

[b] Def. 9.

[c] Theor. 13.

[d] Cor. 1. Def. 18.

[e] 9 El. 11.

[f] Cor. 1. Def. 10.

[g] Cor. 3. Def. 15.

Let BA be the Original Line, and ba its Indefinite Image, I the Place of the Eye, and IK and Ix the Director and Radial of the Original Line. Fig. 15.

Through any Points A, A, in the Projective and Perſpective Parts, and B, B, in the Tranſprojective part of the Original Line, draw AC, AC, BE, BE, parallel to xP; and from the correſponding Images a, a, b, b, *&c.* draw other Lines aD, aD, bF, bF, *&c:* parallel to the Original Line BA, cutting the Parallels firſt drawn in the reſpective Points p, p, π, π. It muſt be ſhewn, that a Curve Line $ppPpp$ drawn through the reſpective Interſections of AC, AC, *&c.* with aD, aD, *&c.* is a Portion of an *Hyperbola*; and that the Curve $\pi\pi\Pi\pi\pi$, drawn through the reſpective Interſections of BE, BE with bF, bF, is a Portion of the oppoſite *Hyperbola*.

Dem. In the Similar Triangles ACI and axI, taking each A, and its reſpective Image a,

$$AC : ax :: IC : Ix.$$

But by the Conſtruction, AC is equal to IK, ax to pC, IC to Dp, and Ix to PK; therefore

$$IK : pC :: Dp : PK.$$

And conſequently the Parallelogram IKPx is equal to the Parallelogram DpCI[a].

Thus every one of the Parallelograms DpCI being equal to the ſame Parallelogram IKPx, the Points p, p, *&c.* are in an *Hyperbola* by the Property of that Curve; and therefore the Curve $ppPpp$ is a Portion of an *Hyperbola*.

Again, in the Similar Triangles IEB and IxB, taking each B, and its reſpective Image b,

$$EB : xb :: IE : Ix.$$

But by Conſtruction, EB is equal to IK, xb to Eπ, IE to Fπ, and Ix to PK; therefore

$$IK : E\pi :: F\pi : PK;$$

that is, every Parallelogram FπEI is equal to the ſame Parallelogram IKPx, and conſequently $\pi\pi\Pi\pi\pi$ is a Portion of the oppoſite *Hyperbola*, according to the Properties of thoſe Curves. *Q. E. D.*

COR.

From the Nature of the *Hyperbola* it follows, that I is the Center, and Ix and IK, or EC and DF produced indefinitely, are the *Aſymptotes* of the Oppoſite *Hyperbolas* pPp and $\pi\Pi\pi$; it being impoſſible that any of the Points p or π ſhould ever fall in thoſe Lines, though they may continually approach nearer to them, unleſs the Original Point were taken at K, or the Image were ſuppoſed to be at x; the firſt of which ſuppoſes the Image to be at an Infinite Diſtance, and the other ſuppoſes the Original Point to be ſo.

SCHOL.

Having in this Section made uſe of ſome Terms of Art different from what have hitherto been commonly employed by other Writers on this Subject, it may not be improper here to give ſome reaſon for that Variation.

The general Situation that Painters give the Picture (except in Paintings on Ceilings or ſuch like) is perpendicular to the Ground or Plane of the Horizon. Hence it ariſes that the chief, and very often the only Original Plane conſidered by them, is the Ground itſelf, on which the Spectator and the Objects are ſuppoſed to ſtand, which they therefore call the *Geometrical Plane*, and ſometimes the *Ground*, *Floor*, or *Pavement of the Picture*; and the Interſecting Line of this Plane they call the *Ground Line*, as being that where, upon this Suppoſition, the Ground and the Plane of the Picture Interſect. Now this Plane being ſuppoſed perpendicular to the Picture, its Vaniſhing Line will paſs through the Center of the Picture, which will alſo be the Center of that Vaniſhing Line[b]; this Vaniſhing Line they call the *Horizontal Line*, as being that which determines the viſible Horizon, ſuppoſing the Ground to be a Plane[c]; and the natural Situation of moſt things ſtanding upon the Ground being perpendicular to it, the Front, Sides, and Walls of Buildings, *&c.* will be in Planes perpendicular to the Ground, and therefore the Vaniſhing Lines of thoſe Planes will be perpendicular to the Horizontal Line, in which Line the Center of thoſe Vaniſhing Lines will alſo fall[d]. And in regard ſuch a Situation of the Picture is generally choſen, as that the Buildings repreſented, are ſuppoſed to have one Face parallel to the Picture, the ſide Faces (if the Buildings be Rectangular, as they moſt commonly are) will be perpendicular to the Picture, and conſequently the Vaniſhing Line of thoſe ſide Faces, as well as the Horizontal Line, will paſs through the Center of the Picture, which Center will be the Common Center of both thoſe Vaniſhing Lines[e]; and the Vaniſhing Point of the Common Interſection of ſuch Planes with the Ground, as well as of all other Lines in any of thoſe Planes parallel to their Interſection with the Ground, will alſo be in the Center of the Picture[f]. From this it ariſes, that the Center of the Picture is the chief

Point

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[a] 16 El. 6.

[b] Cor. 1. Theor. 9.

[c] Def. 25.

[d] Cor. 3. Theor. 16.

[e] Cor. 2. Theor. 16.

[f] Cor. 1. Theor. 16.

Point uſed in ſuch Pieces, and is therefore called the *Principal Point*, or *Point of Sight*, which laſt Name we have given to that Point where the Spectator's Eye is ſuppoſed to be; and the Axe of the Eye being the Radial of all Planes perpendicular to the Picture[a], that Line is the Radial of all thoſe Planes which in this Situation moſt uſually occur, and therefore it is called the *Principal Ray*; and that Diſtance being ſet off on each Side of the principal Point upon the Horizontal Line, the Points where it falls, are called the *Points of Diſtance*, the uſe of which will be ſeen farther on[b]. And in this particular Situation of the Picture with regard to the Objects, the principal Lines being either perpendicular or parallel to the Ground or to the Picture, the Images of thoſe that are parallel to the Picture, will make the ſame Angle with the Ground Line, as the Original Lines themſelves do with the Ground[c]; and thoſe Lines which are parallel to the Ground, but not to the Picture, will have their Vaniſhing Points in the Horizontal Line[d]; and if theſe do not fall in the principal Point, then they call them *Accidental Points in the Horizontal Line*. Laſtly when they have occaſion to repreſent any Lines which are neither parallel nor perpendicular to the Picture, nor to the Ground, they generally find their Images by the help of the Seat of ſuch Lines on the Ground, to which they moſt commonly refer all ſuch Lines as do not come within the above Deſcription, without inquiring after their Vaniſhing Points or Radials.

Now theſe Terms, as above explained, appearing to be particularly applicable to the Situation of the Picture here ſpoken of, and to no other, they ſeem for that reaſon to be too much confined in their Senſe; for with reſpect to Practice, any other Plane may be taken as an Original Plane, as well as the Ground; and the Picture may have any other Situation with reſpect to the Ground or any other Plane, as well as that of being perpendicular to them; and it may often be neceſſary, in the ſame Picture, to conſider ſeveral different Original Planes, each of which will have its own Interſecting and Vaniſhing Lines, and the other Points and Lines thereon depending; by the Help of which ſeveral Problems in *Stereography* may be eaſily ſolved, which without them would be extreamly tedious and difficult, if not impracticable. So that although the Horizontal and Ground Lines are very proper to expreſs the Vaniſhing and Interſecting Lines of the Ground, conſidered as an Original Plane; yet it would be abſurd to call the Vaniſhing or Interſecting Lines of any other Original Plane, beſides the Ground, by thoſe Names.

SECTION V.

Of the Proportions of the Images of determinate Original Lines.

THEOR. XXIII.

A determinate Original Line in a Plane parallel to the Picture, is in the ſame Proportion to its Image, as the Diſtance of the Eye from the Original Plane, is to its Diſtance from the Picture.

Fig. 16. Let EFGH be the Picture, O its Center, I the Place of the Eye, and AB a determinate Original Line in a Plane *efgh* parallel to the Picture, and let IO be the Axe of the Eye, produced till it cut the parallel Plane in *o*. It muſt be proved, that the Original Line AB is to its Image *ab*, as I*o*, the Diſtance between the Eye and the Plane *efgh*, is to IO, the Diſtance between the Eye and the Picture.

Having drawn IB and IA, cutting the Picture in *b* and *a*, and thereby determining the image *ab*, draw B*o* and *b*O.

Dem. Becauſe AB is in a Plane parallel to the Picture, it is therefore itſelf parallel to the Picture, and to its Image *ab*[e]; wherefore the Triangles IAB and I*ab* are Similar; and for the ſame reaſon, the Triangles I*o*B and IO*b*, are alſo Similar.

Therefore in the Similar Triangles IAB, I*ab* $AB : ab :: IB : Ib.$
And in the Similar Triangles I*o*B, IO*b* $IB : Ib :: Io : IO.$
Conſequently $AB : ab :: Io : IO.$

After the ſame manner, if $\alpha\beta$ were an Original Line in a Plane VWXY lying behind the Eye parallel to the Picture; *ab* the tranſprojected Image of $\alpha\beta$, being parallel to its Original, the Triangles $I\alpha\beta$ and I*ab* will be ſimilar, as well as the Triangles $I\omega\beta$

[a] Cor. 1. Theor. 9.

[b] Prob. 5, &c. B. II.

[c] Cor. 3. Theor. 2.

[d] Cor. 3. Theor. 10.

[e] Theor. 2.

Plate 5. Book 1. Sect. 5

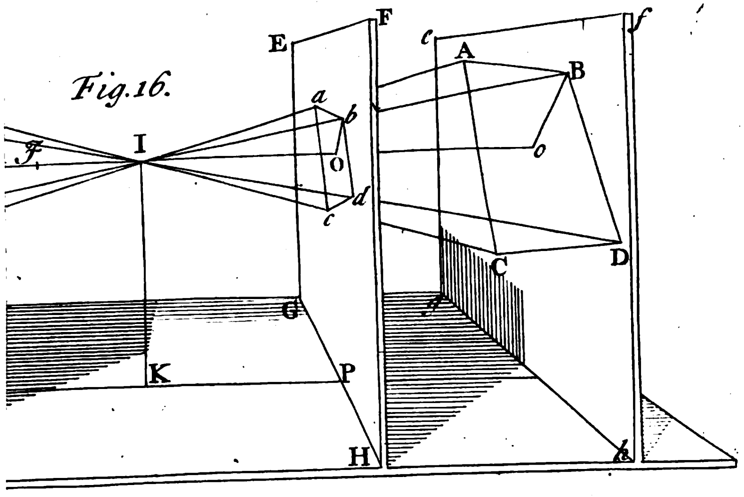
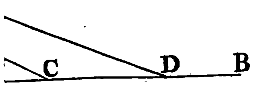


Fig. 16.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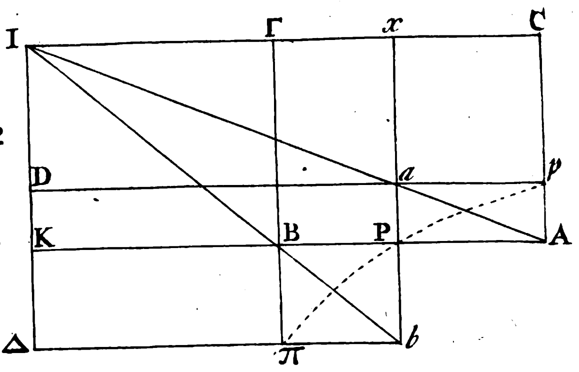
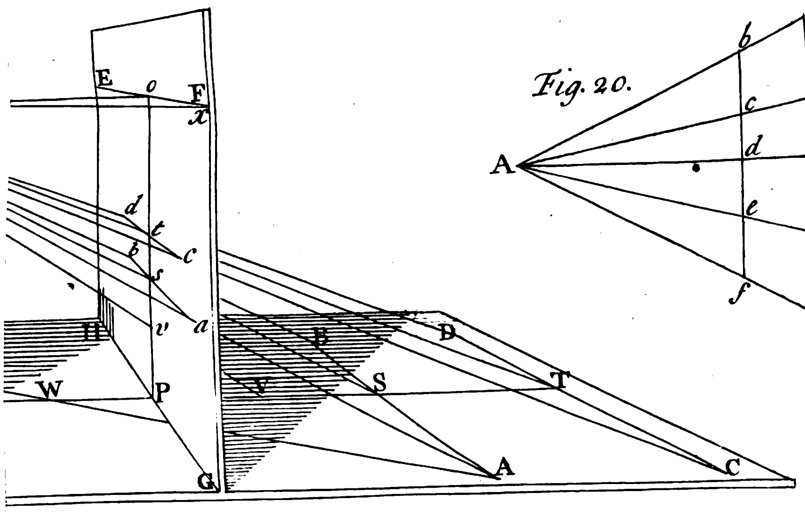


Fig. 18.



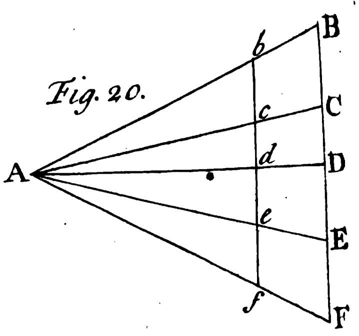


Fig. 20.

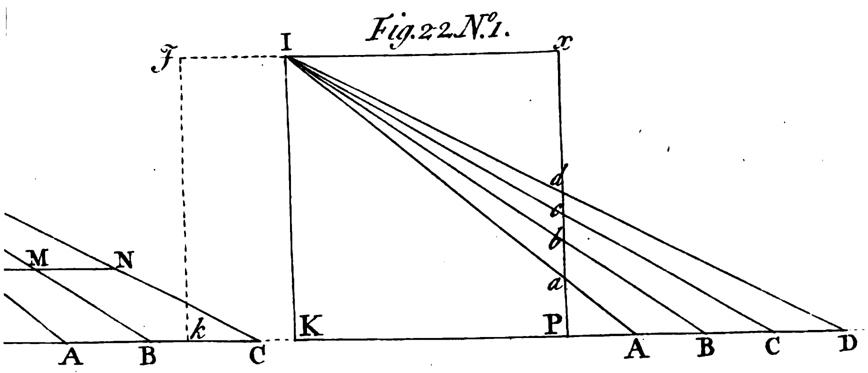


Fig. 22 N°. 1.

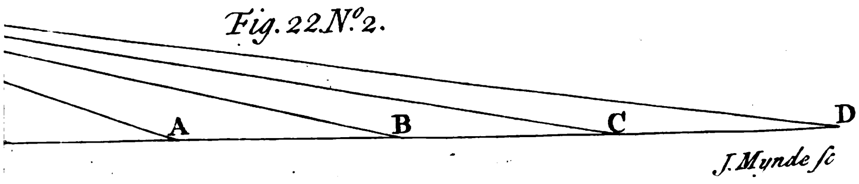


Fig. 22 N°. 2.

J. Mynde sc.

and IO*b*; whence, by the like Analogy as before, it will be found that $\alpha\beta : ab :: I\omega : IO$. *Q. E. D.*

C O R. 1.

If the Original Line AB be anywiſe divided into ſeveral Parts, the Images of thoſe Parts will have the ſame Proportion to each other as their Originals.

Becauſe the Original of every Part is to its Image, as the Diſtance of the Eye from the Original Plane, is to its Diſtance from the Picture, which two Diſtances do not vary.

C O R. 2.

The Images of the Sides of any Figure A B C D, in a Plane parallel to the Picture, have the ſame Proportion to each other, as the correſponding Sides of the Original Figure.

Becauſe every Side of the Original Figure is to its Image, as the Diſtance of the Eye from the Original Plane is to its Diſtance from the Picture.

C O R. 3.

The Image of any Figure, in a Plane parallel to the Picture, is Similar to its Original.

Becauſe the Angles of the Image are equal[a], and its Sides proportional[b] to the correſponding Angles and Sides of the Original Figure.

C O R. 4.

The Image of a determinate Line, in a Plane parallel to the Picture, will be of the ſame Length, wherever the Eye be placed in the Directing Plane.

Becauſe whatever Point in the Directing Plane be taken for the Place of the Eye, the Diſtance between the Eye, the Picture, and the Original Plane are not varied; and therefore the Proportion of the Image to its Original continues the ſame.

C O R. 5.

If the Picture and Original Plane be both on the ſame Side of the Eye, the farther the Eye is removed from the Original Plane, the Image of any determinate Line in that Plane, will become more nearly equal to its Original.

Let the Eye be removed farther from the Original Plane *efgh* to *J* in the Line I*o*, the Picture *E F G H* retaining its Situation.

Then becauſe the Original Line A B is to its Image from the Station I, as I*o* to IO[c], the ſame Line AB will be to its Image at the Station *J*, as *Jo* to *J*O; but *J*I being bigger in Proportion to IO than to I*o*[d], $JI+IO=JO$ will be bigger in Proportion to $JI+Io=Jo$, than IO to I*o*; and conſequently the Image of AB from the Station *J*, will be bigger in Proportion to AB, than from the Station I; and as in this Situation of the Object, with reſpect to the Picture and the Eye, the Image of AB is always leſs than its Original[e], the Image increaſing as the Diſtance of the Eye is increaſed, muſt therefore bring the Original Line and its Image nearer to an Equality.

If E F G H be the Original Plane, and *efgh* the Picture, the ſame thing may be proved after the like manner.

D E F. 27.

It having been ſhewn, that the Directing Point of an Original Line can have no Image[f], it follows, that any Point of an Original Line, which can be repreſented, muſt be at ſome Diſtance from the Directing Plane: the Image of which Point will bound the Indefinite Image of the Original Line at one end, as it is terminated at the other end by the Vaniſhing Point.

The Indefinite Image thus bounded, we ſhall call the *whole Image of the Original Line from the Original of the Point which bounds it*; when the Image of any determinate Part of an Original Line is mentioned, it ſhall be expreſſed to be the *Image of that Part*, and the Remainder of the whole Image ſhall be called the *Complement of that Part:* and in like manner, that Part of an Original Line which lies between its Directing Point and the neareſt Point deſcribed, ſhall be called the *Complement of the Original Line from that Point.*

Thus let I*x*K B repreſent the Radial Plane[g] of an Original Line A B.

Fig. 17. If A be the neareſt Point of A B that is deſcribed, then *ax* is the whole Image of AB, indefinitely produced from A beyond B; if P be the neareſt Point deſcribed, P*x* is the whole Image of PB, produced from P in like manner, the ſame in this Caſe

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[a] Cor. 3. Theor. 2.

[b] Cor. 2.

[c] Theor. 23.

[d] 8 El. 5.

[e] Art. 18. Sec. 3.

[f] Cor. 1. Theor. 4.

[g] Def. 18.

with the whole Perſpective; if C or D be the neareſt Point conſidered, then cx or dx is the whole Image.

Alſo if cd, Pd, or ad be the given Image of any determinate Part of an Original Line, dx is the Complement of that Image; or if ac or Pc be the given Image, cx is its Complement, as Px is the Complement of the Image aP. And in like manner, KA is the Complement of the Original Line AB, KP is the Complement of PB, and KC of CB.

The ſame is to be underſtood of the other Indefinite Part of the Original Line which lies behind the Directing Plane, which is here conſidered as a diſtinct Line from that Part of it which lies before the Directing Plane [a].

THEOR. XXIV.

The whole Image of an Original Line is to its whole Perſpective, or its Director, as the Radial is to the Complement of the Original Line.

Fig. 18. Let IxKA be the Radial Plane of an Original Line KA, bx the Indefinite Image of that Line, and a and b the Images of any two Points A and B in the given Line; it muſt be proved, that ax, the whole Image of the Original Line from the Point A, is to Px its whole Perſpective, which is equal to IK its Director, as the Radial Ix is to KA, the Complement of the Original Line; alſo that bx, the whole Image of the Original Line from the Point B, is to Px, as Ix is to the Complement KB.

Through B and A draw BΓ, AC parallel to IK, and through b and a draw Δb, Da parallel to Ix, cutting BΓ and AC in π and p.

Dem. Becauſe π, P, p are Points in an *Hyperbola*, the Parallelograms $IDpC$, $I\Delta\pi\Gamma$, and $IKPx$ are equal [b].

Therefore $pC = ax : Px = IK :: Ix : Dp = KA$.

And $\pi\Gamma = bx : Px = IK :: Ix : \Delta\pi = KB$. *Q. E. D.*

THEOR. XXV.

The Diſtance of the Image of any Point in an Original Plane from the Vaniſhing Line, is to the Vertical Line, or Eye's Director of that Plane, as the Radial of the Original Plane, is to the Diſtance between the Original Point and the Directing Line.

Fig. 19. Let A be the given Point in the Original Plane LMGH. Through A draw AB parallel to the Interſecting Line GH, cutting the Line of Station KP in S, and draw IS, cutting the Vertical Line oP in s.

Dem. The Point s being the Image of S, it follows, that so, the Diſtance of s from the Vaniſhing Line EF, is to Po the Vertical Line, or IK the Eye's Director, as Io the Radial of the Original Plane, is to KS, the Diſtance between the Original Point S and the Directing Line LM [c]. Now becauſe AB is parallel to GH, its Image will alſo be parallel to GH; wherefore ab drawn through s parallel to GH, is the Image of AB [d], in which a, the Image of A, muſt lie. But the Diſtance of a from EF being equal to so, and the Diſtance of A from LM being equal to KS, and Io and Po being conſtantly the ſame; it follows, that in whatever Point of AB the Point A be taken, the Diſtance of the Image of that Point from the Vaniſhing Line will be to Po, as the Radial Io is to the Diſtance between the given Point and the Directing Line. *Q. E. D.*

COR. 1.

The Diſtance of the Image of any Point in an Original Plane from the Vaniſhing Line, continues the ſame, in whatever Point of the Eye's Parallel the Eye be placed.

For AB, ab, and NR being all in the ſame Plane, a Line drawn from any Point in NR to any Point in AB, muſt cut the Picture ſomewhere in the Line ab, every Point of which Line is equally diſtant from EF.

COR. 2.

If the Height of the Eye be increaſed or diminiſhed, the Eye continuing in the ſame Directing Plane, the Diſtance between the Image of the Original Point and the Vaniſhing Line will be increaſed or diminiſhed in the ſame Proportion.

For by the Theorem, the Eye's Director is to the Diſtance of the Image of the Original Point from the Vaniſhing Line, as the Diſtance between the Original Point and the

Directing

[a] Schol. Theor. 4. and Def. 24.

[b] Theor. 22.

[c] Theor. 24.

[d] Cor. 2. Theor. 15.

Directing Line, is to the Radial of the Original Plane; and theſe two laſt Terms continuing the ſame, in whatever Point of the Directing Plane the Eye be placed, the Proportion between the two firſt Terms muſt alſo continue the ſame; and conſequently if either of them be increaſed or diminiſhed, the other muſt alſo be increaſed or diminiſhed in the ſame Proportion.

COR. 3.

If the Diſtance of the Eye be increaſed or diminiſhed, its Height remaining the ſame; the Diſtance of the Image of the Original Point from the Vaniſhing Line will alſo be increaſed or diminiſhed.

For by the Theorem, $so : Po :: Io = KP : KS$. If then Io or KP be increaſed by any quantity x, $KP + x$ will be bigger in Proportion to $KS + x$ than KP to KS[a]; and conſequently so will alſo be bigger in Proportion to Po than it was before, and Po continuing the ſame, so is therefore enlarged. And for the ſame reaſon, if Io be decreaſed, so will be decreaſed alſo.

THEOR. XXVI.

The Image ab of a determinate Part AB of an Original Line, is to its Complement ax, as the Original Part AB is to its Complement KB. Fig. 18.

Dem. For the Parallelograms $IDpC$ and $I\Delta\pi\Gamma$ being equal[b];
It follows, that $\pi\Gamma = bx : pC = ax :: KA : \Delta\pi = KB$.
Therefore by Diviſion $bx - ax = ba : ax :: KA - KB = BA : KB$. *Q. E. D.*

COR. 1.

The Image ba is to the whole Image bx, as the Original Part AB is to the whole Line AK, continued to its Directing Point K.
For by the Theorem $ba : ax :: BA : KB$.
Therefore by Compoſition $ba : ba + ax = bx :: BA : BA + KB = AK$.

COR. 2,

The whole Image bx is to its Complement ax, as the whole Line KA is to its Complement KB.

THEOR. XXVII.

The Image of a determinate Part of an Original Line, from any one Station of the Eye in the Directing Plane, is to the Image of the ſame part, at any other Station of the Eye in the ſame Directing Plane, as the Director of the Original Line at the firſt Station, is to the Director of that Line at the other Station.

Dem. In the firſt place, the Parts BA, KB, KP, and KA of the Original Line, continue the ſame, whatever the Eye be placed in the Directing Plane, and the Radial Ix of the Original Line being always equal to KP[c], the Proportions of theſe Lines to each other continue the ſame, in whatever Point of the Directing Plane the Eye be placed. Now becauſe $ba : bx :: BA : KA$[d], which laſt Proportion doth not vary, the Proportion of ba to bx is conſtantly the ſame; and becauſe $bx : IK :: Ix : KB$[e], which laſt Proportion doth not vary, the Proportion of bx to IK is conſtantly the ſame; wherefore ſince the Proportion of ba to bx, and of bx to IK is conſtant, the Proportion of ba to IK is conſtant; conſequently, if IK the Director of the Original Line, be increaſed or diminiſhed in any Proportion, the Image ba, of the determinate Part BA of the Original Line, will alſo be increaſed or diminiſhed in the ſame Proportion. *Q. E. D.* Fig. 18.

COR.

The Images bP and Pa, of any two Parts BP and PA of the ſame Original Line, have the ſame Proportion to each other, in whatever Point of the Directing Plane the Eye be placed.

For by the Theorem, the Proportion of bP to IK is conſtant, and for the ſame reaſon, the Proportion of Pa to IK is conſtant; conſequently the Proportion of bP to Pa is conſtant, in whatever Point of the Directing Plane the Eye be placed.

[a] 8 El. 5.

[b] Theor. 24.

[c] Cor. 1. Def. 18.

[d] Cor. 1. Theor. 26.

[e] Theor. 24.

LEM. 2.

Fig. 20. If any Number of Lines AB, AC, AD, *&c.* proceeding from the ſame Point A, cut any two parallel Lines BF and bf, they will cut them proportionally; that is, the Parts BC, CD, DE, *&c.* of the Line BF, will be proportional to the correſponding Parts bc, cd, de, *&c.* of the Line bf.

In the Similar Triangles ABC, Abc $\quad BC : bc :: AC : Ac$
And in the Similar Triangles ACD, Acd $\quad CD : cd :: AC : Ac$
Wherefore $\quad BC : bc :: CD : cd$
Or $\quad BC : CD :: bc : cd.$
The ſame may be ſhewn of any other correſponding Diviſions of BF and bf. *Q. E. D.*

LEM. 3.

If any Geometrically proportional Quantities, be ſeverally multiplied by the like Number of other Quantities in Geometrical Proportion, the Products will alſo be Geometrically proportional.

If $\quad a : b :: c : d$
And $\quad e : f :: g : b$
Multiply the Antecedents of the firſt Proportionals by e, and the Conſequents by f, and that will produce $\quad ae : bf :: ce : df$[a].
Then multiply the Antecedents in the ſecond Proportionals by c, and the Conſequents by d, which will produce $\quad ce : df :: cg : db$
And therefore $\quad ae : bf :: cg : db.$
Q. E. D.

THEOR. XXVIII.

Fig. 21. If an Original Line AC be divided at pleaſure into two Parts by the Point B, whereby its whole Image ax will be divided into three Parts ab, bc, cx; then the Rectangle between ab and cx, the Extremes of the whole Image, will be to the Rectangle between the middle part bc, and the whole Image ax, as AB the nearer part, is to BC the farther part of the Original Line.

Through a the Image of A in the Radial Plane IxKC, draw LN parallel to the Original Line AC, cutting IB and IC in M and N.

Dem. In the Similar Triangles Ixb, baM $\quad aM : Ix :: ab : bc+cx.$
And in the Similar Triangles Ixc, caN $\quad Ix : aM+MN :: cx : ab+bc.$
Therefore multiplying theſe two Series by each other[b], the Product will be
$Ix \times aM : Ix \times aM + Ix \times MN :: ab \times cx : ab \times cx + ab \times bc + bc \times cx + bc^2.$
But $ab \times bc + bc \times cx + bc^2 = bc \times ax$[c].
Therefore $Ix \times aM : Ix \times aM + Ix \times MN :: ab \times cx : ab \times cx + bc \times ax.$
And ſubſtracting the Antecedents from the Conſequents
$Ix \times aM : Ix \times MN :: ab \times cx : bc \times ax$, that is, $aM : MN :: ab \times cx : bc \times ax.$
But becauſe LN and AC are parallel[d], $aM : MN :: AB : BC.$
Therefore $\quad ab \times cx : bc \times ax :: AB : BC.$ *Q. E. D.*

SCHOL.

Although the Figure here referred to, ſuppoſes the Line AC to lie wholly beyond the Picture, yet this Propoſition is univerſally true in whatever Point of the Line AC the Point P be taken, ſo long as Px is made parallel to IK. For whatever Line parallel to Px or IK ſhall cut Ix, IA, IB, and IC, produced if neceſſary, the Parts of that Line intercepted by them, will have the ſame Proportion to each other, as the correſponding Parts ab, bc, and cx of the Line Px[e].

COR. 1.

The Image ab of the nearer part of the Original Line, is to bc the Image of the farther part, as the Rectangle between AB the nearer part, and the whole Line KC produced to its Directing Point K, is to the Rectangle between BC and KA, the Extremes of the whole Line.

Through c draw Qc parallel to AC, cutting IA and IB in R and S.

Then if ac be conſidered as an Original Line, x as its Directing Point, Ix as its Director, and Qc as its whole Image;

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[a] 4 El. 5.

[b] Lem. 3.

[c] 1 El. 2.

[d] Lem. 2.

[e] Lem. 2.

It follows from the Theorem that $RS \times Qc : Sc \times QR :: ab : bc.$

But becauſe Qc and AC are parallel $Qc : QR :: KC : KA$[a]. [a] Lem. 2.

And likewiſe $RS : Sc :: AB : BC$

Therefore[b] $RS \times Qc : Sc \times QR :: AB \times KC : BC \times KA.$ [b] Lem. 3.

Conſequently $ab : bc :: AB \times KC : BC \times KA.$

COR. 2.

If the Parts AB and BC of the Original Line be equal, then *ab* the Image of the nearer part, will be to *bc* the Image of the farther Part, as the whole Image *ax*, is to its Complement *cx*.

Becauſe by the Theorem, $ab \times cx : bc \times ax :: AB : BC.$

If AB and BC be equal, then $ab \times cx = bc \times ax.$

Which gives this Analogy $ab : bc :: ax : cx.$

In this Caſe, the Line ax is Harmonically divided by the Points b and c; the Nature and Properties of which kind of Proportion will be farther conſidered in another place[c]. [c] B. III.

COR. 3.

The ſame thing being ſuppoſed as in the laſt Corollary, *ab* will be to *bc*, as KC to KA.

In the Similar Triangles ILN, I*xc*, $IL = ax : cx :: LN : Ix = La.$

But by the laſt Corollary, $ab : bc :: ax : cx.$

And becauſe LN and KC are parallel[d] $LN : La :: KC : KA.$ [d] Lem. 2.

Therefore $ab : bc :: KC : KA.$

COR. 4.

If the Images *ab* and *bc* be equal, then AB the nearer part of the Original Line, will be to BC the farther part, as KA the Complement of the Original Line, is to KC the whole Line produced to its Directing Point K.

Becauſe by Cor. 1. $ab : bc :: AB \times KC : BC \times KA.$

If *ab* and *bc* be equal, then $AB \times KC = BC \times KA.$

Which gives this Analogy $AB : BC :: KA : KC.$

And here, the Line KC *is Harmonically divided by the Points* B *and* C.

COR. 5.

The ſame thing being ſuppoſed as in the preceeding Corollary; AB will be to BC as *cx* to *ax*.

In the Similar Triangles ILN, I*xc*, $Ix = La : LN :: cx : ax.$

But[e] $La : LN :: KA : KC.$ [e] Lem. 2.

And by the laſt Corollary $KA : KC :: AB : BC.$

Conſequently $AB : BC :: cx : ax.$

DEF. 28.

Harmonical Proportion continual is, when in a Series of Quantities, any three adjoining Terms being taken, the Difference between the firſt and ſecond, is to the Difference between the ſecond and third, as the firſt is to the third: Or, when a Series of Quantities is ſo conſtituted, as to be reciprocally Proportional to a Series of Numbers in Arithmetical Progreſſion: Both which Properties equally belong to all Quantities which are Harmonically Proportional.

Thus, in the following Series of Harmonical Proportionals, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5},$ *&c.*

And in this Series in Arithmetical Progreſſion $1, 2, 3, 4, 5,$ *&c.*

If the three firſt Harmonical Terms be taken, *viz.* $1, \frac{1}{2}, \frac{1}{3}.$

Then $1 - \frac{1}{2} = \frac{1}{2} : \frac{1}{2} - \frac{1}{3} = \frac{1}{6} :: 1 : \frac{1}{3}$; and ſo of any other three adjoining Terms of that Series.

It is evident alſo that the upper Series is reciprocally Proportional to the lower, for $1 : \frac{1}{2} :: 2 : 1$, or $\frac{1}{2} : \frac{1}{3} :: 3 : 2$, &c.

THEOR. XXIX.

If an Original Line AD be divided into any Number of equal Parts AB, BC, CD, *&c.* the whole Images of thoſe Parts, and alſo their Complements, will be in a continual Harmonical Proportion. Fig. 22. N°. 1.

Let *ab*, *bc*, *cd*, be the Images of AB, BC, and CD; the whole Images are *ax*, *bx*, and *cx*, and the Complements of thoſe Images are *bx*, *cx*, and *dx*; it muſt be ſhewn, that *ax*, *bx*, *cx*, and *dx*, are in continual Harmonical Proportion.

[a] Cor. 2. Theor. 28.

Dem. Becauſe AB and BC are equal $ab : bc :: ax : cx$[a].
And becauſe BC and CD are equal $bc : cd :: bx : dx$.
But $ab = ax - bx$, $bc = bx - cx$, and $cd = cx - dx$.
Therefore $ax - bx : bx - cx :: ax : cx$.
And $bx - cx : cx - dx :: bx : dx$.
Conſequently ax, bx, cx, and dx, are all in a continual Harmonical Proportion.
Again, becauſe of the Similar Triangles IKA, axI, $IK : KA :: ax : Ix$.
And for the like Reaſon $IK : KB :: bx : Ix$.
And $IK : KC :: cx : Ix$.
Wherefore $ax : bx :: KB : KA$.
And $bx : cx :: KC : KB$.
And conſequently ax, bx, cx, &c. are reciprocally Proportional to KA, KB, KC, *&c.* which laſt Quantities are in an Arithmetical Progreſſion, their Differences AB, BC, *&c.* being equal. *Q. E. D.*

COR. 1.

The Difference between the Image of a nearer part of the Original Line, and the Image of the part next beyond it, is greater than the Difference between this laſt Image, and that of the next ſucceeding part, and ſo on.

[b] Cor. 3. Theor. 28.

For the Parts AB, BC, CD, being equal $ab : bc :: KC : KA$[b].
And for the ſame Reaſon $bc : cd :: KD : KB$.
Therefore $ab - bc : bc :: KC - KA = AC : KA$.
And alſo $bc - cd : cd :: KD - KB = AC : KB$.
But AC is bigger in Proportion to KA than to KB, therefore $ab - bc$ is alſo bigger in Proportion to bc, than $bc - cd$ is to cd; and bc being greater than cd, $ab - bc$ is therefore ſo much the more greater than $bc - cd$.

COR. 2.

The farther the equal Parts of the Original Line lye from its Directing Point, the Images of thoſe Parts will approach nearer to an Equality.
Becauſe their Differences continually leſſen.

COR. 3.

The farther the Eye is removed from the Picture, the Images of any two adjoining Parts of the Original Line will become more nearly equal.
Let the Eye be removed to *J* in the Radial Ix, and draw *J*k parallel to Px, repreſenting the Director of the Original Line AD at the Station *J*.
Then becauſe AB and BC are equal, the Image of AB is to that of BC, from the
Station I, as KC to KA[c], and from the Station *J*, they are as kC to kA; but kC is
leſs in proportion to kA than KC to KA[d], therefore the Image of AB is leſs in pro-
portion to that of BC, from the Station *J*, than from I. But the Images of AB and
BC are in the ſame proportion to each other, where-ever the Eye be placed in the ſame
Directing Plane[e], therefore whether the Eye be placed at *J*, or any where elſe in a
Plane paſſing through *J* parallel to the Picture, the Images of AB and BC from thence,
will be nearer equal than from I.

[c] Cor. 3. Theor. 28.
[d] Cor. 5. Theor. 23.
[e] Cor. Theor. 27.

COR. 4.

Fig. 22. N°. 2.

If from the Interſecting Point P of any Line PD, ſeveral Diſtances PA, AB, BC,
CD, *&c.* be taken, each equal to KP; the Complements ax, bx, cx, and dx of the
Images of thoſe Parts, will be in proportion to IK or Px, as the following Series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,
$\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, *&c.* and the Images of the Parts themſelves will be as $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{42}$, *&c.*
of Px, the Differences of the Denominators of each of theſe laſt Fractions, increaſing
from 0 in this Series, 2, 4, 6, 8, 10, *&c.*

In the Similar Triangles IKA, aPA $KA : PA :: IK = Px : Pa$.
But by Suppoſition $KA = 2PA$, therefore $Px = 2Pa$, conſequently $Pa = ax = \frac{1}{2}Px$.
Again in the Similar Triangles IKB, bPB, $KB : PB :: IK = Px : Pb$.
But $PB = \frac{2}{3}KB$, therefore $Pb = \frac{2}{3}Px$, conſequently $bx = \frac{1}{3}Px$.
And in the ſame manner it may be proved, that cx is $\frac{1}{4}$, and dx $\frac{1}{5}$ of Px.
Now if we call Px, 1, ax, $\frac{1}{2}$, bx, $\frac{1}{3}$, cx, $\frac{1}{4}$, and dx, $\frac{1}{5}$; the Image Pa, which is the
Difference between Px and ax, will be $\frac{1}{2}$, ab the Difference between ax and bx will
be $\frac{1}{6}$, bc, $\frac{1}{12}$, and cd, $\frac{1}{20}$, *&c.* the Difference of the Denominators of which Fractions
increaſe from 0 in the Series 2, 4, 6, 8, *&c.*

THEOR.

THEOR. XXX.

If an Original Line KC, produced to its Directing Point K, be ſo divided in A and B, as that KA, KB, and KC may be in contitinual Geometrical Proportion, the whole Image ax of that Line will be divided in the ſame Proportion. Fig. 21.

Dem. Firſt $bx : ax :: KA : KB$[a]
And for the ſame reaſon $cx : bx :: KB : KC$
But by the Suppoſition $KA : KB :: KB : KC$
Therefore $bx : ax :: cx : bx.$
And inverting the Terms $ax : bx :: bx : cx.$ *Q. E. D.*

COR.

If KC be ſo divided by A, B, C, and D, that KA may be to KB as KC to KD, then the whole Image ax will be to bx as cx to $dx.$ Fig. 22. N°. 1.

For as before $bx : ax :: KA : KB$
And in like manner $dx : cx :: KC : KD$
But by the Suppoſition $KA : KB :: KC : KD$
Therefore $bx : ax :: dx : cx$
Or inverting the Terms $ax : bx :: cx : dx.$

THEOR. XXXI.

If in an Original Plane LMGH, any determinate Line AB be drawn parallel to the Picture, the Original Line AB will be to its Image ab, as IK or Po, the Eye's Director, or Vertical Line of the Original Plane, is to so, the Diſtance between the Image ab and the Vaniſhing Line EF. Fig. 19.

Dem. Becauſe $AB : ab :: KS : KP$[b]
And $Po : so :: KS : KP$[c]
Therefore $AB : ab :: Po : so.$ *Q. E. D.*

COR. 1.

If through the Line AB, a Plane were imagined to paſs parallel to the Picture, any determinate Line in that Plane, will be to its Image, as the Vertical Line Po, is to so, the Diſtance between ab and EF.

Becauſe any determinate Line, in this ſuppoſed Plane, will be to its Image, as AB is to ab[d].

COR. 2.

If the Diſtance PS were biſected in V, then AB will be to ab, as Pv the Image of PV, is to vs the Image of VS.

Becauſe $Pv : vs :: Po : so$[e].

COR. 3.

If in the Plane LMGH, any Number of determinate Lines AB, CD, *&c.* were drawn parallel to the Picture, and equal between themſelves, the Images ab, cd, *&c.* of theſe Parallels, will be in proportion to each other, as so, to, *&c.* their ſeveral Diſtances from the Vaniſhing Line EF.

For by the Theorem $AB : ab :: Po : so$
Therefore by Permutation $AB : Po :: ab : so$
And for the ſame reaſon $CD : Po :: cd : to$
If then AB and CD be equal $ab : so :: cd : to$
Or $ab : cd :: so : to$

THEOR. XXXII.

If in an Original Plane LMGH, any two Lines AB and KT be drawn, the one parallel, and the other anywiſe inclining to the Picture, and cutting each other in any Point S; and if any Point V or T be taken in the Inclining Line, either nearer to or farther from Fig. 19.

[a] Cor. 2. Theor. 26.

[b] Theor. 23.

[c] Theor. 25.

[d] Cor. 2. Theor. 23.

[e] Cor. 2. Theor. 28.

from its Directing Point K than the Point S; then if a part SB be taken in the parallel Line AB, in the ſame Proportion to IK, the Director of the Inclining Line, as the part SV or ST of this Line, is to VK or TK the Diſtance between the aſſumed Point V or T and the Directing Point K, the Image of SV or ST will be equal to the Image of SB.

Dem. Becauſe VS is a determinate Line inclining to the Picture, and vs is its Image, therefore $sv : so :: SV : VK$[a]
And becauſe SB is a determinate Line parallel to the Picture, and sb is its Image, therefore $sb : so :: SB : Po = IK$[b]
But by the Suppoſition $SB : IK :: SV : VK$
Therefore by parity of reaſon $sb : so :: sv : so$
Conſequently $sb = sv.$
Again, becauſe st is the Image of ST, $st : so :: ST : TK$[c]
And as before $sb : so :: SB : IK$
And alſo by the Suppoſition $SB : IK :: ST : TK$
Therefore by parity of reaſon $sb : so :: st : so$
And conſequently $sb = st.$ *Q. E. D.*

COR. 1.

If the Point V biſect KS, and the Line SB be taken equal to the Director IK, then the Image of SB will not only be equal to the Image of SV, but alſo to the Complement of that Image.

For if $SV = VK$, and $SB = IK$, then $SB : IK :: SV : VK$
And conſequently $sb = sv$[d]
And ſeeing $SV : VK :: sv : so$[e], if $SV = VK$, then $sv = so$
Whence alſo $sb = so.$

COR. 2.

If through any two Points S and A of a Line AB parallel to the Picture, there be drawn any two Lines SK, AQ, interſecting any where in W, and the Radials I*o* and I*x* of theſe two Lines be drawn; then the Radial I*o*, of either of theſe Lines SK, will be to *ox*, the Diſtance between their Vaniſhing Points, as WK the Complement of KS from the Point of Interſection W, is to KQ, the Diſtance between their Directing Points; or as WS to SA.

For the Triangle I*ox* having its Sides parallel reſpectively to the Sides of the Triangles WKQ and WSA, thoſe three Triangles are Similar,
And therefore $Io : ox :: WK : KQ.$
And alſo $Io : ox :: WS : SA.$

THEOR. XXXIII.

Fig. 23. If a determinate Original Line PB, adjoining to the Picture at P, be anywiſe divided into two Parts in the Point A, and there be taken any two Diſtances of the Eye, as I and *J* in the Radial I*x* of the Original Line; Then ab, the Image of the farther part AB of the Original Line at the Station I, will be to *ab*, its Image at the Station *J*, as the Rectangle between Pa and b*x*, the Extremes of the whole Perſpective P*x* at the Station I, is to the Rectangle between P*a* and *bx*, the Extremes of the whole Perſpective at the Station *J*.

Dem. In the firſt place it is evident, that P*x* continues the ſame in whatever Point of I*x* the Eye be placed, and that IK and *Jk* are the Directors of the Original Line at the Stations I and *J*.
Now in the Radial Plane I*x*KB $Pa \times bx : Px \times ab :: PA : AB$[f]
And in the Radial Plane *JxkB* $Pa \times bx : Px \times ab :: PA : AB$
Therefore $Px \times ab : Px \times ab :: Pa \times bx : Pa \times bx.$
Conſequently dividing the two firſt Terms by P*x*; $ab : ab :: Pa \times bx : Pa \times bx.$
Q. E. D.

THEOR.

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[a] Theor. 26.

[b] Theor. 31.

[c] Cor. 1. Theor. 26.

[d] Theor.

[e] Theor. 26.

[f] Theor. 28.

THEOR. XXXIV.

The ſame things being ſuppoſed as before, If the Diſtances I and *J* be taken in ſuch Proportion, that the Radial Jx, or its equal $k\mathrm{P}$, may be to PA the nearer part of the Original Line, as the whole Line PB is to the Radial I*x*, or its equal KP, then the Images ab and *ab* of the part AB of the Original Line at both Stations will be equal. Fig. 23.

Dem. In the Similar Triangles $\mathrm{I}x\mathrm{a}$, aPA $\mathrm{a}x : \mathrm{Pa} :: \mathrm{I}x = \mathrm{KP} : \mathrm{PA}$
And in the Similar Triangles Jxb, bPB $\mathrm{P}b : bx :: \mathrm{PB} : Jx = k\mathrm{P}$.
But by the Suppoſition $\mathrm{KP} : \mathrm{PA} :: \mathrm{PB} : k\mathrm{P}$
Therefore $\mathrm{a}x : \mathrm{Pa} :: \mathrm{P}b : bx$
And by Compoſition $\mathrm{a}x + \mathrm{Pa} = \mathrm{P}x : \mathrm{Pa} :: \mathrm{P}b + bx = \mathrm{P}x : bx$
Conſequently $\mathrm{Pa} = bx$.
Likewiſe in the Similar Triangles $\mathrm{I}x\mathrm{b}$, bPB $\mathrm{Pb} : \mathrm{b}x :: \mathrm{PB} : \mathrm{I}x = \mathrm{KP}$
And in the Similar Triangles Jxa, aPA $ax : \mathrm{P}a :: Jx = k\mathrm{P} : \mathrm{PA}$
But according to the Suppoſition $\mathrm{PB} : \mathrm{KP} :: k\mathrm{P} : \mathrm{PA}$
Therefore $\mathrm{Pb} : \mathrm{b}x :: ax : \mathrm{P}a$
And by Compoſition $\mathrm{Pb} + \mathrm{b}x = \mathrm{P}x : \mathrm{b}x :: ax + \mathrm{P}a = \mathrm{P}x : \mathrm{P}a$
Conſequently $\mathrm{b}x = \mathrm{P}a$
Therefore $\mathrm{Pa} \times \mathrm{b}x = \mathrm{P}a \times bx$
But $\mathrm{ab} : ab :: \mathrm{Pa} \times \mathrm{b}x : \mathrm{P}a \times bx$[a]
Conſequently $\mathrm{ab} = ab$. *Q. E. D.*

[a] Theor. 33.

C O R.

It is evident that P*b* is equal to a*x*, becauſe Pa and *bx* are equal, and a*b* is common to both.

Alſo Pb is equal to *ax*, becauſe P*a* and b*x* are equal, and *a*b is common to both.

L E M. 4.

If four Quantities be Geometrically Proportional, and to each of them the ſame Quantity be added, the Rectangle between the biggeſt and leaſt of thoſe Proportionals thus increaſed, will be larger than the Rectangle between the increaſed means, by a Rectangle under the common added Quantity, and the Difference between the Sum of the Extremes and the Sum of the Means of the Proportionals firſt ſuppoſed.

Thus if $a : b :: c : d$ (ſuppoſing a to be the largeſt of the four) and any Quantity x be added to each of them; It muſt be ſhewn that the Rectangle between $a + x$ and $d + x$, is larger than the Rectangle between $b + x$ and $c + x$, by the Rectangle between x and the Exceſs of $a + d$ above $b + c$.

Firſt $a + x$ multiplied into $d + x$ produces $ad + ax + dx + xx$
And $b + x$ multiplied into $c + x$ produces $bc + bx + cx + xx$
But $ad = bc$[b], therefore $ad + xx = bc + xx$.

And ſubſtracting theſe equal Quantities out of each of the Products, there will remain of the Product of the biggeſt and leaſt $ax + dx$
And of the Product of the Means $bx + cx$.

But $a + d$ is bigger than $b + c$[c], therefore $ax + dx$ is bigger than $bx + cx$.

And if the Difference between $a + d$ and $b + c$ be called y, it is evident that $a + d = b + c + y$, conſequently $ax + dx = bx + cx + yx$.

Therefore the Rectangle between $a + x$ and $d + x$, is larger than the Rectangle between $b + x$ and $c + x$, by the Rectangle yx, which is contained between the common added Quantity x, and y the Difference between $a + d$ and $b + c$. *Q. E. D.*

[b] 16 El. 6.

[c] 25 El. 5.

C O R.

If three Quantities be in continual Geometrical Proportion, and the ſame Quantity be added to each of them, the Rectangle between the Extremes thus increaſed, will be larger than the Square of the Mean increaſed Quantity, by a Rectangle between the added Quantity, and the Exceſs of the Sum of the Extremes of the given Proportionals above the Double of the Mean.

The Demonſtration of this is the ſame as before, if b and c be ſuppoſed equal.

THEOR. XXXV.

Fig. 24. The ſame things being ſuppoſed as in the laſt Theorem, If the Diſtance of the Eye Ix, be taken a mean proportional between the nearer part PA and the whole Line PB, the Image of the farther part AB from that Station, will be larger than from any other Station of the Eye in the Radial of the Original Line; and the Images will become ſmaller, as the other Stations are taken more diſtant from the Station I, either farther from or nearer to the Picture.

Dem. Take two Stations J and i on each Side of I in ſuch Proportion that $kP : PA :: PB : \mathrm{k}P$
And two other Stations Δ and Γ ſo that $\varkappa P : PA :: PB : \chi P$
Then becauſe by the Suppoſition $KP : PA :: PB : KP$
It follows that $kP : KP :: KP : \mathrm{k}P$
And alſo that $\varkappa P : kP :: \mathrm{k}P : \chi P$
Now in the Similar Triangles $Ix\mathrm{a}$, $\mathrm{a}PA$ $P\mathrm{a} : \mathrm{a}x :: PA : Ix = KP$
And in the Similar Triangles $Ix\mathrm{b}$, $\mathrm{b}PB$ $\mathrm{b}x : P\mathrm{b} :: Ix = KP : PB$
And by the Suppoſition $PA : KP :: KP : PB$,
Therefore $P\mathrm{a} : \mathrm{a}x :: \mathrm{b}x : P\mathrm{b}$
And by Compoſition $P\mathrm{a} + \mathrm{a}x = Px : P\mathrm{a} :: \mathrm{b}x + P\mathrm{b} = Px : \mathrm{b}x$
Conſequently $P\mathrm{a} = \mathrm{b}x$.
Again in the Similar Triangles $i\mathrm{k}A$, αPA $i\mathrm{k} = IK : P\alpha :: \mathrm{k}A : PA$
Therefore $P\alpha \times \mathrm{k}A = IK \times PA$
And in the Similar Triangles JkA, aPA, $Jk = IK : Pa = \beta x$ [a] $:: kA : PA$
Therefore $\beta x \times kA = IK \times PA$
And in the Similar Triangles IKA, $\mathrm{a}PA$, $IK : P\mathrm{a} = \mathrm{b}x :: KA : PA$
Therefore $P\mathrm{a} \times KA = IK \times PA$
Conſequently $P\alpha \times \mathrm{k}A = \beta x \times kA = P\mathrm{a} \times KA = \mathrm{b}x \times KA$
Which gives theſe Analogies $P\mathrm{a} : P\alpha :: \mathrm{k}A : KA$
And $\mathrm{b}x : \beta x :: kA : KA$
Conſequently [b] $P\mathrm{a} \times \mathrm{b}x : P\alpha \times \beta x :: \mathrm{k}A \times kA : KA \times KA$.

But it has already been ſhewn, that $kP : KP :: KP : \mathrm{k}P$. If then to each of theſe Proportionals the ſame Quantity AP be added, whereby they become kA, KA and $\mathrm{k}A$; the Rectangle between the Extremes $\mathrm{k}A$ and kA thus increaſed, will be larger than the Square of KA the Mean increaſed Quantity [c],
That is $\mathrm{k}A \times kA$ is larger than $KA \times KA$
Conſequently $P\mathrm{a} \times \mathrm{b}x$ is alſo larger than $P\alpha \times \beta x$.
But $\mathrm{ab} : \alpha\beta :: P\mathrm{a} \times \mathrm{b}x : P\alpha \times \beta x$ [d]. Therefore ab is larger than $\alpha\beta$.
And becauſe $kP : PA :: PB : \mathrm{k}P$, $\alpha\beta$ is equal to ab [e].
Therefore ab is alſo larger than ab.

That is, ab the Image of AB at the Station I, is larger than $\alpha\beta$ or ab the Images of AB at the Stations i and J.

It remains to be proved, that $\alpha\beta$ and ab are alſo larger than de and $\delta\varepsilon$, the Images of AB at the Stations Γ and Δ.

It has been ſhewn in the former part of this Demon. that $P\alpha \times \mathrm{k}A = \beta x \times kA$
After the like manner it may be ſhewn that $P\alpha \times \mathrm{k}A = Pd \times \chi A = P\delta \times \varkappa A$
And $P\delta$ being equal to ex [f], hence theſe Analogies ariſe $P\alpha : Pd :: \chi A : \mathrm{k}A$
And $\beta x : P\delta = ex :: \varkappa A : kA$
Conſequently [g] $P\alpha \times \beta x : Pd \times ex :: \chi A \times \varkappa A : \mathrm{k}A \times kA$.

And it having been ſhewn, that $\varkappa P : kP :: \mathrm{k}P : \chi P$; If to each of theſe Proportionals the ſame Quantity AP be added, whereby they become $\varkappa A$, kA, $\mathrm{k}A$, and χA; the Rectangle between χA and $\varkappa A$, the biggeſt and leaſt of theſe increaſed Quantities, will be larger than that between the Means $\mathrm{k}A$ and kA [h], conſequently $P\alpha \times \beta x$ is larger than $Pd \times ex$.

And $\alpha\beta : de :: P\alpha \times \beta x : Pd \times ex$ [i]; therefore $\alpha\beta$ or its equal ab, is larger than de or its equal $\delta\varepsilon$.

That is, the Images $\alpha\beta$ and ab of AB, at the Stations i and J, are larger than de and $\delta\varepsilon$ the Images of AB at the Stations Γ and Δ. *Q. E. D.*

COR.

[a] Cor. Theor. 34.
[b] Lem. 3.
[c] Cor. Lem. 4.
[d] Theor. 33.
[e] Theor. 34.
[f] Cor. Theor. 34.
[g] Lem. 3.
[h] Lem. 4.
[i] Theor. 33.

COR.

If the Line PB were divided in A, in Extreme and Mean Proportion[a], the ſmaller Segment PA being next the Picture; then if I*x* or KP be taken equal to the larger Segment AB, the Image of AB will be largeſt at the Station I.

For by this Suppoſition PA : AB :: AB : PB

If then AB = KP, it follows that PA : KP :: KP : PB.

THEOR. XXXVI.

Fig. 24. N°. 2. If an Original Line KB, produced to its Directing Point K, be any-wiſe divided into two parts in the Point A, and its Director IK be taken a mean proportional between the whole Line KB, and the part KA adjoining to its Directing Point; then the farther part AB of the Original Line, will appear to the Eye at I, under a larger Angle than from any other Point in the Line IK.

Having drawn IA and IB, circumſcribe the Triangle IAB with a Circle.

Dem. Then becauſe IK meets the Circle in I, and alſo a Chord AB of that Circle in K, and by Suppoſition KA : KI :: KI : KB; therefore IK is a Tangent to the Circle in I[b]; but all the Angles in the Circle which inſiſt on the Chord AB, being equal to the Angle AIB[c], and all the Angles made by Lines from A and B to any other Point *J* in IK different from I, falling without the Circle, and being therefore leſs than the Angle AIB, it follows that the Angle AIB, under which the Line AB appears to the Eye at I, is larger than that under which it would appear, from any other Point *J* in the Line IK. *Q. E. D.*

COR.

If a Point P be taken in KA, ſo that PA may be to KP as KP to PB, and a Line P*o* be drawn parallel to IK, repreſenting the Section of the Plane IKB with a Picture; then the part AB of the Original Line, will not only appear under the largeſt poſſible Angle from I, but its Image *ab* will alſo be the largeſt it can be, on this ſuppoſed Picture, with the Height of the Eye IK[d].

GENERAL COROLLARY.

If a Line not parallel to the Picture, lying in an Original Plane, be divided at pleaſure by any Number of Points, and through each of thoſe Points there be drawn Lines in the Original Plane parallel to the Interſecting Line; the Proportion of the Diſtances of the Images of all thoſe Parallels from each other, and from the Vaniſhing, Interſecting, and Directing Lines of the Original Plane, will be the ſame as that of the Images of the correſponding Parts of the Line firſt ſuppoſed, with reſpect to each other, and the Vaniſhing, Interſecting, and Directing Points of that Line; and conſequently what has been ſhewn at Theor. XXIV, XXV, XXVI, XXVII, XXVIII, XXIX, XXX, XXXI, XXXIII, XXXIV, and XXXV, and their Corollaries, touching the Proportions of the Parts of a Line, and of their Images, on the ſeveral Suppoſitions there mentioned, is equally true of the Diſtances of the Parallels here ſuppoſed, and of their Images in like Circumſtances.

This plainly follows from Theorem XXV.

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[a] 30 El. 6.

[b] 37 El. 3.

[c] 21 El. 3.

[d] Theor. 35.

STEREOGRAPHY,

OR A

COMPLEAT BODY

OF

PERSPECTIVE,

In all its BRANCHES.

BOOK II.

IN this Book we ſhall treat of the various Methods of deſcribing the Images of Points, Lines, and Figures lying in an Original Plane whoſe Vaniſhing and Interſecting Lines are given; the Center and Diſtance of the Picture being conſtantly ſuppoſed to be known.

SECTION I.

Of the Preparation of the Picture and Original Plane.

IN the Figures hitherto uſed, the Picture has been repreſented *Stereographically*, as ſtanding on the Original Plane; but this being unfit for Practice, we muſt now ſeparate them, and let each be drawn out by itſelf in its proper Meaſures, with ſuch Lines in them as will be neceſſary for the Work.

Fig. 25. N°. 1. Let us then ſuppoſe the Diſtance of the Eye, the Center of the Picture, and the Situation of the Original Plane with regard to the Picture, to be given; and let Z be the Original Plane, Y the Picture, X the Directing Plane, V the Vaniſhing Plane, and I*o*KP the Vertical Plane, as deſcribed at Fig. 8. the ſame Letters here repreſenting the ſame Things as in that Figure.

The Picture is prepared in this Manner.

Fig. 25. N°. 2. Upon the Paper or Cloth intended for the Picture, draw at Pleaſure the Interſecting Line GH, and the Vaniſhing Line EF parallel to it [a], at the Diſtance of the Height of the Eye IK, or *o*P [b]; then mark on the Vaniſhing Line, its Center *o*, and on the Interſecting Line, the Point P which is the Foot of the Vertical Line [c]: the Vertical Line P*o* may be alſo drawn, and in it the Center of the Picture O may be marked [d], when the Center of the Vaniſhing Line of the Original Plane does not coincide with it [e]; then on the Vertical Line produced beyond *o* as far as neceſſary, ſet off from *o* the Length of the Radial I*o* at I, and through I draw NR parallel to the Vaniſhing Line EF.

Fig. 25. N°. 1, 2. By this Preparation it appears, that the Picture Y is ſuppoſed to be ſeparated from the Original Plane Z, and laid flat on the Table; and that the Vaniſhing Plane V, remaining fixed to the Picture at the Vaniſhing Line EF, is ſo turned upon that Line as to fall backward, and make one continued Plane YV with the Picture, by which means the Plane V is ſeen on the undermoſt Side; and the Place of the Eye or Point of Sight falls in the Point I, and NR repreſents the Eye's Parallel [f]: for the Vertical Line P*o* being perpendicular to the Vaniſhing Line EF, and paſſing through its Center *o*, and the Radial I*o* being alſo perpendicular to the ſame Line EF [g], when the Plane V is turned upon the Line EF till it comes into the ſame Plane with the Picture,

[a] Cor. 1. Def. 10 B. I.

[b] Cor. 3. Def. 15. B. I.

[c] Def. 13. B. I.

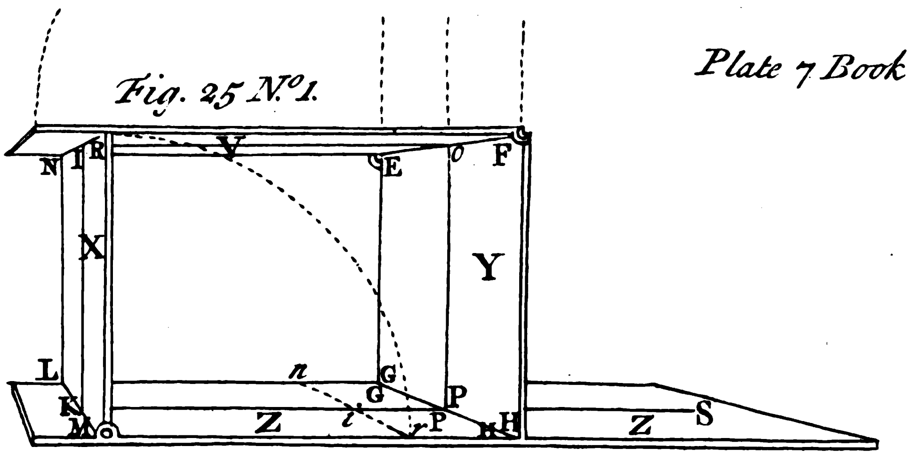
[d] Cor. 2. Def. 15. B. I.

[e] Cor. 1. and 2. Theor. 9. B. I.

[f] Cor. 1. Def. 10. B. I.

[g] Cor. 1. and 2. Def. 15. B. 1.

Fig. 25 N.° 1.



.25.N.° 2

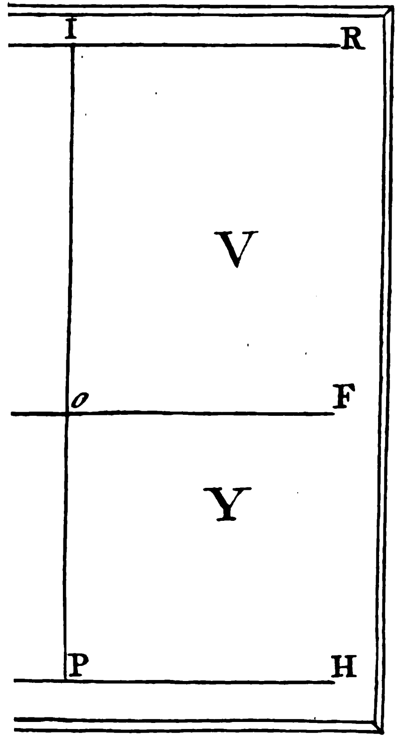
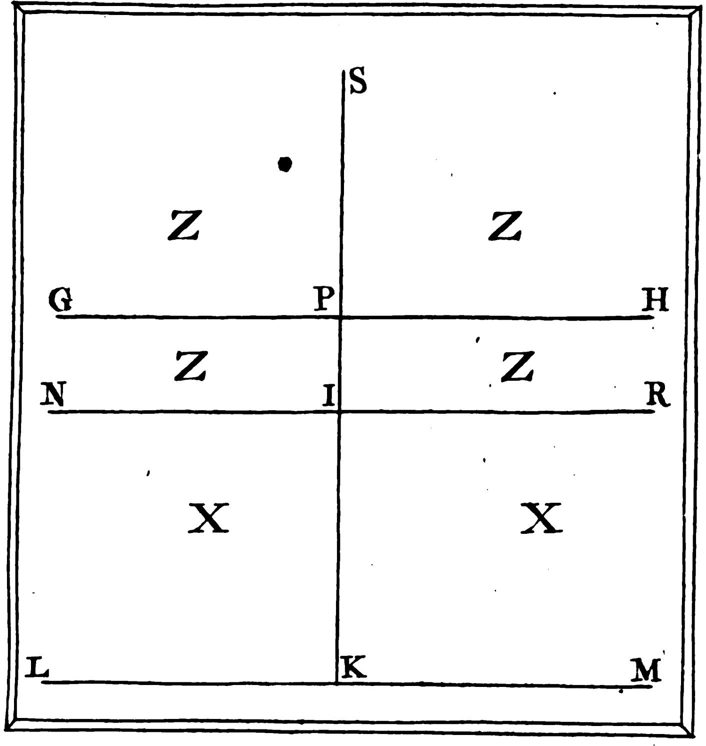


Fig. 25.N.° 3.



.1.

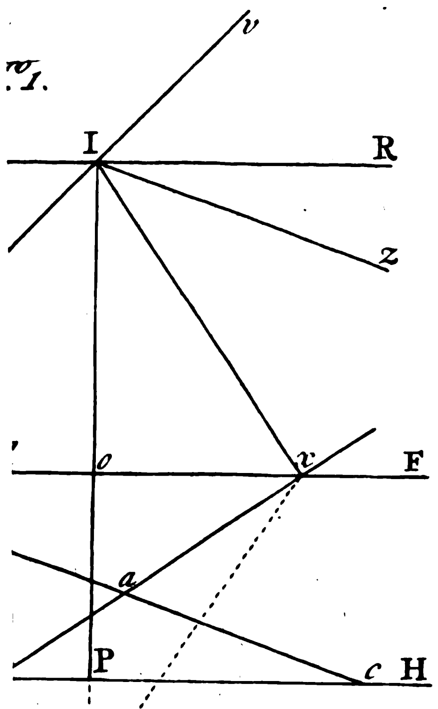


Fig. 26.N.° 2.

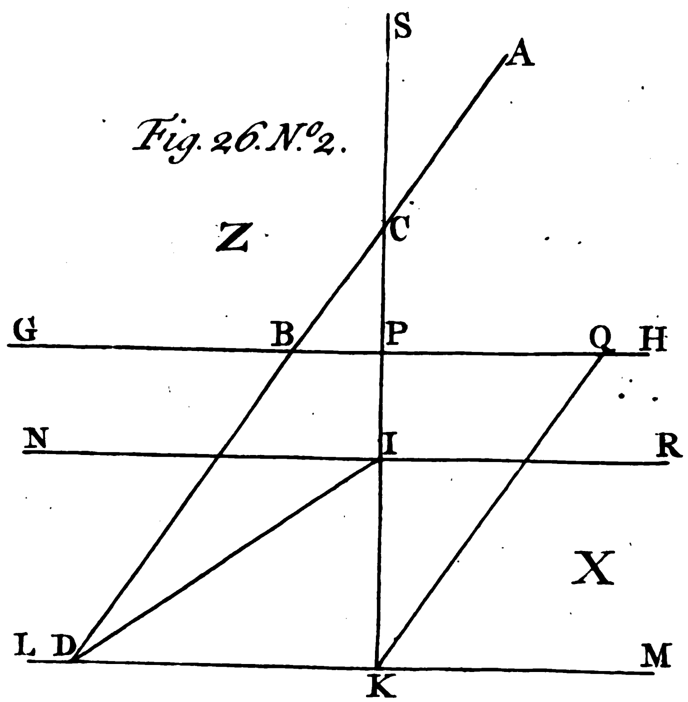
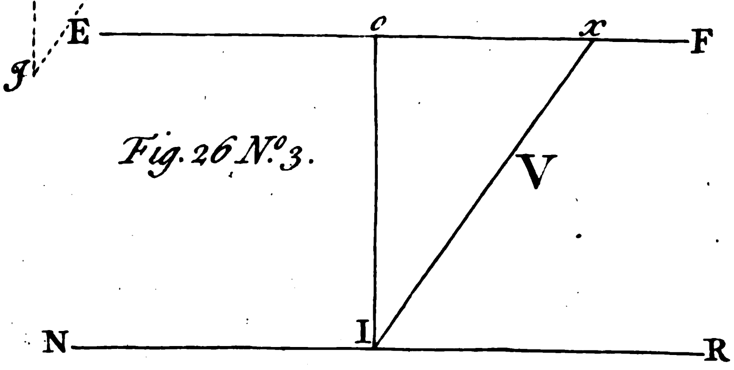


Fig. 26.N.° 3.



J. Mynde sc.

cture, the Line I *o* will be ſtill perpendicular to E F, and muſt therefore neceſſarily coincide with the Vertical Line P *o* produced, and make one continued ſtraight Line with it[a].

The Original Plane is prepared in the following manner.

Fig. 25. N°. 3. Upon the Original Plane firſt draw the Interſecting Line GH, where the Picture is ſuppoſed to cut it, and in that Line mark the Point P, anſwering to the Point P in the Picture, and through P draw KS perpendicular to GH, which will be the Line of Station[b]; then take PK equal to the Radial of the Original Plane, and K will be the Foot of the Eye's Director[c], and through K draw LM parallel to GH, and LM will be the Directing Line[d]; laſtly from K, upon the Line KP, ſet off KI equal to the Height of the Eye, and through I draw NR parallel to LM, then KI will repreſent the Eye's Director, and NR the Parallel of the Eye[e], the ſame with NR in the Picture.

Fig. 25. N°. 1. By this Conſtruction it appears, that the Original Plane Z is alſo ſuppoſed to be ſeparated from the Picture, and laid flat on the Table, the Perſpective Part of the Original Plane being diſtinguiſhed from its Projective Part by the Interſecting Line GH[f]; and that the Directing Plane X, remaining fixed to the Original Plane at the Directing Line LM, is ſo turned upon that Line, as to fall upon and make one continued Plane with the Original Plane, by which means the Directing Plane is ſeen on its backſide, and the Point of Sight or Place of the Eye falls in the Point I in the Line of Station KS, as is evident from what was ſaid of the falling of the Point I in the Picture. Fig. 25. N°. 3.

Theſe two Planes being thus prepared, every thing is ready for the Deſcription of any Lines or Figures lying in the Original Plane, whether that Plane be ſuppoſed perpendicular or anywiſe inclined to the Picture.

Note, *Moſt commonly the Line* N R, *as well in the Picture as in the Original Plane, may be omitted, ſo as the Point* I *be marked. And the Line o* P *in the Picture may alſo be left out, only marking the Points o and* P.

Of the Indefinite Images of Lines in the Original Plane.

P R O B. I.

To find the Indefinite Image of an Original Line not parallel to the Interſecting Line.

METHOD 1.

By the Vaniſhing and Interſecting Points.

Fig. 26. N°. 1, 2. Let YV repreſent the Picture and Vaniſhing Plane, and ZX the Original Plane and Directing Plane prepared as juſt now directed, wherein the ſame Letters denote the ſame things as before, and let AB be the Original Line, the Indefinite Image of which is required.

Produce the Original Line, till it cut the Interſecting Line GH of the Original Plane in B, and take P*b* in the Picture equal to PB in the Original Plane, and towards the ſame ſide of P. Then from I in the Picture draw I*x*, cutting the Vaniſhing Line in *x* in ſuch manner, that the Angle RI*x*, or its equal *ox*I[g], may be equal to the Angle ABH[h], which is the Angle of Inclination of the Original Line to the Interſecting Line, or (which amounts to the ſame thing) that the Angle *o*I*x* may be equal to the Angle SCA or PCB, which is the Angle the Original Line makes with the Line of Station KP: then a Line *bx* drawn in the Picture through the Points *b* and *x*, is the Indefinite Image deſired.

Dem. In the Vaniſhing Plane V, the Angle RI*x* being taken equal to the Angle ABH in the Original Plane, the Line I*x* is the Radial of the Original Line AB, and *x* is its Vaniſhing Point[i]; but GH in the Original Plane and GH in the Picture being both of them to be conſidered as the ſame Interſecting Line, and P*b* in the Picture being taken equal to PB in the Original Plane, it is evident that *b* is the Interſecting Point of the Original Line in the Picture, wherefore *bx* is the Indefinite Image of the Original Line AB[k]. *Q. E. I.*

Note, *The Line* I*x* *muſt be ſo drawn, that the Point x may fall on the ſame ſide of the Point o, to which the Inclination of the Original Line to the Interſecting Line is directed. If the Original Line had inclined the contrary way, x muſt have fallen on the other ſide of o. If it had been perpendicular to the Interſecting Line, the Points x and o would have been the ſame.*

[a] 14 El. 1.

[b] Cor. 2. Def. 15. B. I.

[c] Cor. 3. Def. 15. B. I.

[d] Cor. 1. Def. 10. B. I.

[e] Cor. 1, and 2. Def. 10. B. I.

[f] Def. 26. B. I.

[g] 29 El. 1.

[h] 23 El. 1.

[i] Theor. 11. B. I.

[k] Theor. 4. B. I.

COR. 1.

The Point x may be alſo thus found: From K in the Original Plane draw KQ parallel to the Original Line AB, cutting GH in Q; then make ox in the Picture equal to PQ, and x will be thereby found.

For in the Triangles oIx and PKQ, the ſides Io and PK are equal[a], as alſo the ſides ox and PQ by Conſtruction, and the Angles at P and o are right[b]; therefore thoſe two Triangles are ſimilar and equal, and the Angle PKQ equal to the Angle oIx[c]. But becauſe KQ and AB are parallel, the Angles PKQ and PCB are equal[d]; and therefore the Angle oIx is equal to the Angle PCB, and conſequently the Point x is rightly determined[e].

COR. 2.

The Point Q, and conſequently x, may be found without drawing KQ, if the Original Line be produced to its Directing Point D. For if BQ be made equal to DK, Q will be thereby determined, becauſe DBQK being a Parallelogram, the ſides DK and BQ muſt be equal[f].

METHOD 2.

By the Directing and Interſecting Points.

The ſame things being ſuppoſed as before, from I in the Directing Plane, to D the Directing Point of the Original Line, draw ID the Director of that Line; and from b in the Picture, found as before, draw bx, inclining the ſame way to GH as ID doth to LM, making the Angle xbH equal to IDM, then bx will be the Image ſought, and x its Vaniſhing Point.

Dem. Becauſe the Director of an Original Line makes the ſame Angle with the Directing Line, as its Image makes with the Interſecting Line in the Picture[g]. *Q. E. I.*

METHOD 3.

By the Directing Plane and Interſecting Point.

The Point b being found, and the Director ID drawn as before, apply the Eye's Parallel NR in the Directing Plane to the Interſecting Line GH in the Picture, ſo as to make thoſe two Lines coincide, the Directing Line LM falling below the Interſecting Line of the Picture, and the Picture and Directing Plane thus making together one continued Plane; then from b draw bx parallel to ID in this Situation, and bx will be the Image required.

Dem. This is evident, becauſe while NR and GH coincide, LM will be parallel to GH; and therefore if bx be drawn parallel to DI, the Angles IDK and xbH will be equal[h]. *Q. E. I.*

Note, *In this laſt Method the Vaniſhing Plane is not concerned, and this way may be uſed when the Vaniſhing Point is out of reach.*

PROB. II.

Having the Indefinite Image of a Line given, thence to find its Original.

METHOD 1.

By the Directing and Interſecting Points.

Fig. 26. N°. 1, 2. The ſame things being ſuppoſed as before, let bx be the given Image whoſe Original is required.

From I, in the Directing Plane X, draw ID, making the Angle IDM equal to the Angle xbH; and having taken PB in the Interſecting Line of the Original Plane, equal to Pb in the Interſecting Line of the Picture, through D and B draw DA, and that Line will be the Indefinite Original of the given Image bx.

Dem. For B is the Interſecting Point of the Original Line, and DI being the Director of that Line[i], D is its Directing Point, through which two Points the Original Line muſt neceſſarily paſs. *Q. E. I.*

COR.

The Directing Point D may be alſo found, by placing the Line NR of the Directing Plane X, on the Interſecting Line GH of the Picture, and then drawing from I, in the Directing Plane thus placed, a Line ID parallel to bx, which will determine the ſame Point D. Or if from b a Line be drawn perpendicular to EF cutting it in d,

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[a] Cor. 3. Def. 15. B. I.

[b] Cor. 1. Def. 15. B. I.

[c] 4 El. 1.

[d] 29 El. 1.

[e] Cor. 1. Theor. 11. B. I.

[f] 34 El. 1.

[g] Theor. 12. B. I.

[h] 29 El. 1.

[i] Theor. 12. B. I.

from K in the Directing Line, ſet off KD equal to *dx*, and D will be thereby found; it being evident, that the Triangles *bdx* and IKD are Similar, and conſequently the Angles IDK and *bxd* equal, which laſt is equal to the Angle *xb*H.

METHOD 2.

By the Vaniſhing Plane and Interſecting Point.

Place the Vaniſhing Plane V on the Original Plane, ſo that the Vaniſhing Line EF may coincide with GH in the Plane Z, and the Eye's Parallel NR in the Plane V may coincide with LM in the Plane X (for ſo they will do, the Lines I*o* and KP being equal[a],) the Vaniſhing Plane, thus placed, being ſeen on the upper ſide, and making one continued Plane with the Plane Z; then through B, found as before, draw BA parallel to I*x* in this Situation, and that will be the Original of the Image propoſed.

Dem. Becauſe the Original Line is parallel to its Radial[b]. *Q.E.I.*

Note, *Here the Directing Plane is not concerned, ſo that this Method may be uſed when the Directing Point is out of reach.*

SCHOL.

This Problem being the reverſe of the preceding, the different Rules there given may be applied here, by uſing the Directing Plane and Directing Line here, as the Vaniſhing Plane and Vaniſhing Line were uſed there, and *vice verſa*. And hence it will be eaſy to apply any Rule given for finding the Image from its Original, to the finding the Original from its Image; by ſuppoſing the Picture to be the Original Plane, the Original Plane to be the Picture, the Vaniſhing Plane to be the Directing Plane, and the Directing Plane to be the Vaniſhing Plane, and the other Lines and Points in thoſe Planes to change their Names accordingly, except only the Parallel of the Eye and the Interſecting Line, which upon either Suppoſition continue the ſame.

But in this, regard muſt be had to the different Situation given to the Vaniſhing Plane with reſpect to the Picture, to that which the Directing Plane hath with reſpect to the Original Plane: the firſt being ſuppoſed to be ſeen on the under ſide, whereby the Inclination of I*x* to the Vaniſhing Line EF is towards the contrary ſide, that the Original Line AB inclines on GH in the Original Plane, although if the Vaniſhing Plane were Rectified, or turned round the Line EF till it came into its proper Situation, the Lines I*x* and AB would then become parallel; whereas the Directing Plane being ſuppoſed to be laid down on the Original Plane, the Directors make the ſame Angle, and incline the ſame way on the Directing Line, as the Images do on the Interſecting Line of the Picture. And the Radials would have fallen in the ſame manner in the Vaniſhing Plane, if inſtead of its being turned upwards, as before directed, it had been turned downwards on the Line EF, ſo as to make the Point I fall at *J* below P on the Line *o*P; for then it is evident, the Line *Jx* will incline the ſame way to GH the Interſecting Line in the Picture, that the Original Line AB doth to GH and LM, the Interſecting and Directing Lines of the Original Plane. But by this laſt Method, the Picture would be too much incumbered with the Radials drawn from *J* to the Vaniſhing Line, for which reaſon the other Method is preferred.

However, it may be found ſometimes convenient to ſeparate the Vaniſhing Plane intirely from the Picture, when the part where it ſhould lie, is otherwiſe taken up with Figures, and ſeveral Vaniſhing Points are required to be found, and to draw out the Vaniſhing Plane apart by itſelf, and then it may have its natural Situation given it; as in Figure N°. 3. where NR repreſents the Eye's Parallel, EF the Vaniſhing Line, I*o* the Diſtance of that Vaniſhing Line, *o* its Center (the ſame with *o* in the Picture) and I*x* the Radial of the Original Line, which will then make the ſame Angle, and incline the ſame way on NR, as the Original Line doth on the Interſecting Line of the Original Plane; and then the Diſtance *ox* in Figure N°. 3. may be transferred from *o* to *x* on the Vaniſhing Line of the Picture, and the Vaniſhing Point *x* will be thereby rightly determined, and the Picture diſincumbered of the Vaniſhing Plane and the Lines drawn in it. Fig. 26. N°. 3.

And this in effect is the ſame with the ſecond way propoſed for finding *x*, at Cor. 1. and 2. Method 1. Prob. I. where LMGH in the Original Plane is uſed inſtead of Figure N°. 3. the Line KP being by Conſtruction equal to I*o*, and LM and GH repreſenting NR and EF, and the Point Q repreſenting the Point *x* of that Figure.

The ſame is to be underſtood of making a ſeparate Directing Plane when there is occaſion.

[a] Cor. 3. Def. 15. B. I.

[b] Cor. 1. Def. 18. B. I.

PROB. III.

Having the Common Vaniſhing Point of any parallel Lines in the Original Plane given, thence to find the Vaniſhing Point of all other Lines in that Plane, which make a given Angle with the Lines firſt propoſed.

Fig. 26. N°. 1. The ſame things being ſuppoſed as before, let x be the given Vaniſhing Point; and let it be required to find another Vaniſhing Point y, ſo that all Original Lines whoſe Vaniſhing Point is y, may make a given Angle with thoſe whoſe Vaniſhing Point is x.

From the given Vaniſhing Point x draw the Radial Ix, then from I, towards that ſide of x, to which the Original Lines, whoſe Vaniſhing Point is required, are ſuppoſed to incline, draw Iy, making the Angle xIy equal to the given Angle, and y will be the Vaniſhing Point ſought.

Dem. Becauſe the Original Lines, whoſe Radials are Ix and Iy, make together the ſame Angle as thoſe Radials do[a]. *Q. E. I.*

SCHOL.

If any Lines drawn in the Picture through x and y, croſs each other in the Perſpective Part of the Picture, then their Originals will meet and make the given Angle in the Perſpective Part of the Original Plane. If the Images croſs in the Projective or Tranſprojective Part of the Picture, the Originals will meet in the correſponding part of the Original Plane; and if the Images be drawn parallel to each other, the Interſections of their Originals will be in the Directing Line[b]; ſo that although the Images, whoſe Vaniſhing Points are x and y, ſhould not croſs in the Picture, yet they may be ſaid to contain a Stereographical Angle equal to yIx, becauſe their Originals muſt Interſect ſomewhere, and form that Angle.

CASE 2.

If the Original Lines propoſed, be parallel to the Interſecting Line of the Original Plane, the Vaniſhing Point of Lines which make a given Angle with the propoſed Original Lines, may be found in this manner.

Through I draw Ix, cutting the Vaniſhing Line in x, towards that ſide of o, to which the Originals required are ſuppoſed to tend, ſo that the Angle RIx, or its equal Ixy, may be equal to the Angle propoſed, and x will be the Vaniſhing Point ſought.

Dem. Becauſe the Radial Ix makes the ſame Angle with the Eye's Parallel NR, as the Originals whoſe Vaniſhing Point is x, make with the Interſecting Line in the Original Plane[c], or any Line parallel to it. *Q. E. I.*

CASE 3.

If the Original Lines propoſed, incline ſo much to the Interſecting Line of the Original Plane, that their Vaniſhing Point is out of reach, yet if the Angle which the Originals make with the Interſecting Line be known, the Vaniſhing Point of Lines which make a given Angle with the propoſed Original Lines, may be found thus.

Through I draw Iz towards that ſide to which the Originals are ſuppoſed to tend, making the Angle RIz equal to the Angle of Inclination of the Original Lines to the Interſecting Line; then draw another Line Iy cutting the Vaniſhing Line in y, ſo that the Angle zIy may be equal to the Angle propoſed, and y will be the Vaniſhing Point deſired.

Dem. Becauſe Iz is the imperfect Radial of the given Original Lines, therefore the other Lines whoſe Vaniſhing Point is y, will make an Angle with theſe, equal to the Angle zIy[d], which was taken equal to the Angle propoſed. *Q. E. I.*

SCHOL.

The Reverſe of this Problem, *viz.* from a Directing Point given, thence to find another Directing Point, ſo that the Images of all Original Lines which have thoſe Points for their Directing Points, may make in the Picture an Angle equal to any Angle propoſed, is very eaſy; only by uſing the Directing Plane and Directors in this Caſe, as the Vaniſhing Plane and Radials were uſed in the other; regard being had to the placing of the Directing Point ſought, which muſt fall on the contrary ſide of the

given

[a] Cor. 3. Theor. 11. B. I.

[b] Theor. 7. and Cor. 1, 2. B. I.

[c] Theor. 11. B. I.

[d] Cor. 2. Theor. 11. B. I.

given Directing Point to that, to which the propoſed Images are intended to incline. The Demonſtration of which Practice is deduced from *Theor.* XII. and its Corollaries, as thoſe of this Problem follow from *Theor.* XI. and its Corollaries.

Note, *The Angles determined by this Problem, when neither of the Original Lines are ſuppoſed parallel to the Interſecting Line, are thoſe comprehended between the two Vaniſhing Points, or the two Interſecting Points of the Images, as* yax *or* bac, *or the correſponding Angles of the Originals; and not the Angles which the Originals or their Images make ſidewiſe, as the Angles* yab *or* xac.

DEF. 1.

The Angles yax or bac, or any others in the like Situation, and their correſponding Originals, ſhall be called *Inward Angles*, to diſtinguiſh them from the Angles yab and xac, or ſuch like, which ſhall be called *Outward Angles*.

COR.

Having the Vaniſhing Point of any parallel Lines given, thence to find the Vaniſhing Point of other Lines, which make with the firſt, an Outward Angle equal to an Angle propoſed.

Let x be the given Vaniſhing Point; from I draw Iv beyond I, towards the ſame ſide, on which the propoſed Angle is intended to lie, making the Angle vIx equal to the given Angle; and the Line vI, being produced till it cut the Vaniſhing Line in y, will determine y the Vaniſhing Point required.

For the Inward Angle yax repreſenting an Angle equal to yIx, the Outward Angle xac, which is the Complement to two Rights of the Angle yax, muſt repreſent an Angle equal to vIx, which is the Complement to two Rights of the Angle xIy, and was taken equal to the Angle propoſed.

PROB. IV.

A Vaniſhing Point being given, thence to find two other Vaniſhing Points, ſo that all Lines drawn in the Picture from thoſe three Points on the ſame ſide of the Vaniſhing Line, may by their mutual Interſections form Triangles, whoſe Originals ſhall be Similar to an Original Triangle given.

Fig. 27. N°. 1.

The ſame things being ſuppoſed as before, let x be the given Vaniſhing Point.

Having drawn the Radial Ix, take in it from I any part IB, and make on that Line a Triangle IBC Similar to the Original Triangle propoſed[a], having either of its Angles at I; then produce IC till it cut the Vaniſhing Line in z, and from I draw another Line Iy parallel to CB, cutting the Vaniſhing Line in y, and z and y will be the two Vaniſhing Points required; and all Lines drawn in the Picture from the three Points x, y, and z, on the ſame ſide of the Vaniſhing Line, will by their mutual Interſections form Triangles, the Originals of which will be Similar to the Triangle propoſed.

Draw from z any Lines zb, zc, and from x the Lines xf, xb, and from y other Lines ye, yf, yb, *&c.* making by their mutual Interſections any Triangles abc, ade, aef, afb, *&c.* it muſt be ſhewn that the Originals of all theſe Triangles are Similar to the Original Triangle propoſed.

Dem. Becauſe of the Vaniſhing Points z, x, and y, the Originals of ac, ab, and cb are reſpectively parallel to the Radials Iz, Ix, and Iy[b]; but Iy is by Conſtruction parallel to CB, therefore the Original of bc is alſo parallel to CB[c]: and thus the Originals of the three Sides ab, ac, and cb of the Triangle abc, being reſpectively parallel to the three Sides IB, IC, and CB of the Triangle ICB, their correſponding Angles are equal[d]; and therefore the Original of the Triangle abc is Similar to the Triangle ICB, which was made Similar to the Original Triangle propoſed.

The ſame may be ſhewn in like manner of each of the other Triangles ade, aef, *&c.* *Q. E. I.*

CASE 2.

Fig. 27. N°. 2.

If in making the Triangle ICB Similar to the Original Triangle given, the Line CB ſhould be parallel to the Vaniſhing Line EF, then the Original Triangle will have but two Vaniſhing Points, from whence any two Lines being drawn to cut each other, and theſe being again cut by any Line parallel to the Interſecting Line, a Triangle will be thereby formed, whoſe Original will be Similar to the Triangle propoſed.

Dem. Thus if the Side CB of the Triangle ICB be parallel to EF or NR, a Line

[a] 22 El. 1.

[b] Cor. 1. Def. 18. B. I.

[c] 9 El. 11.

[d] 10 El. 11.

drawn from I parallel to CB muſt coincide with NR, ſo that the Side of the Original Triangle correſponding to CB, can have no Vaniſhing Point, and therefore will be parallel to the Interſecting Line [a]; to which its Image will alſo be parallel [b]; therefore if any two Lines xg and zf be drawn from the two Vaniſhing Points x and z, cutting each other any where in a, any Lines bc, de, drawn parallel to the Interſecting Line GH, or the Interſecting Line itſelf, will, by their Interſections with the Lines xg and zf, form Triangles abc, ade, afg, *&c.* whoſe Originals will be Similar to the Original Triangle given. *Q. E. I.*

COR.

Hence if the three Vaniſhing Points of a Triangle be given, or if it have but two Vaniſhing Points, the Species of the Original Triangle may be found.

Fig. 27. N°. 1. Thus if z, x, and y be given, draw the Radials Iz, Ix, and Iy; then draw a Line through either of the two adjoining Radials parallel to the third, and that will determine the Species of the Original Triangle. If through Iz and Ix a Line CB be drawn parallel to Iy, the Triangle ICB will be Similar to the Original Triangle; or if through Ix and Iy a Line BD be drawn parallel to Iz, then IBD is the Species of the Original Triangle, the Triangles ICB and IBD being Similar, as is ſufficiently evident; Fig. 27. N°. 2. or if x and z be the only two Vaniſhing Points of the Original Triangle, it is apparent that CB drawn parallel to NR, determines ICB the Species of that Triangle, which is alſo Similar to the Triangle Ixz.

SCHOL.

It is limited in this Problem, that the Lines from the three Vaniſhing Points of the Original Triangle, ſhall be drawn all on the ſame Side of the Vaniſhing Line, to the end that the Triangle formed by the Interſections of thoſe Lines, may repreſent a Triangle Similar to the Original; for if thoſe Interſections fall ſome on one ſide and ſome on the other of the Vaniſhing Line, the Original of the Triangle thereby formed, will be two diſtinct and ſeparate Indeterminate Figures, the one in the Projective and Perſpective Parts, and the other in the Tranſprojective Part of the Original Plane.

Fig. 27. N°. 3. Thus if through the Vaniſhing Points z, x, and y, three Lines za, xc, and ya were ſo drawn, that their Interſections a and c ſhould fall on one ſide of the Vaniſhing Line EF, and their Interſection b on the other, thereby forming a Triangle abc; the Original of this Triangle is not one Figure, but two diſtinct and indeterminate Figures, if they may be ſo called, the Original of the Side ba being two ſeparate Indefinite Lines, as is alſo the Original of bc, lying part on one ſide and part on the other of the Directing Line in the Original Plane [c]; ſo that the Original of the Part zbx of the Triangle abc, will be two Indefinite Lines cutting each other in the Original of b, and making together an Angle repreſented by zbx, that is, an Angle equal to CIB; and the Remainder $zacx$ of the Triangle abc, will be a Figure having one determinate Side correſponding to ac, but thoſe which correſpond to az and cx, will be Indefinite Lines; and the Original of za will make an Angle with the Original of ac, equal to the Angle zIy made by the Radials Iz and Iy, that is, an Angle equal to zCB; but the Angle xca being an outward Angle [d], it will be equal to the Complement to two Rights of the Angle xIy made by the Radials Ix and Iy [e], that is, the Angle TIy, to which the Angle CBx is equal, Iy and CB being by Conſtruction parallel.

On the other hand, if the Image za be produced indefinitely from b and a contrarywiſe beyond g and d, and the Line xb be in like manner produced beyond f and e, the Originals of the two Indefinite Figures fbg and $dace$ will form a Triangle Similar to the Triangle ICB, lying part on one ſide and part on the other ſide of the Directing Line.

For the Original of bg and ad indefinitely produced beyond g and d, is a determinate Line paſſing through the Directing Line, and joining the Originals of b and a; alſo the Original of bf and ce indefinitely produced, is a determinate Line joining the Originals of b and c [f], and the Original of the Line ac is a Line joining the Originals of a and c, wherefore thoſe three Original Lines make a Triangle. Now the Angle of this Triangle repreſented by fbg is equal to the Angle zIx, and it having been ſhewn, that the Angles zac and xca repreſent Angles equal to zCB and xBC, the Angles dac and ace, which are the Complements to two Rights of the Angles zac and xca, will repreſent Angles equal to ICB and IBC, which are the Complements to two Rights of the Angles zCB and xBC: wherefore the Original Triangle which produces the two indeterminate Figures fbg and $dace$, having its three Angles at

the

[a] Theor. 1. B. I.

[b] Theor. 2. B. I.

[c] Schol. Theor. 4. B. I.

[d] Def. 1.

[e] 29 El. 1.

[f] Schol. Theor. 4. B. I.

Plate 8 Book 2 Sect. 1. 2.

Fig. 27. N° 1.

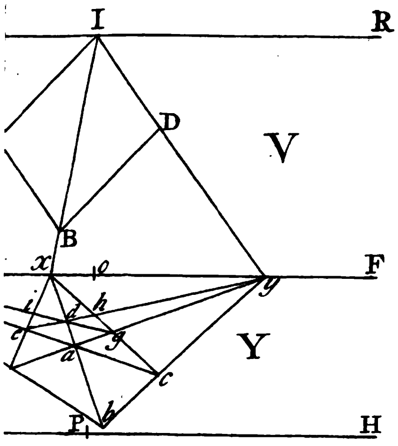


Fig. 27 N° 2.

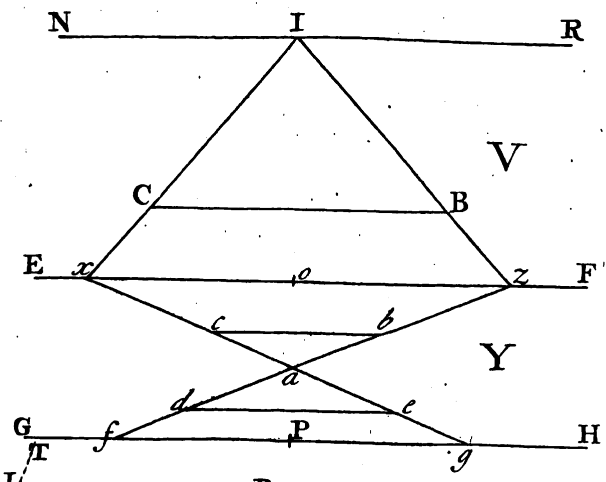


Fig. 27. N. 3.

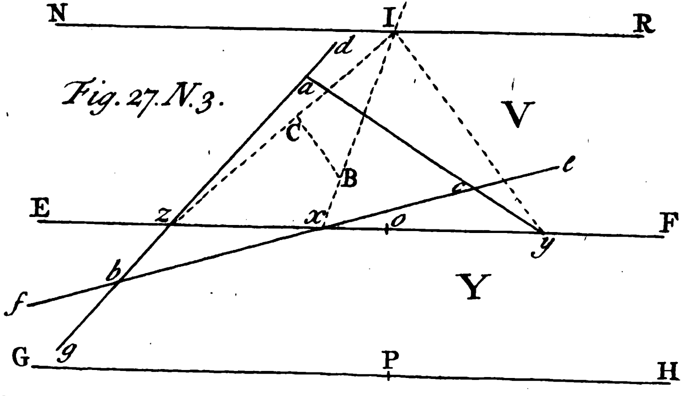


Fig. 28. N° 1.

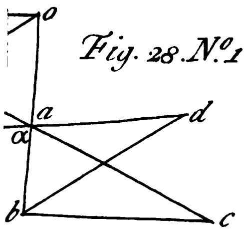
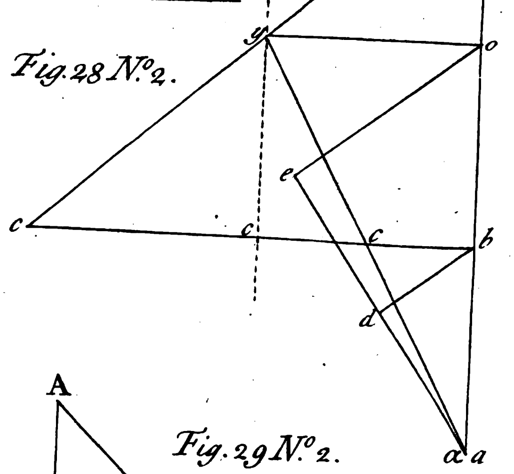


Fig. 28 N° 2.



N° 1.

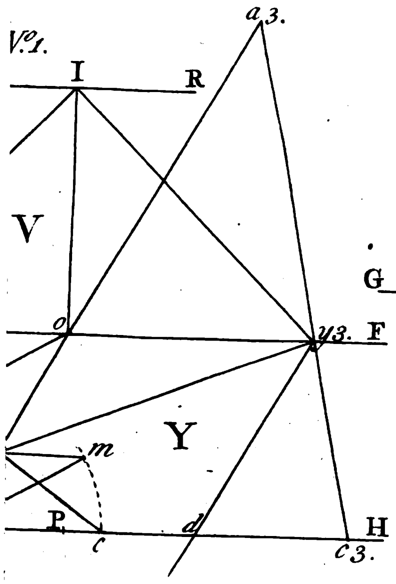
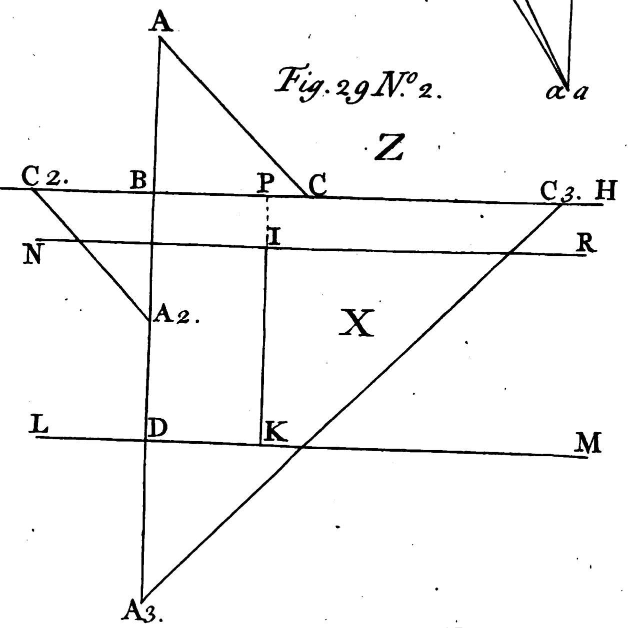


Fig. 29 N° 2.



J. Mynde sc.

the Originals of *b*, *a*, and *c* reſpectively equal to the three Angles of the Triangle I C B, thoſe two Triangles are Similar.

GENERAL COROLLARY.

The Reverſe of this Problem, *viz.* from a Directing Point given, to find two other Directing Points, ſo that the Images of all Original Lines drawn from thoſe three Directing Points, may, by their mutual Interſections, form in the Picture Triangles Similar to a given Triangle, is performed by uſing the Directing Plane and Directors of the Original Lines, as the Vaniſhing Plane and Radials are directed to be uſed here, as was obſerved at the laſt *Scholium* of Problem III. which being ſo evident, it will be needleſs to inſiſt farther on Operations of this laſt ſort.

SECTION II.

Of the determinate Images of Points and Lines in the Original Plane.

LEM. I.

IF from any two Points *o* and *b*, in a given Line *ob*, any two parallel Lines *oy* and *bc* be drawn, and through the Extremities *y* and *c* of theſe Parallels, there be drawn *yc* cutting *ob* in *a*; if then the Lines *oy* and *bc* be anywiſe turned round *o* and *b*, ſo as they may ſtill remain parallel, a Line joining their Extremities in this new Poſition will cut *ob* in the ſame Point *a*. Fig. 28. N°. 1, 2.

Let *oy* and *bc* be turned on the Points *o* and *b*, till they come into the Poſition *oe* and *bd* parallel to each other, and draw *ed*; it muſt be proved that *ed* will cut *ob* in the ſame Point *a* where it is cut by *yc*.

If *ed* do not cut *ob* in *a*, let it cut it any where elſe in α.
Then in the Similar Triangles abc, aoy $\quad bc : oy :: ba : ao$
And in the Similar Triangles αbd, αoe $\quad bd : oe :: b\alpha : \alpha o$
But by Suppoſition $bc = bd$, and $oy = oe$.
Therefore $\quad ba : ao :: b\alpha : \alpha o$
And $\overline{ao+ba} = bo : \overline{\alpha o + b\alpha} = bo :: ao : \alpha o$
Wherefore $ao = \alpha o$, that is, the Points *a* and α are the ſame. *Q. E. D.*

COR. I.

If *oe* and *bd* be not taken equal, but only in the ſame Proportion to each other as *oy* to *bc*, the Line *ed* will ſtill cut *ob* in the ſame Point *a*.

For if $bc : oy :: bd : oe$, then $ba : ao :: b\alpha : \alpha o$, as before.

So that the place of *a* doth not depend on the Angles made by *bc* and *oy* with *ob*, nor on the Length of *bc* and *oy*, but only on the Proportion of *bc* to *oy*; and while that Proportion is kept, the Point *a* will always fall in the ſame Place of *ob*.

COR. 2.

If the Lines *bc* and *oy* be taken, one on the one ſide and the other on the other ſide of *ob*, the Point *a* will fall ſomewhere between *o* and *b*; but if they be taken both on the ſame ſide of *ob*, the Point *a* will fall ſomewhere beyond *o* or *b*, according as *bc* or *oy* is the larger; if they be equal, *ob* and *yc* will be parallel, and ſo can never meet to determine the Point *a*. Fig. 28. N°. 1. Fig. 28. N°. 2.

PROB. V.

To find the Image of a given Point in the Original Plane.

This is done by finding the Indefinite Images of any two Lines in the Original Plane which paſs through the given Point, the Interſection of thoſe Images being the Image of the Point required[a].

Now the Original Point may be either in the Perſpective, Projective, or Tranſprojective Part of the Original Plane. How to find the Image of the given Point in each of theſe Caſes, we ſhall propoſe the following Methods, and for the Conveniency of the Demonſtrations, the Letters which relate to the Projections and Tranſprojections, ſhall

[a] Theor. 7. B. I.

ſhall be diſtinguiſhed from each other, and from thoſe which concern the Perſpectives, by the Numeral Figures 2 and 3 annexed to them reſpectively.

METHOD I.

Fig. 29. N°. 1, 2. Let VY and ZX repreſent the Picture and Original Plane as before; and let A be a Point in the Perſpective Part of the Original Plane, A 2 a Point in the Projective Part, and A 3 a Point in the Tranſprojective Part of that Plane, the Images of which are ſought.

Firſt, To find the Perſpective of the Point A.

Draw A B perpendicular to the Interſecting Line of the Original Plane, cutting it in B, and take Pb in the Interſecting Line of the Picture, equal to P B in the Original Plane, and from b to o the Center of the Vaniſhing Line, draw bo; then from o ſet off a Diſtance oy, on the Vaniſhing Line, on either ſide of o, equal to Io the Radial of the Original Plane, and from b ſet off a Diſtance bc, on the Interſecting Line, equal to A B the Diſtance from A to B, but taken the contrary way from b, that y was taken from o; then draw yc, cutting bo in a, and a will be the Perſpective of the Original Point A.

Secondly, To find the Projection of the Point A2.

Draw A 2 B perpendicular to the Interſecting Line of the Original Plane, cutting it in B, and having taken b in the Picture and drawn bo, as in the former Caſe, take oy in the Vaniſhing Line, on either ſide of the Center o, equal to Io, and take bc 2 on the Interſecting Line, equal to A 2 B the Diſtance between A 2 and B, on the ſame ſide of b as y was taken from o, and through y and c 2 draw a Line till it cut bo, produced beyond the Interſecting Line, in a 2, and a 2 will be the Projection of the Original Point A 2.

Thirdly, To find the Tranſprojection of the Point A3.

Draw A 3 B perpendicular to the Interſecting Line of the Original Plane, cutting it in B, and having taken b in the Picture and drawn bo, and ſet off the Diſtance Io on either ſide of o on the Vaniſhing Line, as at y 3, as before, take bc 3 in the Interſecting Line of the Picture, equal to A 3 B the Diſtance from A 3 to B, on the ſame ſide of b as y 3 is of o, as in the laſt Caſe, and through c 3 and y 3 draw a Line till it cut bo produced beyond the Vaniſhing Line, in a 3, and a 3 will be the Tranſprojection of the Original Point A 3.

Dem. Take B C in the Interſecting Line of the Original Plane equal to B A, and draw A C, and in the Vaniſhing Plane draw the Radials Iy and Iy 3.

Then becauſe the Original Line AB is perpendicular to the Interſecting Line, and conſequently parallel to the Line of Station, its Vaniſhing Point is therefore in o the Center of the Vaniſhing Line [a], and b being the Interſecting Point of A B, bo is its Indefinite Image: and becauſe the Triangles ABC, Ioy are Similar, they being both Iſoſceles, and Rectangular at B and o, the Angles B A C and oIy are equal; wherefore y is the Vaniſhing Point of AC [b], and c being its Interſecting Point, bc and B C being by Conſtruction equal, cy is therefore the Indefinite Image of CA; wherefore a the Interſection of bo with cy, is the Image of A the Interſection of B A with C A, which is the Original Point propoſed.

The ſame Conſtruction and Demonſtration will ſerve for the other Points A 2 and A 3 and their reſpective Images a 2 and a 3, as is evident from the Figures. *Q. E. I.*

SCHOL.

Hence the only Difference between the Method of finding the Perſpective, the Projection, and the Tranſprojection of a Point is, that for the Perſpective, the Diſtances AB and Io, before directed to be ſet off on the Interſecting and Vaniſhing Lines of the Picture, are to be taken the contrary way from b and o; but for Projection and Tranſprojection, thoſe Diſtances are both to be ſet off on the ſame ſide of thoſe Points; with a due regard to which Difference, all the Rules that ſhall be given for finding the Image of a Point in the Perſpective Part of the Original Plane, will be equally applicable to the finding the Images of Points in the Projective or Tranſprojective Parts of that Plane.

Note alſo, That in Projection, the Diſtance A 2 B of any Point A 2, or its equal bc 2 is always leſs than Io, or its equal oy, ſeeing the Original Point A 2 muſt lie ſomewhere between the Interſecting Line G H and the Directing Line L M, and therefore the Lines ob and yc 2 muſt Interſect in a 2, below the Interſecting Line of the Picture;

In

[a] Cor. 2. Theor. 11. B. I.

[b] Prob. I.

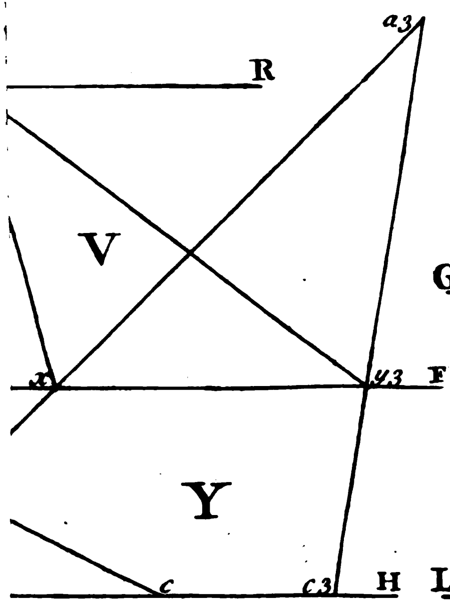
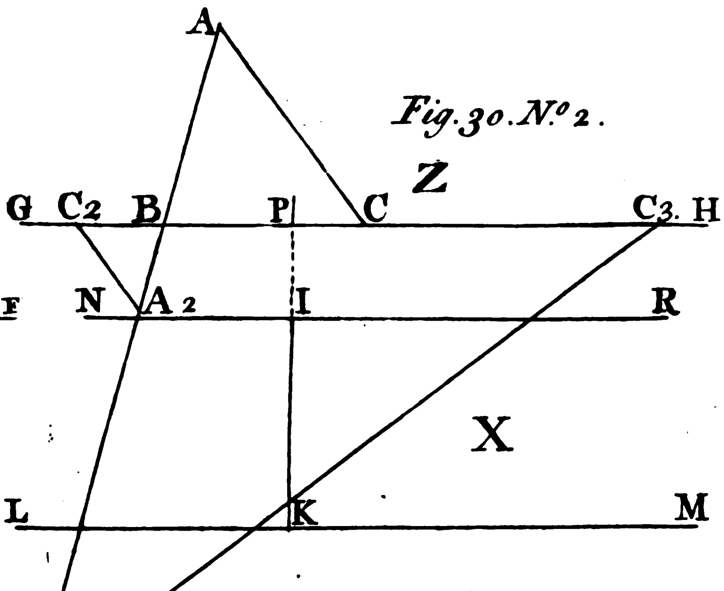


Fig. 30. N.° 2.



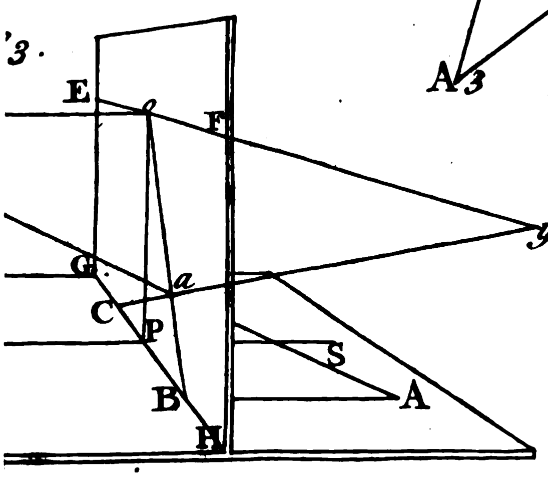


Fig. 31. N.° 1.

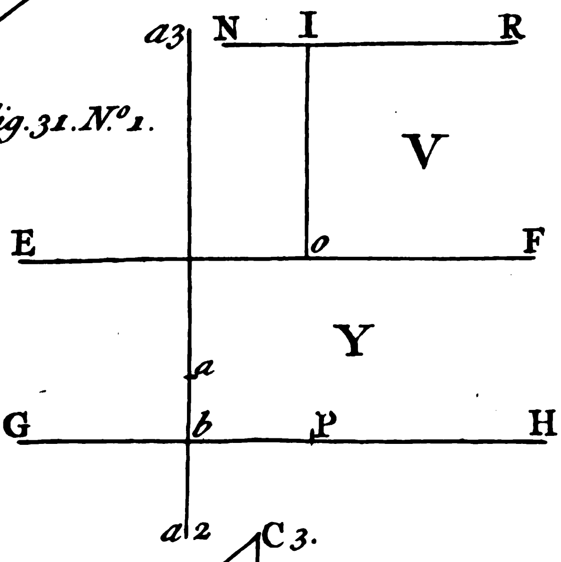


Fig. 31. N.° 2

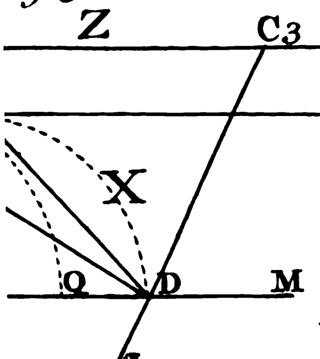
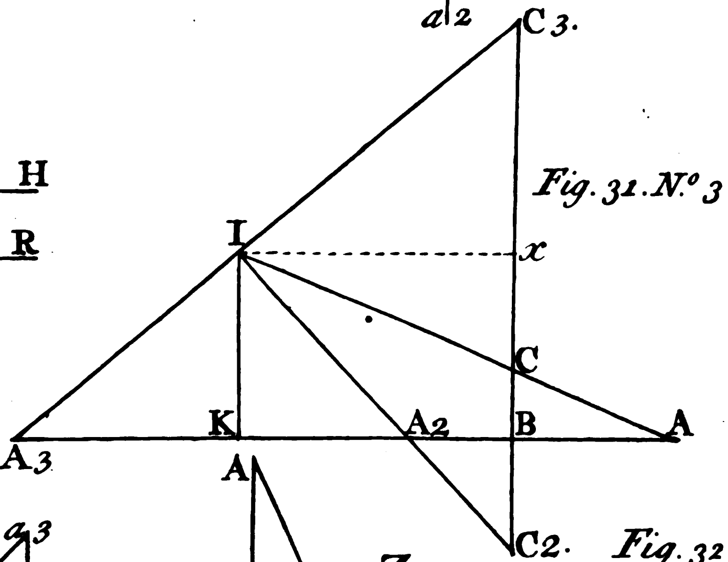


Fig. 31. N.° 3.



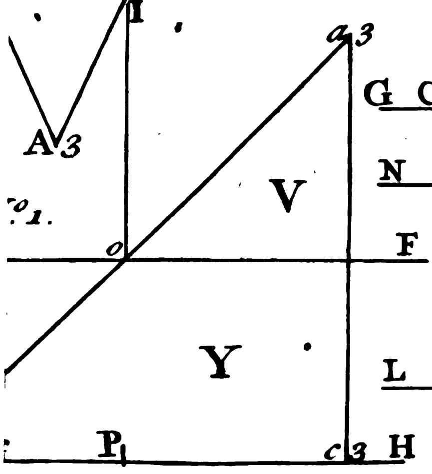
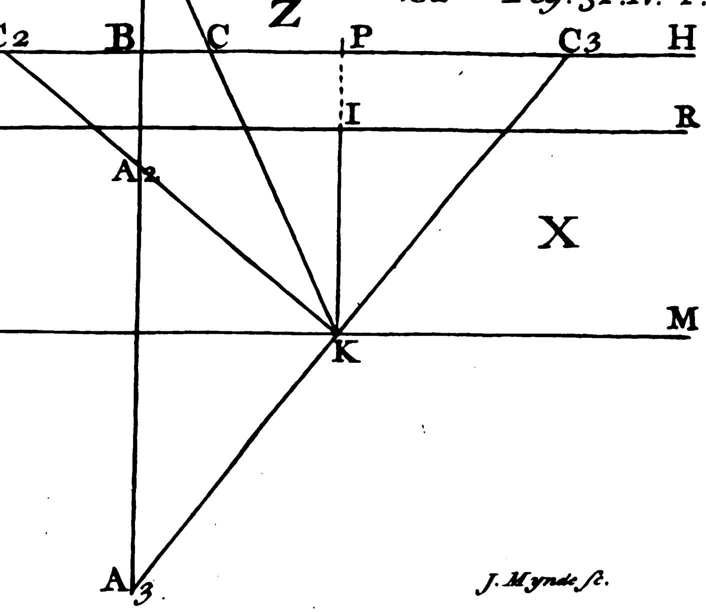


Fig. 32. N.° 2.



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In Tranſprojection, the Diſtance A3B muſt always be greater than PK, or its equal Io, becauſe the Original Point A3 muſt lie ſomewhere behind the Directing Line, wherefore bo and $c3y3$ muſt interſect in $a3$ ſomewhere beyond the Vaniſhing Line; but in Perſpective, the Diſtance AB may be either bigger or leſs than Io, for the two Diſtances being to be ſet off contrarywiſe at y and c, the Line yc muſt always cut ob in ſome Point between b and o, at whatever Diſtance the Point c be taken from b.

If the Diſtance DB be equal to PK, which can only be when the Original Point D is in the Directing Line LM, then if bd be made equal to PK, $oy3$ and bd will alſo be equal, and the Line $dy3$ will be parallel to bo, and ſo can never cut it to determine the Image of that Point; ſo that a Point D in the Directing Line can have no Image, as has already been ſhewn [a].

COR. 1.

If the Diſtance bc be taken on the other ſide of b, as at b, and the Diſtance Io be ſet off the contrary way at $y3$, a Line $by3$ will cut bo in the ſame Point a [b].

COR. 2.

If the Point b be known, the Image of A may be found without drawing bo, by ſetting off the Diſtance AB both ways from b, at c and b, and the Diſtance Io both ways from o, at y and $y3$, and drawing $y3b$ and yc.

For each of theſe Lines cutting bo in a, they muſt therefore cut each other in the ſame Point.

COR. 3.

If bo be given, the Point a may be found, by drawing from o and b any two Lines ol and bm parallel to each other, and making ol equal to Io, and bm equal to AB, for ſtill a Line lm will cut bo in the ſame Point a [c].

COR. 4.

If inſtead of the Diſtances Io and AB, any other Diſtances were ſet off from o and b, either on the Vaniſhing and Interſecting Lines, or on any other parallel Lines drawn from o and b, not equal to, but only bearing a like Proportion to each other as Io doth to AB; a Line drawn through the Extremities of the Diſtances thus taken, will ſtill cut bo in the ſame Point a [d].

SCHOL.

The Practices in theſe Corollaries are uſeful in ſeveral Inſtances; for if by ſetting off the Diſtances at b and $y3$, the Line $y3b$ cut bo ſo obliquely, that the Point a cannot be exactly determined, the Points y and c may be uſed; and it may be laid down as a general Rule, that the Diſtance of the Original Point from b the Interſecting Point of AB, ought, in Caſes of Perſpective and Tranſprojection, to be ſet off on that ſide of b towards which bo inclines, as at c or $c3$; but for Projection, it ought to be taken the contrary way, as at $c2$: for then the Lines, whoſe Interſections determine the Image of the Point propoſed, will not cut ſo obliquely, as when the Diſtances are taken the other way.

But if yc and $y3b$ do both cut bo too obliquely, ſo that neither of them can conveniently be uſed with bo, then they may be uſed together without bo [e].

And laſtly, if neither of theſe Methods will prevent the obliquity of the Interſection of the Lines which determine the Image, the Practices in the third and fourth Corollaries will effectually do it. Thus if the Diſtances be too large to be ſet off on the Vaniſhing and Interſecting Lines, a half, a third, or any other part of thoſe Diſtances may be taken, and ſet off on the Vaniſhing and Interſecting Lines, or on any other parallel Lines as may be moſt convenient; and in like manner, if the Diſtances be too ſmall, they may be doubled or tripled, to prevent the obliquity of the Interſection of the Lines which are to determine the Image.

METHOD 2.

Fig. 30. N°. 1, 2.

The ſame things being ſuppoſed as before, the Images of the Points A may be found in this manner.

Through A draw any Line AB at pleaſure, cutting the Interſecting Line in B, and having found b and x, the Interſecting and Vaniſhing Points of AB, draw bx its Indefinite Image [f], and Ix its Radial; then take xy in the Vaniſhing Line equal to Ix, and bc in the Interſecting Line equal to AB, obſerving the Rules for ſetting off thoſe

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[a] Cor. 1. Theor. 4. B. I,

[b] Lem. 1.

[c] Lem. 1.

[d] Cor. 1. Lem. 1.

[e] Cor. 2.

[f] Prob. 1.

Diſtances as directed in the preceeding Method of this Problem; laſtly draw yc, which will cut bx in a the Image required.

Dem. From B in the Original Plane ſet off BC equal to AB, and draw AC; and from I to y in the Picture draw Iy.

Then in the Triangles Ixy and ABC, x being by Suppoſition the Vaniſhing Point of AB, the Angle Ixy is equal to the Angle ABC[a], and the Sides xI and xy of the Triangle Ixy being taken equal, as alſo the Sides AB and BC of the Triangle ABC, theſe two Triangles are Similar[b]; and conſequently the Angles xIy and BAC are equal: wherefore Iy is the Radial, and y the Vaniſhing Point of AC[c]; and c being its Interſecting Point, bc and BC being by Conſtruction equal, yc is therefore the Indefinite Image of AC; and conſequently the Point a, where xb and yc interſect, is the Image of the Original Point A. *Q. E. I.*

Note, *All the Corollaries of the preceeding Method of this Problem are equally applicable to this, only uſing the Point x inſtead of the Point o, and the Radial* Ix *inſtead of* Io.

SCHOL.

Although the foregoing Methods are demonſtrated from the Conſideration of the Angles, made by the Lines which paſs through the Original Point, with the Line of Station and Interſecting Line, and with each other; without regard to the viſual Ray, or Line which paſſes from the Eye to the Original Point, which is the real Line which naturally cuts the Picture in the Image ſought: yet it may not be amiſs to ſhew how theſe Methods may be alſo deduced and proved from the Conſideration of the viſual Ray. To which end we muſt once more ſuppoſe the Picture to be placed on the Original Plane, in its proper Situation, as in Fig. N°. 3. where the ſame Letters repreſent the ſame things as uſual.

Fig. 30. N°. 3. Let A be the Original Point, and AB a Perpendicular from it to the Interſecting Line, cutting it in B, and Bo the Indefinite Image of that Line, and Io its Radial. It is evident, a Line IA, drawn from the Eye to the Original Point, cuts Bo in a the Image of A: now ſuppoſing the Lines Io and BA were ſo turned on the Points o and B, as to come into the Plane of the Picture (continuing ſtill parallel) ſo as I might fall at y, and A at C, then a Line Cy drawn through the Points C and y, will cut Bo in the ſame Point a, where it is cut by the Line IA[d]: and this is the firſt Method propoſed.

'Tis evident the ſame thing would happen, if o were ſuppoſed to be the Vaniſhing Point of a Line AB, not perpendicular to the Interſecting Line; which anſwers to the ſecond Method. And if either of theſe Methods be applied for finding the Projection or Tranſprojection of an Original Point, the ſame Demonſtration will hold good.

DEF. 2.

Fig. 30. N°. 1. If on either Side of a Vaniſhing Point x, a Diſtance xy be taken on the Vaniſhing Line, equal to Ix the Radial of that Vaniſhing Point, the Point y is called *the Point of Diſtance of the Vaniſhing Point x.*

METHOD 3.

The Picture and Original Plane being again ſeparated and prepared, the Images of the Points A may be found in this manner.

Fig. 31. N°. 1, 2. Through K, the foot of the Eye's Director in the Original Plane, and the Point A, draw AK, cutting the Interſecting Line in B, and from K ſet off KD on either ſide upon the Directing Line LM, equal to IK the Height of the Eye, and through D and the Original Point A, draw DA till it cut the Interſecting Line in C; then take Pb in the Interſecting Line of the Picture, equal to PB in the Original Plane, as before directed, and draw ba in the Picture perpendicular to the Interſecting Line; laſtly make ba equal to BC, and a will be the Image deſired; regard being had to the placing of the Point a, with reſpect to b, according to the Nature of the Image ſought.

Dem. The Line AB having K for its Directing Point, its Image is therefore perpendicular to the Interſecting Line[e], and b being the Interſecting Point of that Line in the Picture, ba drawn perpendicular on the Interſecting Line, is the Indefinite Image of AB.

Fig. 31. N°. 3. Now let IKBx be the Radial Plane of the Original Line AB, and Bx its Indefinite Image, and let the Points A be taken at the ſame Diſtances from K in this Figure, as they lie in the Original Plane; it is evident, that Lines drawn from I through the Points A, will cut the Indefinite Image Bx in C, the Images of the Original Points.

It

[a] Theor. 11. B. I.

[b] 6 El. 6.

[c] Prob. 1.

[d] Lem. 1.

[e] Cor. 3. Theor. 12. B. I.

Plate 10. Book 2. Sect. 2.

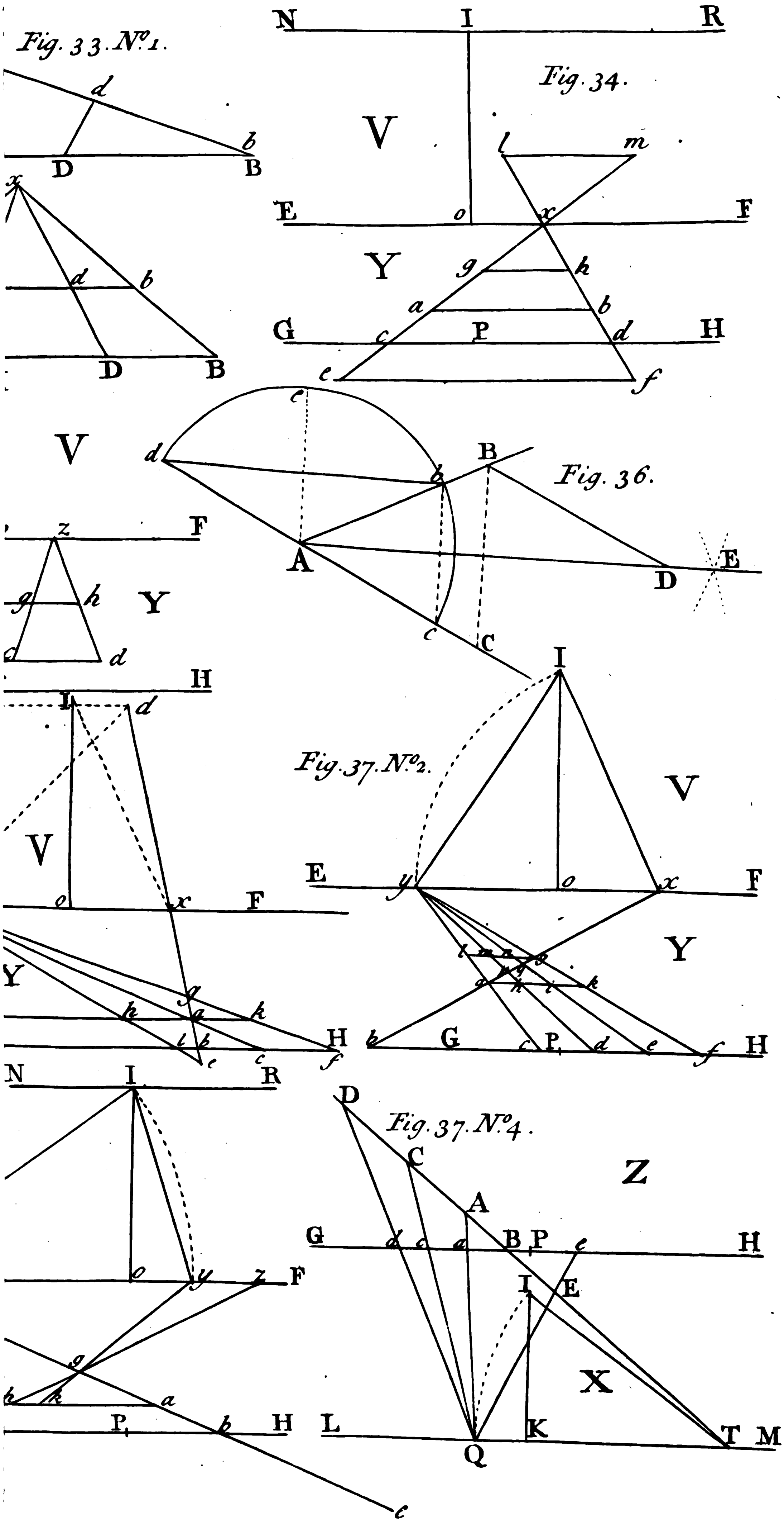

Fig. 33. No. 1.

Fig. 34.

Fig. 36.

Fig. 37. No. 2.

Fig. 37. No. 4.

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It muſt therefore be ſhewn, that the Diſtances BC in this Figure are the ſame with the correſponding Diſtances marked on the Interſecting Line of the Original Plane. Fig. 31. N°. 2.

In the Similar Triangles DKA, CBA DK : CB :: KA : BA Fig. 31.
And in the Similar Triangles IKA, CBA IK : CB :: KA : BA. N°. 3.

But in Figure N°. 3. IK, KA, and BA are by Conſtruction equal reſpectively to DK, KA, and BA in Figure N°. 2. wherefore CB in the one, is equal to CB in the other; and conſequently *ba* in the Picture, being made equal to BC in the Original Plane, it is alſo equal to BC in the Radial Plane, and *a* is therefore the Image of A. *Q. E. I.*

COR.

If inſtead of drawing from A, a Line AB to the foot of the Eye's Director, it were to be drawn to any other Point T in the Directing Line, the Images of A may ſtill be found by this Method; only that inſtead of ſetting off the Diſtance TQ equal to IK, it muſt be made equal to IT the Director of the Original Line, and the Line *ab*, on which the Diſtances BC are to be marked, muſt not then be perpendicular to the Interſecting Line of the Picture, but muſt make an Angle with it equal to the Angle ITK.

The Demonſtration of this is the ſame with that of the foregoing Method, if IKB*x* be taken as the Radial Plane of the Original Line, and IK as its Director, equal to IT in the Original Plane. Fig. 31. N°. 3.

METHOD 4.

The Images of the Points A may be alſo found in this manner.

Through A draw AB perpendicular on the Interſecting Line, cutting it in B, and through the ſame Point A draw another Line to K, the Foot of the Eye's Director, cutting the Interſecting Line in C. Then on the Interſecting Line of the Picture, take P*b* and P*c* equal to PB and PC in the Original Plane, repreſenting the Interſecting Points of AB and AC as before, and from *b* to the Center *o* draw *ob*, and from *c* draw *ca* perpendicular on the Interſecting Line, and the Point *a*, where theſe two Lines interſect, will be the Image ſought. Fig. 32. N°. 1, 2.

Dem. The Line *bo* is the Indefinite Image of AB[a], and *ca* is the Indefinite Image of AC[b], wherefore *a*, the Interſection of *bo* with *ca*, is the Image of A the Point propoſed. *Q. E. I.*

If the Point A *be in the Line of Station or very near it, this Method cannot be uſed, becauſe the Images of the Lines* AB *and* AC *would either coincide, or croſs ſo obliquely, that the Point a could not be thereby determined.*

As to the Reverſe of this Problem, the Methods here propoſed are eaſily applicable to it, after what has been already ſaid at the four firſt Problems.

LEM. 2.

To divide a given determinate Line *ab* in the ſame Proportion as any other given Line AB is divided. Fig. 33. N°. 1.

Place either Extremity *b* of the Line *ab* on either Extremity B of the Line AB, ſo as thoſe Lines may make together any Angle at pleaſure, and join their other Extremities A and *a* by a Line A*a*, to which draw Parallels through the Diviſions C and D of the Line AB, and theſe will cut *ab* in *c* and *d*, the proportional Diviſions required[c].

This may be likewiſe done by placing the propoſed Line *ab* parallel to AB, and drawing A*a* and B*b* till they meet in ſome Point *x*, from whence Lines drawn to C and D will cut *ab* in *c* and *d*, the Diviſions ſought[d]. Fig. 33. N°. 2.

PROB. VI.

To find the Image of a Line in the Original Plane parallel to the Interſecting Line, and from any given Point in that Image to ſet off a part which ſhall repreſent a given part of the Original Line.

Having found *a* the Image of either Extremity of the Original Line[e], through *a* draw *ab* parallel to the Vaniſhing Line; then through *a* draw any Line *xc*, cutting the Vaniſhing and Interſecting Lines in *x* and *c*, from *c* ſet off *cd* on the Interſecting Line, equal to the propoſed part of the Original Line, and towards the ſame ſide of *a* on which the Original Part is ſuppoſed to lie, laſtly from *d* to *x* draw *dx*, cutting *ab* in *b*, and *ab* will be the Image deſired. Fig. 34.

Dem. The Original Line being by Suppoſition parallel to the Interſecting Line, and

[a] Method 1.

[b] Method 3.

[c] 10 El. 6.

[d] Lem. 2. B. I.

[e] Prob. 5.

therefore alſo to the Picture, its Image is parallel to it[a], and conſequently to the Vaniſhing Line E F[b]; and a being the Image of a Point in the Original Line, ab parallel to E F is the Indefinite Image of that Line. And becauſe of the Vaniſhing Point x, the Originals of cx and dx are parallel[c], as are alſo the Originals of cd and ab, therefore the Originals of cd and ab are equal[d]; and cd in the Interſecting Line being both its own Original and Image, and being taken equal to the propoſed part of the Original Line, the Original of ab is alſo equal to that part; and a being the Image of one Extremity of the Original Line, the Part ab of its Indefinite Image is therefore the determinate Image of the Original Part propoſed. *Q. E. I.*

COR. 1.

All Lines drawn in the Picture, on either ſide of the Vaniſhing Line EF and parallel to it, and bounded both ways by the Lines xc and xd produced at pleaſure, as ef, gb, lm, *&c.* repreſent Original Lines parallel and equal to cd. And the Length of any one of thoſe Images will be to that of any other of them, as are their ſeveral Diſtances from the Vaniſhing Line.

Becauſe the Originals of ef, cd, ab, gb, lm, *&c.* are all parallel, and bounded by the Originals of xc and xd which are alſo parallel, they are therefore equal[e]. And it is evident from the Similitude of the Triangles gbx, abx, cdx, *&c.* that gb, ab, cd, *&c.* are in the ſame Proportion to each other as bx, bx, dx, *&c.* which laſt are in the ſame Proportion to each other, as are the ſeveral Diſtances of b, b, and d from the Vaniſhing Line E F[f].

COR. 2.

Hence if it be not convenient to mark the Length of the Original Line on the Interſecting Line, any other Line parallel to the Interſecting Line may be uſed; but then the meaſure ſet off on that parallel, muſt not be equal to the propoſed Original, but muſt be made to bear the ſame Proportion to it, as the Diſtance of the aſſumed parallel from the Vaniſhing Line, doth to the Diſtance of the Interſecting Line from the Vaniſhing Line, or (which is the ſame) as a determinate Image in that parallel, bears to its Original[g].

DEF. 3.

If on any Line parallel to the Interſecting Line, a Meaſure be taken in the ſame Proportion to a given Original Line, as any determinate Part of that Parallel bears to its Original, the Meaſure thus taken is called *the Proportional Meaſure of the Original Line on that Parallel.*

COR. 3.

If the Original Line be anywiſe divided into ſeveral Parts, the correſponding Diviſions of its Image are found, by dividing it in the ſame Proportion as the Original[h]. *This may be done by the foregoing* Lemma.

COR. 4.

If the Image ab be given, the true Length of its Original may be found, by taking any convenient Point x in the Vaniſhing Line, and thence drawing xa and xb till they cut the Interſecting Line in c and d, whereby cd, the true Meaſure of the Original of ab, will be determined. And if the Image ab be anywiſe divided, Lines from x through thoſe Diviſions will mark their reſpective Lengths on cd in the Interſecting Line.

GENERAL COROLLARY.

Hence the Interſecting Line in the Picture, or at leaſt its Diſtance from the Vaniſhing Line, is abſolutely neceſſary to be known; it being that on which the true Meaſures of the Original Lines are to be ſet off, according to the Scale uſed in the Geometrical Draught or Plan, and by which the proportional Meaſures to be uſed on any other Line parallel to the Vaniſhing Line are to be guided, ſo that the Images may have their proper Diminution given them, in proportion to the ſeveral Diſtances of their Originals from the Picture: but when the proportional Meaſures to be uſed on any Parallel are once found, this Parallel will then ſerve all the Purpoſes of the Interſecting Line, and may frequently be the more convenient of the two to be uſed, eſpecially when the Original Line is at a great Diſtance from the Picture, or of ſo great a Length, that its true Meaſure cannot commodiouſly be ſet off on the Interſecting Line.

PROB.

[a] Theor. 2. B. I.

[b] Cor. 1. Theor. 15. B. I.

[c] Cor. 1. Theor. 5. B. I.

[d] 34 El. 1.

[e] 34 El. 1.

[f] Cor. 3. Theor. 31. B. I.

[g] Theor. 31. B I.

[h] Cor. 1. Theor. 23. B. I.

PROB. VII.

The Image ab of a determinate Part of an Original Line parallel to the Picture being given, from any other given Point e in the Picture, to draw a Line, whoſe Original ſhall be parallel and equal to the Original of ab. Fig. 35.

Through the given Point e and the correſponding Extremity a of the given Line ab, draw ae, cutting the Vaniſhing Line in x, and through b the other Extremity of ab, and the ſame Vaniſhing Point x draw bx; then through e draw ef parallel to ab, cutting bx in f, and ef will be the Image deſired, the Original of which will be parallel and equal to the Original of ab.

CASE. 2.

If the given Point were g, from whence a Line paſſing through a, would cut the Vaniſhing Line at an inconvenient Diſtance, any other Point x in the Vaniſhing Line may be taken, and drawing xa and xb as before, through g draw eg parallel to ab, and take in it gb equal to ef, and gb will be the Image ſought.

Or it may be done thus: Produce ab at pleaſure, and in it take cd equal to ab, ſo that a Line cg may cut the Vaniſhing Line conveniently in z, then draw dz, and through g draw gb parallel to cd cutting dz in b, and gb will be the Line required as before.

Theſe Practices evidently follow from *Cor.* 1. *Prob.* VI. *Q. E. I.*

LEM. 3.

To biſect a given Angle.

Let BAC be the given Angle which is propoſed to be biſected. Fig. 36.

Take any Point B in either ſide AB of the given Angle, and draw BD parallel to the other ſide AC, make BD equal to AB, and draw AD, which will biſect the Angle propoſed.

For in the Triangle ABD the ſides AB and BD being equal, the Angles BAD and BDA are equal[a], and becauſe BD and AC are parallel, the Angles BDA and DAC are equal[b], which laſt is therefore equal to the Angle BAD; wherefore the Line AD biſects the given Angle BAC.
[a] 5 El. 1.
[b] 29 El. 1.

This may alſo be done by taking AC equal to AB, and from C and B as Centers, with any opening of the Compaſſes greater than the half of CB, deſcribing two Arches, which will interſect ſomewhere at E in the Line AD[c]. [c] 9. El. 1.

Or it may be done thus: From A as a Center with any Radius Ac, deſcribe a Semicircle cbd cutting AB in b and AC produced in d, then draw db, and parallel to it draw AD which will biſect the Angle BAC.

For the Angle bdA being the half of BAC[d], and the Angles bdA, DAC being equal[e], the angle DAC is half the Angle BAC.
[d] 20 El. 3.
[e] 29 El. 1.

In the ſame manner if bc be drawn, Ae parallel to it, biſects the Angle bAd.

PROB. VIII.

Having the Indefinite Image of an Original Line not parallel to the Picture given, thence to find the Image of any determinate part of that Line.

CASE 1.

When the Original Plane is not drawn out.

Let bx be the Indefinite Image given, x and b its Vaniſhing and Interſecting Points, and Ix its Radial. Fig. 37. N°. 1.

METHOD 1.

From x ſet off xy on the Vaniſhing Line EF equal to the Radial Ix, and from b in the Interſecting Line, ſet off bc equal to the Diſtance between either extremity of the propoſed part of the Original Line and its Interſecting Point, and draw yc cutting bx in a, obſerving to ſet off the Diſtances xy and bc according to the Rules already given[f]; and a will be the Image of one Extremity of the Original Line propoſed[g]. Then from c ſet off cf or cl on the Interſecting Line, equal to the propoſed Part of the Original Line, according as it lies either farther from or nearer to the Eye, than
[f] Schol. Meth. 1 prob. 5.
[g] Meth. 2. prob. 5.

the Original of a, and draw yf or yl cutting bx in g or e, and ag or ae will be the determinate Image required.

Dem. Becauſe of the Vaniſhing Point y, the Originals of yc, yf, and ye are parallel, wherefore the Originals of the Triangles abc, gbf, lbe are Similar, and conſequently the Originals of the Parts be, ba, ag of the Line eg, are proportional to the Parts bl, bc, cf of the Line lf[a], and the Original of ba being equal to bc, the Originals of ag and ae are equal to cf and cl, which were taken equal to the propoſed Part of the Original Line. *Q. E. I.*

C O R. 1.

If the Point a be given, the Points g or e may be found, by drawing through a the Line bk parallel to GH, and making ak or ab to repreſent a Line equal to cf[b], for then yk and yb will cut bx in the ſame Points g and e.

C O R. 2.

Fig. 37. N°. 2. If the Original Line be anywiſe divided into ſeveral Parts, the correſponding Diviſions of its Image are found by dividing cf or ak in the ſame Proportion, and drawing Lines from y to thoſe Diviſions, which will cut ag in their Images.

For becauſe of the Vaniſhing Point y, the Originals of yd, ye, yf being parallel, the Originals of the Sides ag and ak, of the Triangle gak, will be divided in the ſame Proportion[c].

C O R. 3.

If the derminate Image ag of the Original Line be given, then to find the Images of its Diviſions, it is not neceſſary that yx ſhould be taken equal to Ix, but any other Point in the Vaniſhing Line may be uſed, which, to ſave drawing another Figure, we ſhall ſuppoſe to be y; having therefore drawn from a or g, a Line ak or gl parallel to GH, from y draw yg or ya, cutting ak or gl in k or l, then divide ak or gl in the ſame Proportion as the Original Line is ſuppoſed to be divided[d], and Lines drawn from y through thoſe Diviſions, will divide the Image ag in the manner required.

For although ak or gl will not repreſent a Line equal to the Original Line propoſed, when xy is not equal to xI; yet y being the Vaniſhing Point of ya, yb, yi, and yk, their Originals are parallel, and conſequently ak, ag, and gl will repreſent Lines divided in the ſame Proportion; and ag being the determinate Image of the Original Line, its Diviſions are therefore the Images of the correſponding Diviſions of that Line.

Or if on the Line ak indefinitely produced, there be ſet off from a any Diviſions ab, bi, ik in the ſame Proportion to each other, and in the ſame order as are the Diviſions of the Original Line, and ending any where as at k; from k through g draw kg, cutting the Vaniſhing Line in any Point y, and Lines from y through the ſeveral Diviſions of ak, will mark on ag the ſame Diviſions as before.

S C H O L.

When the given Image ag lies between the Vaniſhing Line and the aſſumed Parallel ak, the Images of its Diviſions are found as in Caſes of Perſpective; but when the aſſumed Parallel lg is taken between the Vaniſhing Line and the given Image ag, then its Diviſions are found as in Caſes of Projection[e]: the former way is generally the better when it can be done, eſpecially when the Diviſions are ſmall, the Diviſions on ak being larger than thoſe on lg, and ſo ſerving better to determine the Points p and q; beſides an Error in the Place of b or i in the Line ak, hath not ſo great an effect on the Places of p and q, as a like Error in the Places of m and n in the Line lg would have, the Errors of the firſt being leſſened, but thoſe of the latter increaſed.

C O R. 4.

Fig. 37. N°. 3. If the determinate Image ag be given, and a Point z be taken in the Vaniſhing Line, different from the true Point of Diſtance y, and zg be drawn cutting ak in b, then the Original of ab will be to the Original of ag or ak, as xz is to xy.

For in the Similar Triangles zgx, agb $ab : zx :: ag : gx$
And in the Similar Triangles ygx, agk $ak : yx :: ag : gx$
Conſequently $ab : ak :: zx : yx$
And ab and ak are in the ſame Proportion to each other as their Originals[f].

C O R.

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[a] 2 El. 6.

[b] Cor. 1. Prob. 6.

[c] 2 El. 6.

[d] Lem. 2.

[e] Schol. Meth. 1. Prob. 5.

[f] Cor. 1. Theor. 23. B. I.

COR. 5.

If xz be taken in the ſame Proportion to xy, as the Original of any determinate Part of ak is to its Image, then the true Meaſures of the Original Line and its Parts being ſet off on ak, Lines from z to thoſe Diviſions will cut ag in their true Images.

For by the laſt Corollary $ab : ak :: xz : xy$.

If then xz be to xy as the Original of ak is to ak, it follows, that ab muſt be equal to the Original of ak.

The Length of xz may be found by taking al equal to xy, and drawing xl till it cut GH in p, and making xz equal to bp, this laſt being equal to the Original of al.

SCHOL.

If the Original of ag were divided into ſeveral unequal Parts, it may be leſs trouble to alter the Diſtance xy in the Proportion mentioned in this Corollary, than to be obliged to find the proportional Meaſures of the ſeveral Diviſions of the Original Line to be ſet off on ak; for the Diſtance xz being taken, the true Meaſures of the Parts of the Original Line may be ſet off on ak, which will ſerve the purpoſe deſired.

METHOD 2.

If the Vaniſhing Point x of the Original Line were ſo far diſtant, as that it could not be conveniently marked on the Vaniſhing Line, ſo that the Diſtance Ix could not be ſet off from it at y, yet the Point of Diſtance y may be found, the Angle of Inclination of the Original Line to the Interſecting Line being known.

Fig. 37. N°. 3.

Let bx be the given Indefinite Image, and ſuppoſe x to be out of the Bounds of the Picture, and let A be the Angle of Inclination of the Original Line to the Interſecting Line.

From I draw an Indefinite Radial Ix towards that Side where the Vaniſhing Point x is ſuppoſed to lie, making the Angle NIx equal to the Angle A; then biſect the Angle xIR, the Complement to two Rights of the Angle NIx, by the Line Iy[a], and y will be the Point of Diſtance required.

Dem. Becauſe NR and EF are parallel, the alternate Angles RIy, IyE are equal; but by Conſtruction, the Angles RIy and xIy are equal, therefore the Angle xIy is equal to the Angle IyE, and conſequently the Triangle xIy is Iſoſceles, and hath its Sides xI and xy equal[b]; but the Angle NIx being made equal to the Angle A, Ix is the Indefinite Radial of the Line bx[c], wherefore y is the Point of Diſtance required. *Q. E. I.*

[a] Lem. 3.

[b] 5 El. 1.

[c] Theor. 11. B. I.

SCHOL.

When either the Vaniſhing or Interſecting Point of the Original Line is out of reach, ſo that its intire Indefinite Image cannot be had, ſo much of that Image as is requiſite may be obtained, by finding the Images of any two Points of the Original Line (the more diſtant from each other the better) by any of the Methods of Prob. V. and a Line drawn through the Images of thoſe two Points will be the Indefinite Image ſought; and the Images of any determinate Parts of the Original Line may then be found in its Indefinite Image by the preceeding Method: and as finding the Diviſions of a Line, is no more than finding the Images of the Points by which it is divided, ſuch of the Rules of Prob. V. may be applied to this purpoſe, as may be moſt convenient, if thoſe here mentioned ſhould not be ſufficient.

CASE 2.

When the Original Plane is drawn out.

Let AB be the Original Line in the Original Plane, divided anywiſe in C, D, and E, the Images of which Parts are required to be ſet off on the Indefinite Image given.

Fig. 37. N°. 4.

Produce the Original Line AB to its Directing Point T, and draw the Director IT; then take TQ on the Directing Line equal to TI, and from Q draw QE, QA, QC, QD, cutting the Interſecting Line in e, a, c, d; then on the Indefinite Image given, ſet off from its Interſecting Point, ſeveral Diſtances equal to Ba, Bc, Bd, Be, either above or below the Interſecting Point, according to the Situation of the Original Points with reſpect to B, and thoſe Diſtances will determine the Images of the correſponding Parts of the Original Line.

This follows from Cor. Meth. 3. Prob. V. *Q. E. I.*

SCHOL.

SCHOL.

The ſeveral Rules laid down in the firſt Caſe of this Problem, where only the Vaniſhing Plane and Picture were uſed, without regard to the Original Plane, may with great eaſe be applied to the Original Plane and Directing Plane; the Directing Plane having the ſame relation to the Original Plane, as the Vaniſhing Plane hath to the Picture, as has already been often obſerved.

PROB. IX.

Fig. 37. N°. 2. **The determinate Image ag of a Line divided into any Number of Parts being given, thence to find the true Meaſures of the Originals of thoſe Parts.**

Produce ag to its Vaniſhing Point x, and from y its Point of Diſtance, draw Lines through the Extremities and the ſeveral Diviſions of the given Image ag, till they cut the Interſecting Line GH in c, d, e, and f; and cd, de, ef, will be the true Meaſures of the Parts ap, pq, qg of the Image ag[a]. *Q. E. I.*

COR. 1.

If through either Extremity a or g of the given Image, a Line ak or gl be drawn parallel to the Vaniſhing Line, Lines drawn from y through the Diviſions of ag, will mark on ak or gl, the proportional Meaſures of the Parts of ag[b], ſo that the Originals of the Parts of ak or gl will be equal to thoſe of ag.

COR. 2.

If inſtead of the Point of Diſtance y, any other Point be taken in the Vaniſhing Line EF, Lines drawn from thence through the Diviſions of ag, only mark the Proportions of the Parts of ag on the Lines ak or gl[c]; but the Originals of theſe laſt will not be equal to the Originals of the Parts of ag, but only in the ſame Proportion to thoſe Parts, as the Diſtance between x and the aſſumed Vaniſhing Point, is to the true Diſtance xy[d].

COR. 3.

If in the Indefinite Image bx, any determinate Part ag be taken, and the proportional Meaſure ak of that part be found on a Line parallel to xy, drawn through its nearer Extremity a; then if the Complement gx of the aſſumed Part be equal to, or bigger, or leſs than the Radial or Diſtance yx, the aſſumed Part ag will alſo be equal to, or bigger, or leſs than its proportional Meaſure ak.

For the Triangles yxg, gak being Similar $gx : xy :: ag : ak$. Wherefore if gx be equal to, or bigger, or leſs than xy, ag will alſo be equal to, or bigger, or leſs than ak.

PROB. X.

Fig. 37. N°. 1. **The Indefinite Image bx of a Line, and the Image of any Point a in that Line being given, thence to find the true Meaſure of the Complement of the Original Line[e], or of ſo much of that Line as lies between the Original of a and its Directing Point.**

From y the Point of Diſtance of the Vaniſhing Point x, draw ym parallel to bx, cutting GH in m, then draw ya cutting GH in c, and mc will be the true Meaſure required.

Dem. Becauſe mb is equal to yx, which is equal to the Radial Ix of the Original Line, it is therefore equal to ſo much of that Line as lies between its Interſecting and Directing Points[f]; and becauſe bc is equal to the Original of ba[g], the whole Line mc is therefore equal to ſo much of the Original Line as lies between the Original of a and its Directing Point. *Q. E. I.*

COR.

If through a there be drawn na parallel to EF, cutting ym in n, na will be the proportional Meaſure of the Complement of the Original of xa.

For the Originals of na and mc are equal[h].

PROB.

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[a] Cor. 2. Prob. 8.

[b] Cor. 2. Prob. 8.

[c] Cor. 3. Prob. 8.

[d] Cor. 4. Prob. 8.

[e] Def. 27. B. I.

[f] Cor. 1. Def. 18. B. I.

[g] Prob. 9.

[h] Cor. 1. Prob. 8.

PROB. XI.

If a determinate Line in the Original Plane paſſing through the Directing Line be anywiſe divided by it into two Parts; having the Images a and d of the Extremities of the Original Line given, thence to find the true Meaſures of the Parts of that Line.

Fig. 37. N°. 1.

Draw ad the Complement of the Image of the Original Line[a], cutting EF in x its Vaniſhing Point, and through y the Point of Diſtance of x, draw ym parallel to ad, cutting GH in m, and draw yd and ya, cutting GH in p and c; then mc will be the true Meaſure of ſo much of the Original Line as lies between the Original of a and its Directing Point, or the Complement of the Original of xa, and mp will be the true Meaſure of the Complement of the Original of xd.

Dem. Through d draw dq parallel to EF, cutting ym in q, then dq is the proportional Meaſure of the Complement of the Original of xd[b]; but becauſe of the Vaniſhing Point y, the Original of qd is equal to pm[c], therefore pm is the true Meaſure of the Complement of the Original of xd, and mc is the true Meaſure of the Complement of the Original of xa[d]. *Q. E. I.*

COR. 1.

If through a, a Line ra be drawn parallel to EF, rn and na will be the proportional Meaſures of the Complements of the Originals of xd and xa[e].

COR. 2.

The true as well as the proportional Meaſures of the Parts of the Original Line, and conſequently the Original Parts themſelves, are reciprocally proportional to the Complements of the Images of thoſe Parts, that is, $rn : na :: xa : xd :: pm : mc$.

For in the Similar Triangles rny, yxd $\quad rn : yx = na :: ny = xa : xd$
And becauſe of the Parallels ra, pc $\quad rn : na :: pm : mc$[f]
Therefore $\quad pm : mc :: xa : xd$.

PROB. XII.

The Images a and d of the Extremities of a Line in the Original Plane, which paſſes through the Directing Line, being given, thence to find the Image of a Point which divides the Original Line in any given Proportion.

Fig. 37. N°. 1.

Having found ra the proportional Meaſure of the Complement of the Original of ad[g], divide ra in b in the given Proportion, and draw yb, cutting ad in e, and e will be the Image of the Point deſired.

Dem. For ra being the proportional Meaſure of the whole Original Line, and ba the proportional Meaſure of the Original of ae[h], br is the proportional Meaſure of the Remainder of the Original Line; wherefore the Original of e divides the whole Original Line in the Proportion of ab to br, which was taken in the Proportion required. *Q. E. I.*

COR.

If the Point b ſhould fall in n, then the Original of the Point required is the Directing Point of the Original Line.

For yn and xa being parallel, their Interſection, which ſhould determine the Image of the Point ſought, is at an infinite Diſtance[i].

PROB. XIII.

Having the Indefinite Image bx of a Line, and a determinate Part ag of that Image given; from any Point c in that Image, to ſet off a Part, the Original of which may be equal, or in any other given Proportion to the Original of the given Part.

Fig. 38. N°. 1.

Through a draw ab parallel to the Interſecting Line, and having taken any Vaniſhing Point y, draw yg cutting ab in k, then through c draw ce parallel to ab, and from x draw xk cutting ce in i; laſtly from y draw yi cutting bx in d, and cd will be the Part required, when the Original of that Part is ſuppoſed equal to that of ag:

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[a] Def. 24. B. I.

[b] Cor. Prob. 10.

[c] Cor. 1. Prob. 6.

[d] Prob. 10.

[e] Cor. 1. Prob. 6.

[f] Lem. 2.

[g] Cor. 1. Prob. 11.

[h] Cor. 1. Prob. 9.

[i] Cor. 1. Theor. 7. B. I.

But if the Part whoſe Image is ſought, be bigger or leſs than the Original of ag, then take on the Line ab, a Meaſure ab in the ſame Proportion to ak, as the Original of the Part required bears to the Original of ag, and draw xb cutting ce in e; then a Line ye will cut bx in f, ſo that the Original of cf ſhall be to the Original of ag in the Proportion required.

Dem. Becauſe of the Vaniſhing Points x and y, the Originals of the Triangles agk, cdi, are Similar[a], therefore the Originals of ag and cd are in the ſame Proportion, as the Originals of ak and ci; but theſe repreſent equal Lines[b], therefore ag and cd alſo repreſent equal Lines.

Again, becauſe of the Vaniſhing Point y, the Originals of cd and cf are in the ſame Proportion as ci to ce[c], which have the ſame Proportion as ak to ab, and therefore the Original of cf is to the Original of cd or ag, as ab to ak, which were taken in the Proportion required. *Q. E. I.*

C O R.

Hence if an Original Line were divided into any Number of equal Parts, and each of thoſe Parts were ſubdivided alike in any given Proportion; having the Indefinite Image of the Original Line, and in it the Image of any one of the equal Parts given, the reſt with their Subdiviſions may be found, without the trouble of marking the true or proportional Meaſures of the whole Original Line and its Subdiviſions on the Picture.

Fig. 38. N°. 2. Thus if bx were the Indefinite Image, and ac the Image of one of the equal Parts of the Original Line, and each of the equal Parts of that Line were ſuppoſed to be ſubdivided in the ſame given Proportion.

Having drawn ab parallel to the Interſecting Line, from any Point y in the Vaniſhing Line, draw yc cutting ab in b, and divide ab in r and s in the given Proportion, and draw yr, ys, whereby the correſponding Subdiviſions of ac will be found[d]. Then draw xb, and from c draw cf parallel to ab, cutting xb in f, and from y draw yf cutting bx in d, and cd will repreſent another part of the Original Line equal to the Original of ac. Then from d thus found, draw dg parallel to cf, cutting xb in g, and a Line yg will determine de the Image of another equal Part of the Original Line; and after the ſame manner the Images of as many more equal Parts of the Original Line may be found as are deſired.

Laſtly, draw xr, xs, which will divide cf, dg, &c. in the ſame Proportion with ab, and conſequently Lines drawn from y to the ſeveral Diviſions of cf, dg, &c. will cut the Images cd, de, ſo as to repreſent Lines divided in the ſame Proportion as the Original of ac; and thus the Images of as many equal Parts of an Original Line may be found as are deſired, together with the Subdiviſions of thoſe Parts, only by ſetting off the Meaſure of one of thoſe equal Parts with its Subdiviſions, on a Line parallel to E F.

Note, *The ſeveral Rules and Obſervations at Prob.* VIII. *are equally applicable here.*

P R O B. XIV.

Fig. 38. N°. 1. Having the Indefinite Images xb and xb of two parallel Lines, and a determinate Part ag in one of them given, from any Point e in the other, to ſet off a part which ſhall repreſent a Line equal, or in any other Proportion to the Original of the given Part ag.

Through the given Point e, draw ec parallel to the Vaniſhing Line E F, cutting xb in c, from c ſet off cd or cf, repreſenting a Line either equal or in the given Proportion to the Original of ag[e], and draw dm or fn parallel to ec, cutting xb in m or n; then em or en will be the Part ſought, according as the Original of that Part is required to be either equal, or in any other Proportion to the Original of ag.

Dem. Becauſe the Originals of ce, dm, and fn are parallel, as are alſo the Originals of cx and ex, therefore the Originals of cd and em, or of cf and en are equal[f]; and cd or cf repreſenting a Line in the given Proportion to the Original of ag, em or en repreſents a Line in the ſame Proportion. *Q. E. I.*

P R O B. XV.

Having the Images of two Lines not parallel to each other, and of a Part in one of them, adjoining to their common Interſection, given;

From

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[a] Prob. 4.

[b] Cor. 1. Prob. 6.

[c] Cor. 3. Prob. 8.

[d] Cor. 3. Prob. 8.

[e] Prob. 13.

[f] 34 El. 1.

Fig. 38. N.° 1.

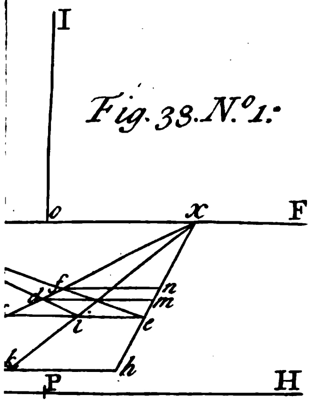
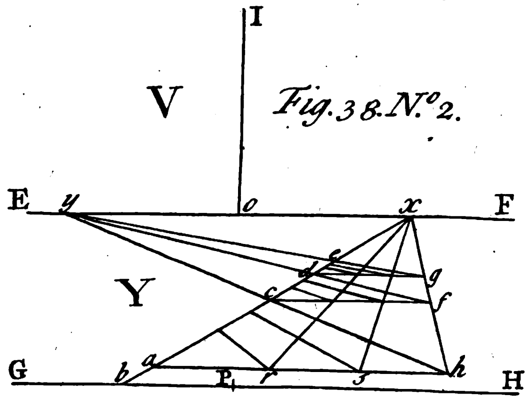


Fig. 38. N.° 2.



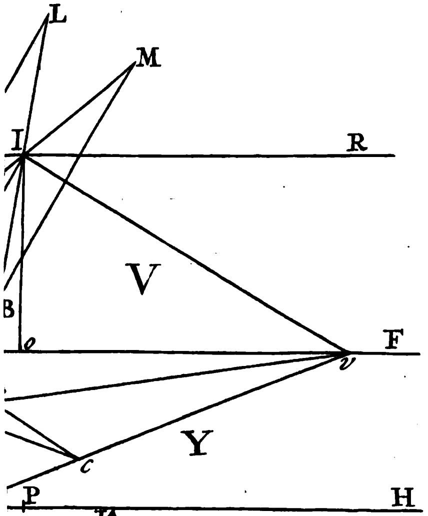
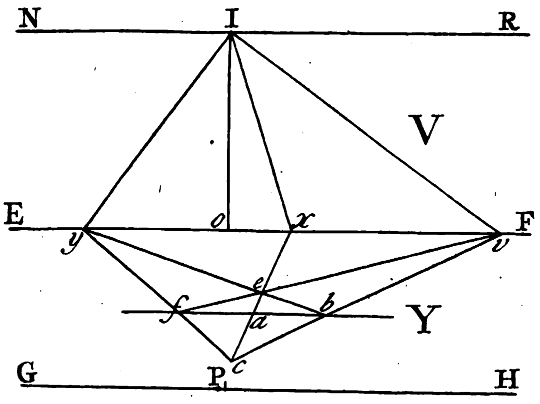


Fig. 39. N.° 2.



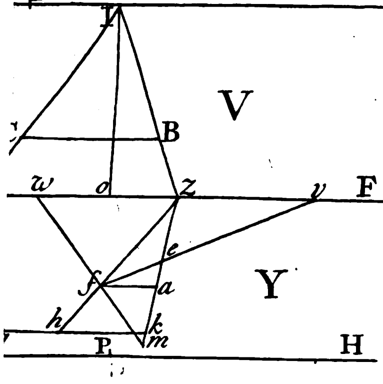


Fig. 41 N.° 1.

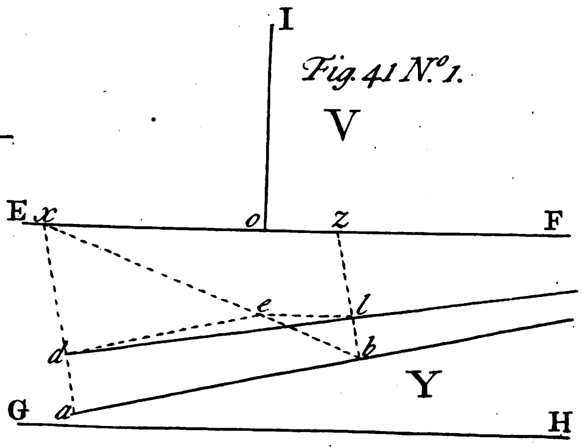


Fig. 41. N.° 2

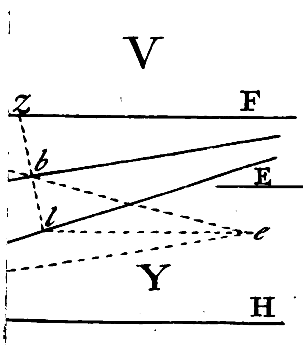
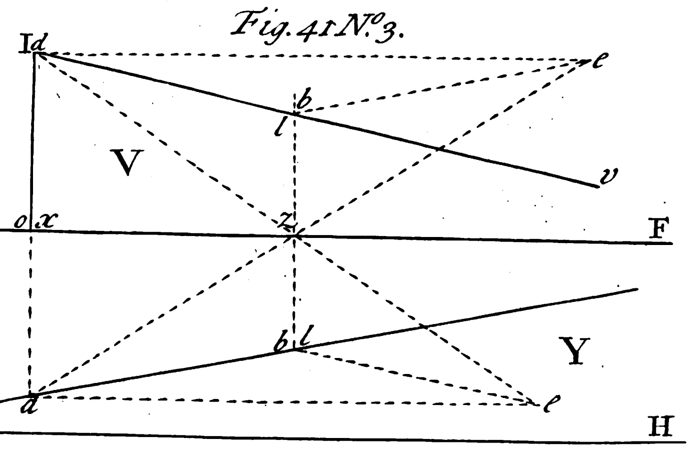


Fig. 41 N.° 3.



J. Mynde sc.

From the ſame common Interſection, to ſet off a Part of the other Line, which ſhall repreſent a Line equal, or in any other Proportion to the Original of the given Part.

CASE 1.

When the Part given, and the Part required, make together an Inward Angle.

Let zb and xc be the two given Images Interſecting in a, from whence it is required to ſet off a Part ac on the Line xc, which ſhall repreſent a Line equal, or in any other Proportion to the Original of ab, a given Part of zb. Fig. 39. N°. 1.

Having drawn Iz and Ix the Radials of the given Lines, from I on Iz the Radial of that Line in which the Part ab is given, ſet off any Diſtance IB, and on Ix the Radial of the other Line, take IC in the ſame Proportion to IB, as the Original of the Part required bears to the Original of the given Part, and draw CB, and from I draw Iv parallel to CB, cutting the Vaniſhing Line in v; laſtly draw vb cutting xc in c, and ac will be the Part required: or if af were the given Part, and ae the Part required, a Line vf will cut cx in e, which will determine the Part ae.

Dem. Becauſe of the Vaniſhing Points x, z, and v, the Originals of the Triangles abc, fde, are Similar to the Triangle IBC[a], and therefore the Originals of the Sides ab and ac of the Triangle abc, or of the Sides af and ae of the Triangle eaf, are in the ſame Proportion to each other, as the correſponding Sides IB and IC of the Triangle IBC, which were taken in the Proportion required. *Q. E. I.*

CASE 2.

When the Part given, and the Part required, make together an Outward Angle.

Let ab be the given Part, and ae the Part required; produce Iz the Radial of the given Part, beyond I, and take any Diſtance IL on it, and on Ix, the Radial of the Part required, take a Diſtance IC in the ſame Proportion to IL, as the Original of the Part required bears to the Original of the given Part, and draw LC, and parallel to it draw Iy, cutting the Vaniſhing Line in y; then a Line yb will cut cx in e, and ae will be the Part ſought: or if af were the given Part, and ac the Part required, a Line yf will determine the Point c, and conſequently ac the Part deſired.

Dem. Becauſe the Angle bae is an outward Angle, its Original is equal to the Complement to two Rights of the Angle xIz[b], that is, the Angle CIL, and the Originals of the Angles bea, eba being reſpectively equal to the Angles xIy, yIz, which laſt are equal reſpectively to the Angles ICL and CLI, therefore the Original of the Triangle eab is Similar to the Triangle LIC, and conſequently the Originals of ab and ae are in the ſame Proportion to each other as IL to IC, which were taken in the Proportion required. It is evident alſo, that the Originals of the Triangles bae and afc are Similar, the Originals of be and fc being parallel, and therefore that the Originals of fa and ac are in the ſame Proportion as the Originals of ab and ae. *Q. E. I.*

COR. 1.

If the Part given and the Part required be propoſed to repreſent equal Lines, then the Point v or y may be found, by biſecting the Angle BIM or BIC by the Line Iv or Iy; and Iy and Iv will be perpendicular.

Thus if IB and IC be equal, the Angles ICB, IBC will be equal[c], to both which Angles the outward Angle BIM will alſo be equal[d]; and conſequently if the Angle BIM be biſected by the Line Iv, the Angle BIv will be equal to the Angle CBI, wherefore CB and Iv will be parallel[e], as they were directed to be taken. In the ſame manner, if IL and IC be equal, the Angles ILC, ICL will be equal, to both which the Angle CIB is equal; therefore CIy, the half of this Angle, is equal to the Angle ICL, and conſequently CL and Iy are parallel. Laſtly, on this Suppoſition, ICB being an Iſoſceles Triangle, the Line Iy which biſects the Angle CIB, is perpendicular to CB, and conſequently to its parallel Iv.

COR. 2.

When the Angle xIz, made by the Radials of the given Lines cx and bz, is biſected by the Line Iy, then Lines from y make the Parts ae and ab, or af and ac, which contain the outward Angles bae, caf, repreſent equal Lines; but when the Angle zIM, the Complement to two Rights of the Angle xIz, is biſected by the Line

[a] Prob. 4.

[b] Cor. Prob. 3.

[c] 5 El. 1.

[d] 32 El. 1.

[e] 29 El. 1.

Line Iv, Lines from v make the Parts af and ae, or ac and ab, which contain the inward Angles eaf, bac, repreſent equal Lines.

COR. 3.

If the Line ba or af were ſuppoſed to be divided in any Proportion, Lines drawn from v to theſe Diviſions, will mark correſponding Diviſions on ac or ae.

For the Originals of all Lines, whoſe Vaniſhing Point is v, being parallel to the Originals of bc and ef, the Baſes of the Triangles abc and eaf; the Originals of the Sides of thoſe Triangles will be cut proportionally by the Originals of the Lines drawn from v[a]. The ſame is to be underſtood of the Diviſions of ab and ae, or of ac and af, by Lines proceeding from the Point y.

[a] 2 El. 6.

CASE 3.

When one of the given Images is parallel to the Interſecting Line.

Fig. 39. N°. 2. Let ab and cx be the given Lines Interſecting in a, and let ab be the given Part in the parallel Line, to which, a Part ae or ac is to be ſet off from a on the Line cx, in any given Proportion.

Take xy on the Vaniſhing Line, in the ſame Proportion to Ix the Radial of cx, as the Original of ab is propoſed to have to the Original of ae, and yb will determine the Part ae; and if xv be taken the contrary way from x, equal to xy, vb will determine the Part ac: the Point c may alſo be found by the Vaniſhing Point y, if af be made equal to ab, for then yf will determine that Point.

The ſame Points y and v will ſerve to determine the Parts ab or af, if the Parts ac or ae were thoſe given.

This evidently follows from Cor. 4. Prob. VIII.

COR. 1.

Hence an eaſy way is given, for increaſing or diminiſhing a given Part ae of any Line cx in any given Proportion, by diminiſhing or increaſing the Diſtance xy in the Proportion required; which Method may be uſed inſtead of that of increaſing or diminiſhing the Line ab, when it is more convenient.

COR. 2.

If the Part given and the Part required be propoſed to repreſent equal Lines, the Points y and v may be found by biſecting the Angles NIx and xIR by the Lines Iy and Iv, and yI and Iv will be perpendicular to each other[b]; and the Points y and v thus found, are the ſame with the Points of Diſtance of the Vaniſhing Point x.

[b] Cor. 1. Caſe 2.

For here NR may be taken as the Radial of the Line ab, which is parallel to the Interſecting Line, and NR and EF being parallel, the alternate Angles NIy, Iyx are equal; conſequently if the Angles NIy, yIx be made equal, the Angles Iyx and yIx will be equal, and therefore xy will be equal to xI; and for the like reaſon xv and xI will alſo be equal, wherefore y and v are the true Points of Diſtance of the Vaniſhing Point x: and the Angles NIx, xIR being together equal to two Rights, the Angles yIx, xIv, which are the Moieties of thoſe two Angles, are equal to one Right Angle, that is, yI and Iv are perpendicular.

GENERAL COROLLARY.

Fig. 39. N°. 1, 2. If the Image abc of any Triangle be given, and any two of its Sides be propoſed to be divided, ſo as they may repreſent Lines divided in the ſame Proportion; if the Diviſions of one of the Sides be found, the correſponding Diviſions of the other are determined by Lines drawn from the Vaniſhing Point of the Baſe.

Thus if the Diviſions of ab be given, and thoſe of ac required; Lines from v the Vaniſhing Point of the Baſe bc, through the Diviſions of ab, will determine thoſe of ac; or if the Diviſions of bc from thoſe of ab be deſired, they are had by Lines drawn from x, the Vaniſhing Point of the Baſe ac; or laſtly, if the Diviſions of ac be given, and thoſe of bc deſired, they are found by Lines drawn from z the Vaniſhing Point of ab; or elſe parallel to ab, when that Side is parallel to the Interſecting Line, and hath no Vaniſhing Point.

And in general, whatever Vaniſhing Point ſerves to determine the Image of any Line by the Image of another, the ſame Point will equally ſerve to determine the Images of the Diviſions of the one Line, by the correſponding Diviſions of the other.

PROB.

PROB. XVI.

The Indefinite Images xb and zm of two Lines not parallel, and of a determinate Part bc in one of them being given; from a given Point a in the other, to ſet off a Part, which ſhall repreſent a Line either equal, or in any other Proportion to the Original of bc.

Fig. 40.

Through b draw bk parallel to the Interſecting Line, cutting zm in k, and from any Point y in the Vaniſhing Line, draw yc cutting bk in g; take kh in the Line bk in the ſame Proportion to bg, as the Original of the Part required is to have to the Original of the given Part, and draw zh, and through the given Point a draw fa parallel to bk, cutting zh in f; then on Ix, the Radial of xb, take IC equal to xy, and draw CB parallel to the Vaniſhing Line, cutting the Radial Iz in B; laſtly, take zv or zw equal to IB on either Side of z, according as the Part ſought is to fall above or below a, and vf will determine the Part ae, and wf the Part am, the Original of either of which will be to the Original of bc in the Proportion required.

Dem. The Original of bg is to the Original of bc, as yx or its equal IC is to Ix, and the Original of fa is to the Original of ae, as zv or its equal IB is to Iz[a]; but becauſe CB is parallel to xz, IC is to Ix as IB to Iz, therefore the Original of bg is to that of bc, as the Original of fa is to that of ae; but the Originals of fa and hk being equal[b], the Original of bg is to that of fa in the Proportion required, and conſequently the Originals of bc and ae are in the ſame Proportion: laſtly, becauſe zw and zv are equal, the Originals of ae and am are alſo equal[c]. *Q. E. I.*

[a] Cor. 4. Prob. 8.

[b] Cor. 1. Prob. 6.

[c] Cor. 4. Prob. 8.

Note, *The Points y and v are here taken nearer to x and z than their true Points of Diſtance, only that they might not run out too far; for the Points of Diſtance of x and z, if within reach, would have equally ſerved to anſwer the Problem.*

PROB. XVII.

The Indefinite Image bx of an Original Line AB, and the Image a of a Point A in that Line being given; from thence to ſet off a Part ae, which ſhall repreſent AE, any given Proportion of AT, the Complement[d] of the Original Line from A.

Fig. 37. N°. 3, 4.

[d] Def. 27. B. I.

Take ae in the ſame Proportion to ax the Complement of the given Image, as the Original Part AE is to ET the Difference between the Original Part propoſed and AT, or the Complement of the Original Line from E, and ae will be the Image deſired.

Dem. Becauſe $ae : ax :: AE : ET$[e]. *Q. E. I.*

[e] Theor. 26. B. I.

COR. 1.

Thus if it be propoſed, that ae ſhould repreſent a half of AT, then AE and ET being equal, ae muſt be made equal to ax; if the Image of a third Part of AT be wanted, then AE being one half of ET, ae muſt be taken equal to one half of ax, and ſo on; all which may therefore be done without having the Original Plane drawn out, the Proportion of AE to ET being given.

COR. 2.

Thus alſo, if in an Original Line AT, from a given Point E, it be required to ſet off a Part EA, the Image of which ea, may be in any given Proportion to ax the Complement of that Image;

Take on the Original Line, from the given Point E, a Diſtance EA, in the ſame Proportion to ET, as ae is propoſed to bear to ax, and EA will be the Original Part deſired. For $EA : ET :: ea : ax$ as before.

PROB. XVIII.

Having the Image ab of a Line given, whoſe Vaniſhing Point is out of reach; through any other given Point d in the Picture, to draw a Line which ſhall repreſent a Line parallel to the Original of ab.

Fig. 41. N°. 1. 2.

METHOD 1.

Through the given Point d draw any Line dx, cutting the given Line ab in a, and the Vaniſhing Line EF in x, and from any other Point b in ab (the farther from a

the better) draw bz parallel to ax, cutting EF in z, thereby forming with EF and ab a Trapezium $xabz$; and having drawn the Diagonal xb, from d draw de parallel to ab, cutting xb in e, and from e draw el parallel to EF, cutting zb in l, then a Line dl drawn from d through l, will be the Line required.

Dem. For in the Similar Triangles xab, xde, $ax : dx :: bx : ex$
And in the Similar Triangles bel, bxz, $bx : ex :: bz : lz$
Therefore $ax : dx :: bz : lz$

Conſequently ax and bz which are parallel, being cut proportionally by the three Lines ab, dl, and EF, theſe three Lines muſt proceed from the ſame Point[a], which being a Point in EF, is therefore the common Vaniſhing Point of ab and dl, which two Lines therefore repreſent Parallels. *Q. E. I.*

[a] Lem. 2. B. I.

COR. 1.

Fig. 41. N°. 3. If the Image ab be given, its Indefinite Radial Iv may be thence found, by uſing I, as the Point through which a Line Iv muſt be drawn, tending to the ſame inacceſſible Point with EF and ab; or if the Indefinite Radial Iv of an Original Line, and the Image a of any Point of that Line be given, its Indefinite Image ab may be thence found, by drawing through a, the Line ab tending to the ſame Point with Iv and EF, as in the Figure, where the neceſſary Lines for both Caſes are drawn.

SCHOL.

Either of the Diagonals xb or az of the *Trapezium* $xabz$ will equally ſerve, provided the Line drawn from d to that Diagonal, divide it in e, in the ſame Proportion as xa is divided in the Points x, a, and d; which Diviſion depends on the Similitude of the Triangles to be formed.

In Fig. N°. 1. the propoſed Point d is between ab and EF, and in Fig. N°. 2. it lies without them; in both theſe the Diagonal xb is uſed; but in Fig. N°. 3. where the propoſed Point alſo lies on the outſide of the two given Lines, the Diagonal az is made uſe of for finding the Radial Iv, and the Diagonal Iz for finding the Indefinite Image ab. But the Demonſtration in all, depends on the ſame Reaſoning.

COR. 2.

Fig. 41. N°. 1, 2. If it were required to draw any Number of Lines from ſeveral given Points, all tending to the Interſection of ab with EF, the ſame Points x and b may be retained for all of them; for drawing a Line from x through any one of the propoſed Points, and a Parallel to that Line through b, the Diagonal xb will continue the ſame, in which therefore the Point correſponding to e is to be found; and the Point which anſwers to l, will lie in the new Parallel drawn through b: and thus Lines being drawn from x to all the propoſed Points, and Parallels to each of them being drawn through b, the Diagonal xb will be the place of all the Points e; and the reſpective Parallels through b, will be the Places of the correſponding Points l.

METHOD 2.

Fig. 41. N°. 4, 5. The ſame things being ſuppoſed as before, from any Point x in either of the given Lines EF, to the given Point d draw dx, and another Line xa, making any Angle with dx, and cutting the other given Line ab in a; from any Point u within reach in the Line EF, draw uf parallel to ab, cutting xa in f, and having drawn ad, and fe parallel to it, cutting dx in e, draw eu; then a Line dl drawn through d parallel to eu will be the Line required.

Let the Interſection of ab with EF be called m, and that of dl with EF, n; it muſt be proved that the Points m and n coincide.

Dem. Becauſe ab and fu are by Conſtruction parallel, the Triangles xam, xfu, are Similar, wherefore $ax : fx :: xm : xu$
And in like manner the Triangles xdn, xeu being Similar $dx : ex :: xn : xu$
But in the Similar Triangles axd, fxe $ax : fx :: dx : ex$
Therefore $xm : xu :: xn : xu$

And conſequently xm and xn are equal, and the Points m and n are therefore the ſame. *Q. E. I.*

COR. 1.

Fig. 41. N°. 6. If the Image ab be given, its Indefinite Radial Iv may be found; or if Iv be given together with the Point a, the Image ab may be determined by this Method, as is ſufficiently evident from the Figure.

Plate 12. Book 2. Sect. 2. 3.

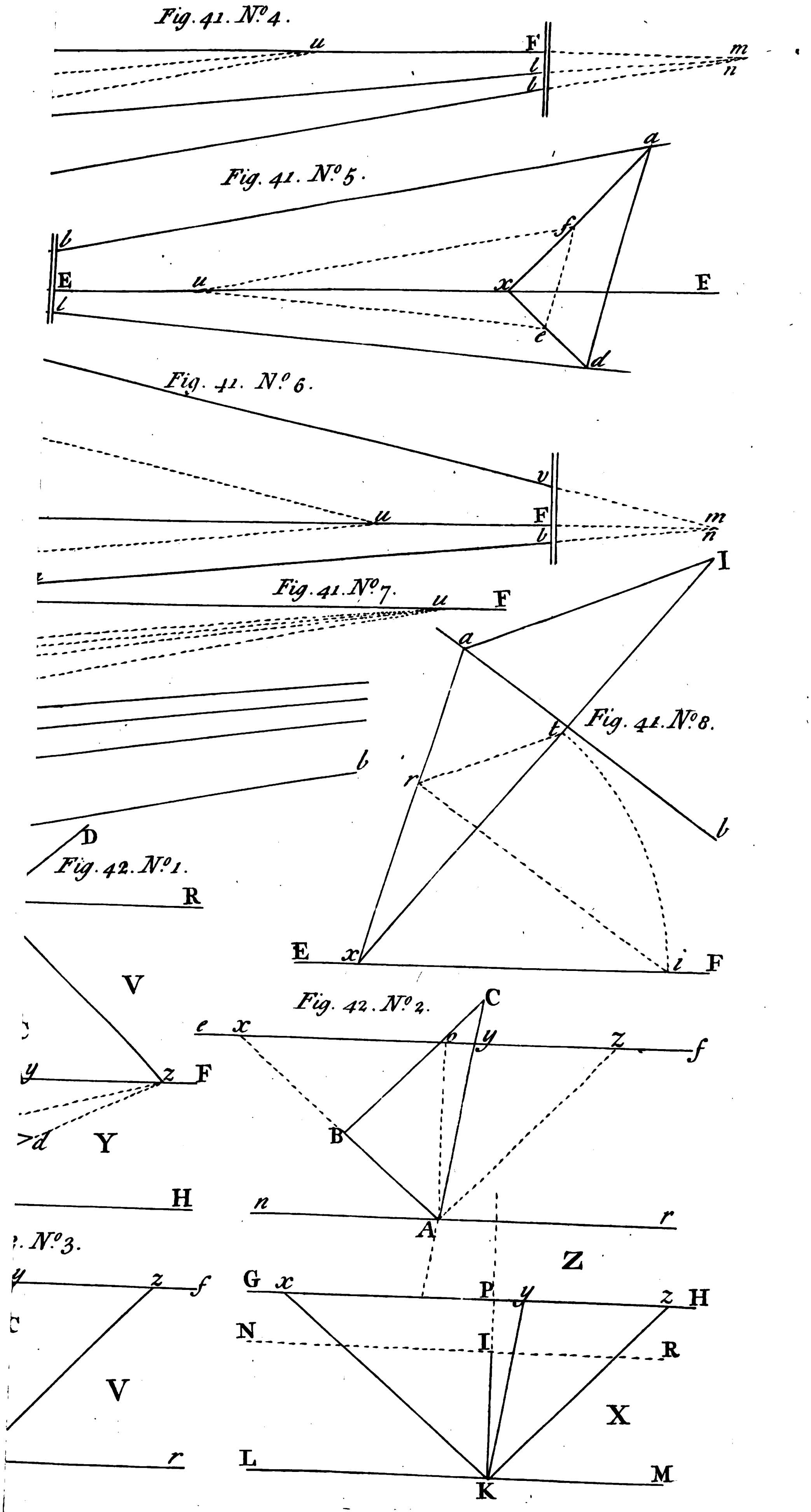


Fig. 41. No 4.

Fig. 41. No 5.

Fig. 41. No 6.

Fig. 41. No 7.

Fig. 41. No 8.

Fig. 42. No 1.

Fig. 42. No 2.

No 3.

C O R. 2.

If it be required through any Number of given Points c, d, e, in the ſame ſtraight Line ce, to draw Lines tending to the ſame Vaniſhing Point with a given Line ab, it is thus done. Fig. 41. N°. 7.

Having produced ce till it meet EF in x, draw any Line xa cutting ab in a, and having from any convenient Point u in EF, drawn uf parallel to ab, cutting xa in f, from a to the ſeveral given Points c, d, e, draw ac, ad, ae, and through f draw $f\gamma$, $f\delta$, $f\varepsilon$ parallel to them reſpectively, cutting cx in γ, δ, and ε; and having drawn γu, δu, εu, Lines drawn through c, d, and e, parallel reſpectively to γu, δu, and εu, will be the Lines required.

C O R. 3.

If a Line EF be given, and it be propoſed through any given Point a without that Line, to draw another, which ſhall tend to an inacceſſible Point in EF, the Diſtance of which from ſome known Point x in that Line is given, it may be done in this manner. Fig. 41. N°. 8.

Join the Points x and a, and from x draw any Line xI, making any Angle with xa, ſo as that a Diſtance xI may be taken on it, equal to that between x and the propoſed Point in EF, to which the Line through a is to tend, and draw Ia; then having taken any Diſtance xt on the Line xI, which can be ſet off from x on the Line EF within reach, as at i, draw tr parallel to Ia, cutting xa in r; and having drawn ir, a Line ab drawn through a, parallel to ir, will be the Line required.

For in the Similar Triangles xIa, xtr, $xr : xa :: xt : x$I. If then xi be made equal to xt, and a Line ab be drawn through a parallel to ir, it is evident, that Line will form a Triangle with xa and xF Similar to the Triangle xri, and then xr will be to xa, as $xi = xt$ is to the Diſtance between x and the Interſection of ab with xF, which Diſtance is therefore equal to xI, which was taken equal to the Diſtance propoſed; and conſequently ab thus drawn, tends to the propoſed inacceſſible Point in EF.

S C H O L.

The Methods here propoſed, ſerve likewiſe to find a Line tending to the ſame inacceſſible Point with any other two given Lines whatſoever.

For the Demonſtrations are the ſame, whether EF be the Vaniſhing Line, or any other Line tending to the ſame Point with ab and dl, or Iv.

SECTION III.

Of the Images of Figures in the Original Plane.

P R O B. XIX.

An Original Triangle ABC, and the Image of either of its Sides ac, being given; thence to find the Image of the whole Triangle. Fig. 42. N°. 1, 2.

METHOD 1.

Make on Iy, the Radial of the given Side, a Triangle IBC Similar to the Original, placing the Angle correſponding to the neareſt Angle BAC at I, produce IB to x, and draw Iz parallel to BC; then xa and zc Interſecting in b, will give abc the intire Image required.

Dem. For by reaſon of the Vaniſhing Points x, y, and z, the Original of the Triangle abc is Similar to the Triangle IBC, which was made Similar to the Original Triangle ABC [a]; and ac being the Image of AC, the Triangle abc is therefore the Image of ABC. *Q. E. I.*

Note, xa and zc *muſt be ſo drawn, as to cut the ſame correſponding Extremities of the given Side* ac, *as their Originals do; for if Lines were drawn from* x *to* c, *and from* z *to* a, *Interſecting in* d, *a Triangle* acd *would be thereby formed on the Side* ac, *Similar indeed to the Original* ABC, *but in a contrary Poſition.*

C O R.

If either of the outward Radials Ix be produced at pleaſure beyond I to D, the three

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[a] Prob. 4.

three Angles xIy, yIz, and zID will be reſpectively equal to the three Angles of the Triangle propoſed; and if that Triangle be Equilateral, thoſe three Angles will be equal.

[a] 29 El. 1. For the Angle zID is equal to IBC [a]: the reſt is evident.

METHOD 2.

By the help of a ſeparate Vaniſhing Plane.

Fig. 42. N°. 3. Having any where a-part drawn a ſeparate Vaniſhing Plane $efnr$ [b], in it draw Iy the Radial of the given Side ac, by tranſporting the Diſtance oy in the Vaniſhing Line of the Picture, to the Vaniſhing Line ef of the ſeparate Vaniſhing Plane; and having on the Radial Iy thus found, drawn a Triangle IBC Similar to the Original Triangle ABC, and alike poſited with reſpect to nr, as the Original Triangle is with reſpect to GH the Interſecting Line of the Original Plane, draw Iz parallel to BC, and produce IB to x, whereby the Points z and x will be found; which being tranſferred to the Vaniſhing Line of the Picture, the Triangle abc may be compleated as before. *Q. E. I.*

[b] Schol. Prob. 2.

SCHOL.

By this Method, the Triangle IBC in the ſeparate Vaniſhing Plane, being to have the ſame Poſition as the Original Triangle, this Practice is leſs liable to a Miſtake than the preceeding, where, by reaſon that the Vaniſhing Plane is ſeen on the underſide when it is joined to the Picture, the Triangle IBC, by which the Vaniſhing Points are determined, hath a contrary Poſition to that of the Original.

METHOD 3.

By the help of the Original Plane.

Fig. 42. N°. 2. Through the neareſt angular Point A of the Original Triangle ABC, draw nr parallel to the Interſecting Line GH of the Original Plane, and having drawn Ao perpendicular to nr, and equal to Io in the Picture, through o draw ef parallel to nr, and $efnr$ will then ſerve the purpoſe of a ſeparate Vaniſhing Plane; and AB and AC produced, if neceſſary, to ef, and a Line Az drawn parallel to BC, will, by their Interſections with ef, give all the three Vaniſhing Points x, y, and z; which being thence transferred to the Vaniſhing Line EF in the Picture, the Image of the Original Triangle may be thence compleated, the Image of either of its Sides being given.

Fig. 42. N°. 3. For the Figure $efnr$ in the Original Plane, is every way Similar to $efnr$ in the ſeparate Vaniſhing Plane, which is Similar to EFNR in the Picture, though in a contrary Poſition, the Point o in each of them repreſenting the Center of the Vaniſhing Line. *Q. E. I.*

COR. 1.

Fig. 42. N°. 2. If from K in the Directing Line of the Original Plane, Lines be drawn to GH, parallel reſpectively to the Sides of the Original Triangle ABC, thoſe will cut GH in x, y, and z at equal Diſtances reſpectively from P, as the correſponding Points in the other Figures are from o. And thus LMGH in the Original Plane, will ſerve inſtead of a ſeparate Vaniſhing Plane, Io and PK being always equal [c].

[c] Cor. 3. Def 15. B. I.

COR. 2,

Fig. 42. N°. 2. If the Original Plane were not drawn out, yet if any Triangle ABC be given, Similar to the Original Triangle propoſed, and the Situation of either of its Sides with reſpect to the Interſecting Line of the Original Plane be known; through A, the neareſt Angle, draw nr, making an Angle with AB or AC, equal to that which the correſponding Sides of the Original Figure are ſuppoſed to make with the Interſecting Line, and uſing A as the Place of the Eye, thereby a ſeparate Vaniſhing Plane may be compleated, and uſed as before directed; by which the Image of the Original Triangle may be found by the Image of either of its Sides.

Or if the neareſt Side of the Original Figure be parallel to the Interſecting Line of the Original Plane, that Side itſelf may be made to ſerve inſtead of the Line nr, and either Extremity of that Side may be taken as the Place of the Eye.

GENERAL COROLLARY.

The two preceeding Methods of uſing a ſeparate Vaniſhing Plane, or making the Original Plane ſerve the ſame purpoſe, for finding the Image of a Triangle, are equally applicable for the finding the Images of any other right lined Figures in the Original Plane, thoſe Images being generally found by reſolving the Originals into Triangles, and

and finding the Images of thoſe Triangles, whereby the Image of the whole Figure is determined.

PROB. XX.

An Original Parallelogram ABCD, and the Image ab of either of its Sides AB being given; thence to find the Image of the whole Figure.

Fig. 43. N°. 1, 2.

METHOD 1.

Having drawn Ix the Radial of the given Side ab, draw Iz, making the Angle xIz equal to the inward Angle DAB or DCB of the Original Figure; then on the Radials Ix and Iz, take any Diſtances IB, and ID, in the ſame Proportion to each other, as AB is to AD in the Original Figure, and having compleated the Parallelogram IDCB, draw the Diagonal IC cutting EF in y: then from z draw za and zb, and from y draw ya cutting zb in c, and from x through c draw xc cutting za in d, and $abcd$ will be the Image of the Original Figure ABCD.

Dem. Becauſe of the Vaniſhing Points x, y, and z, the Originals of the Triangles abc, adc are Similar to the Triangles IBC, IDC[a]; wherefore the Original of the Figure $abcd$ is Similar to the Parallelogram IBCD, which by Conſtruction is Similar to the Original Parallelogram ABCD; and ab being the Image of AB, the Figure $abcd$ is therefore the Image of ABCD. *Q. E. I.*

METHOD 2.

Fig. 43. N°. 1.

Having drawn Iz, za, and zb as before, biſect the Angle xIz by the Line Iv[b], and through a draw bk parallel to EF; from v draw vb cutting bk in k, and make ab to ak as AD to AB in the Original Figure[c], and draw vb cutting za in d, from whence to x draw dx, which will cut zb in c, and thereby determine $abcd$, the Image of the Figure deſired.

Dem. The Original of ab is to the Original of ak, as Ix to xv, and the Original of ad is to that of ab, as Iz to zv[d]; but becauſe the Angle xIz is biſected by Iv, Ix is to Iz as xv to zv[e]; therefore the Original of ab is to that of ad, as the Original of ak is to that of ab, which laſt were taken in the ſame Proportion as AB to AD. The reſt is evident. *Q. E. I.*

CASE 2.

Fig. 43. N°. 3.

If the Original of the given Side ab be parallel to the Interſecting Line, the Vaniſhing Point x being then infinitely diſtant, find Iz the Radial of the inclining Sides of the Figure[f], and having drawn zb and za, take zy in the ſame Proportion to Iz as the Original of ab is to the Original of bc, and draw ya, which will determine the Point c[g], and cd drawn parallel to ab will compleat the Image $abcd$. Or if ab be produced to k, until ak be to ab as the Original of bc is to that of ab, take zv equal to Iz, and vk will give the Point d[h], whence cd is found as before.

The Point y may be alſo found, by taking on NR any Diſtance IB, and making on that Line a Parallelogram IBCD Similar to the Original; for then the Diagonal IC will cut EF in the ſame Point y, ſeeing ID : DC :: Iz : zy. *Q. E. I.*

And here the Line NR *is uſed as the Radial of* ab, *whoſe Vaniſhing Point is infinitely diſtant.*

COR.

Fig. 43. N°. 1, 3.

If the Original Figure be equilateral as AMLD. to which IMLD in the Vaniſhing Plane is Similar, the ſame Line Iv, which biſects the Angle xIz, or NIz, will alſo be the Diagonal; and then the Vaniſhing Point v may be uſed, either as the Point y in the firſt Method, or as the Point v in the ſecond.

For if the Image am of the Side AM be given, and zm and za be drawn as before, a Line va will cut zm in l, from whence a Line being drawn either to x, or parallel to EF, when x is infinitely diſtant, the Point d, and conſequently the intire Image $amld$ will be determined[i].

Or if vm (Fig. N°. 1.) be drawn cutting ak in n, make ab equal to an, and vb will give the ſame Point d. Or (Fig. N°. 3.) make ak equal to am, and vk will give d, whence the reſt may be compleated as before[k].

CASE. 3.

Fig. 43. N°. 1.

If the intire Image $abcd$ of the Parallelogram be given, and the Original be ſuppoſed to be anywiſe divided into ſmaller Parallelograms by Lines parallel to its Sides, as in

[a] Prob. 4.

[b] Lem. 3.

[c] Lem. 2.

[d] Cor. 4. Prob. 8.

[e] 3 El. 6.

[f] Caſe 2. Prob. 3.

[g] Cor. 4. Prob. 8.

[h] Cor. 1. Prob. 8.

[i] Method 1.

[k] Method 2.

in the Figure ABCD, and it be required to divide the given Image in like manner; Through a the neareſt angular Point of the given Image, draw bk, and having produced the Sides cb and cd till they cut kh in p and q, divide ap and aq reſpectively in the ſame Proportion as AB and AD are divided, and Lines drawn from z and x through the Diviſions of ap and aq, will divide the given Image in the manner deſired[a].

Or if zb and xd ſhould cut bk at an inconvenient Diſtance, the intermediate Point v may be uſed, and vd and vb being drawn cutting bk in b and k, and ak and ab being divided in the Proportion required, Lines drawn from v to thoſe Diviſions, will cut ab and ad in correſponding Diviſions[b], from whence Lines to z and x will divide the Image $abcd$ in the manner ſought.

Fig. 43. N°. 3. Or laſtly, if one of the given Sides ab be parallel to the Interſecting Line, produce it till it be cut by vd in k, then divide ab and ak in the Proportion required, and Lines from z to the Diviſions of ab, will give the Images of the Diviſions of the Figure, which are parallel to the Original of the Side ad; and Lines from v through the Diviſions of ak, will mark correſponding Diviſions on ad, from whence Lines drawn parallel to ab will compleat the Diviſion propoſed. *Q.E.I.*

COR.

If the Sides AB and AD of the Original Figure be divided in the ſame Proportion, ſo as the Diviſions of AB from A to B, may be proportional to the correſponding Diviſions of AD from A to D; then the Diviſions of either Side of the Image being found, the Diviſions of the other Side may be determined by the Image of the Diagonal ya.

Fig. 43. N°. 1, 3. Thus the Diviſions of ab being found by any of the Methods before propoſed, and thence Lines drawn to z cutting the Diagonal ya, through theſe Interſections, draw other Lines to x the Vaniſhing Point of ab, or elſe parallel to the Interſecting Line when ab is ſo, and theſe will compleat the Diviſion required.

For the Originals of ab and ac are divided in the ſame Proportion by the Originals of the Lines which proceed from z, and the Originals of ac and ad are divided in the ſame Proportion by the Originals of the Lines whoſe Vaniſhing Point is x, as in the firſt Caſe[c], or which are parallel to the Interſecting Line, as in the ſecond.

Note, *A Square may be conſidered as a Rectangular Equilateral Parallelogram, and therefore comes within the Rules of this Problem.*

PROB. XXI.

Upon a given determinate Line in the Original Plane, to deſcribe a Figure, the Image of which ſhall be Similar to a given Parallelogram.

Fig. 43. N°. 2, 4. Let ABCD be the propoſed Parallelogram divided into ſmaller Parallelograms at pleaſure, and let XZ be the Original Plane prepared as uſual, and in it ab the given Line, on which a Figure is to be deſcribed, the Image of which ſhall be Similar to ABCD.

Produce ab to its Directing Point X; and draw the Director IX, and on it make a Parallelogram ITQS Similar to ABCD, and having produced IS and the Diagonal IQ to their Directing Points Z and Y, draw Za, Zb, and from Y draw Ya cutting Zb in c, and draw Xc cutting Za in d, then $abcd$ will be the Figure required, the Image of which will be Similar to ABCD.

Then to find the Subdiviſions, divide mn in the Interſecting Line, where it is cut by Za and Zb, in the ſame Proportion as AB is divided, and likewiſe pq in the ſame Proportion as AD, and from Z and X through the Diviſions of mn and pq, draw Lines, which will divide the Figure $abcd$ in the manner deſired.

Dem. Becauſe of the Directing Points X, Y, Z, the Images of the Sides of the Triangle abc being parallel to their reſpective Directors[d], the Image of that Triangle is Similar to the Triangle QSI, and for the ſame reaſon the Image of the Triangle adc is Similar to QTI, conſequently the Image of $abcd$ is Similar to QSIT, which was made Similar to ABCD. And becauſe of the Directing Point Z, the Images of ab and mn, in the Triangle bqn, are divided in the ſame Proportion by the Images of the Lines from Z, theſe laſt being parallel[e]; and for the ſame reaſon the Images of ad and pq are divided in the ſame Proportion by the Images of the Lines from X. But mn and pq are their own Images, and were divided in the ſame Proportion as the Sides AB and AD of the Parallelogram ABCD; wherefore the Images of ab and

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[a] Cor. 2. Prob. 8.

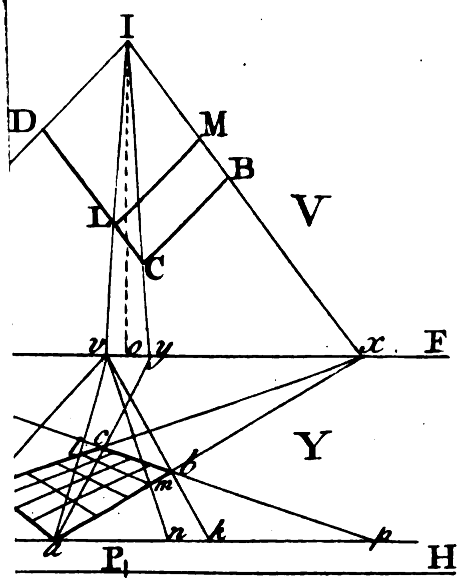
[b] Method 2.

[c] Gen. Cor. Prob. 15.

[d] Cor. 1. Def. 18. B. I.

[e] Theor. 6. B. I.

Plate 13. Book 2. Sect. 3.



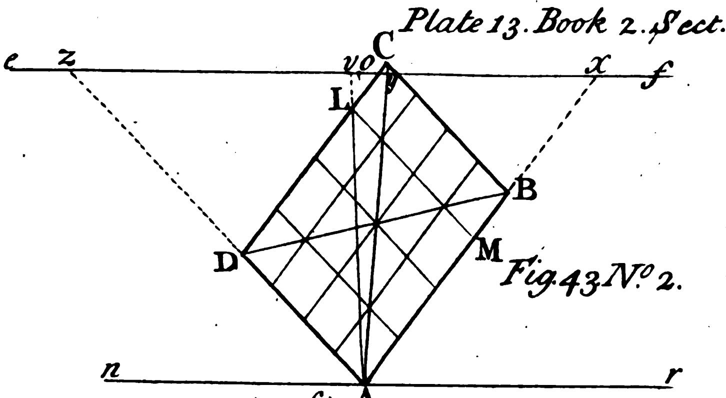
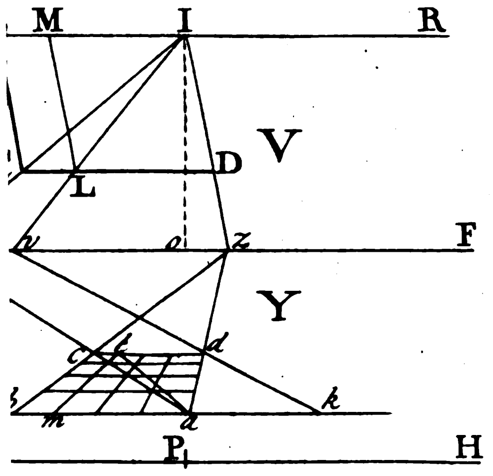
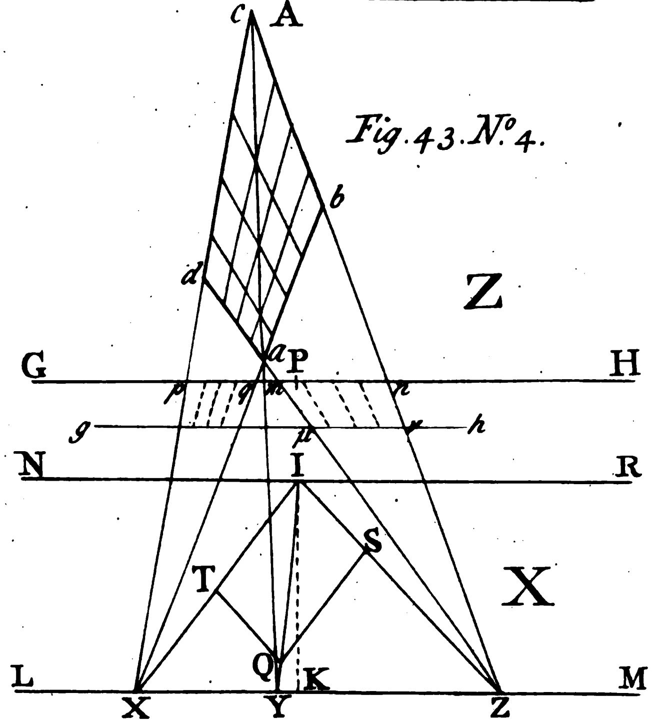


Fig.43.No2.





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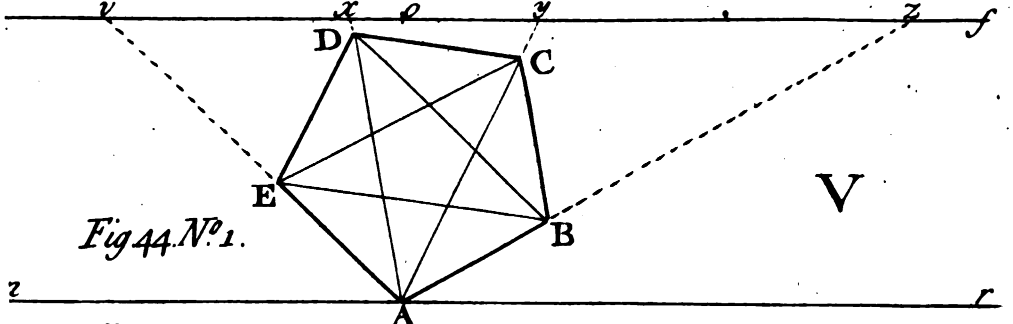

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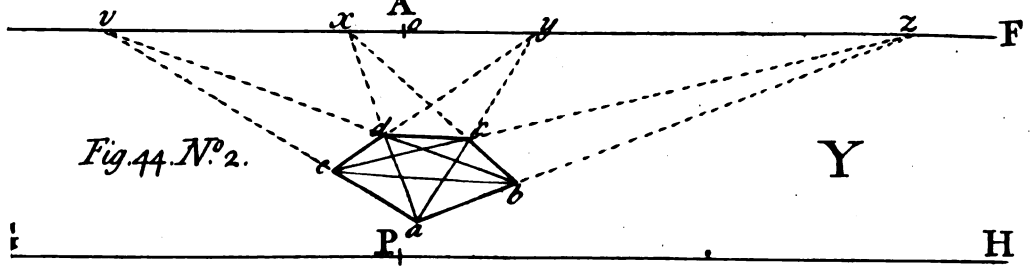

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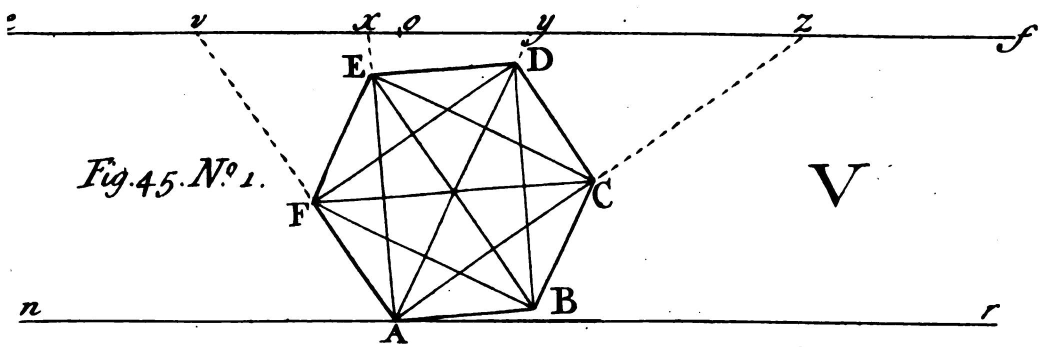

Fig.45.No1.

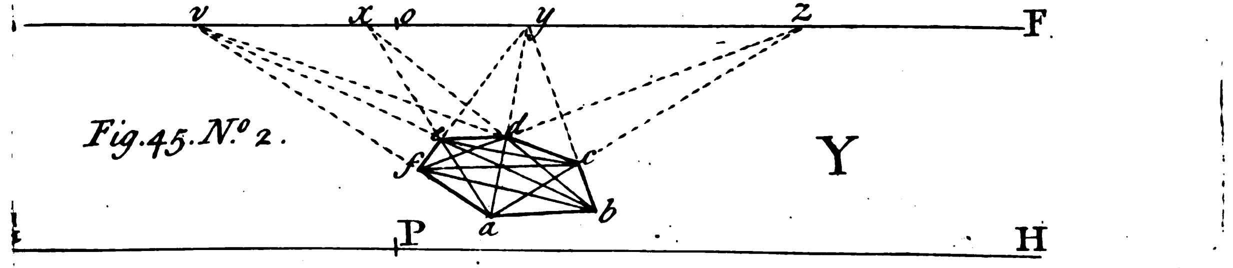

Fig.45.No2.

J. Mynde sc.

ad being divided in that Proportion, the Image of $abcd$ and its Subdiviſions, is Similar to the propoſed Parallelogram ABCD and its Subdiviſions. *Q. E. I.*

COR. 1.

Inſtead of uſing the Interſecting Line GH, any other Line gb parallel to it may be taken, and the Parts of that Line intercepted between Za and Zb, and between Xa and Xd being divided in the ſame Proportion as mn and pq, Lines from Z and X drawn through theſe Diviſions, will ſubdivide the Figure $abcd$ in the ſame manner as before.

For it is evident, that Lines from Z and X, through the Diviſions of GH, will cut the correſponding Parts of gb in the ſame Proportion[a].

COR. 2.

If in an Original Line ab, a Part be required to be ſet off from a, the Image of which ſhall be equal to, and divided in the ſame manner with any propoſed Line: Produce ab to its Directing Point X, and draw its Director IX; from X ſet off XZ on the Directing Line equal to IX, and draw Za cutting GH in m; then take mn equal to the propoſed Line, and divide it in the manner required, and Lines from Z through thoſe Diviſions will cut ab ſo, that its Image ſhall be equal to, and alike divided with the Line propoſed.

For IX and XZ being ſuppoſed equal, the Triangle IXZ will be Iſoſceles; wherefore the Image of the Triangle nqb will alſo be Iſoſceles; and the Images of ma, nb, and of the other intermediate Lines from Z being all parallel, the Images of the Sides qn and qb will be equal, and divided in the ſame Proportion; but qn is its own Image, therefore the Image of qb being equal to and alike divided with qn, the Image of ab will be equal to and alike divided with mn, which was taken equal to the Line propoſed.

DEF. 4.

And here as mn is the true Meaſure of the Image of ab, ſo $\mu\nu$ is *the proportional Meaſure of that Image*; that is, $\mu\nu$ is to the Image of ab, as the Diſtance between the Directing Line LM and the aſſumed Parallel gb, is to the Diſtance between the Directing and Interſecting Lines of the Original Plane[b].

SCHOL.

The Demonſtration of this Problem is much the ſame with that of the preceeding, the Directing Plane and Directors being here uſed with regard to the Original Plane, as the Vaniſhing Plane and Radials were there with reſpect to the Picture; and as the Directors have the ſame Relation to the Images, as the Radials have to the Original Lines, it will be very eaſy to apply any of the Rules and Practices in the foregoing Problems, for finding Images whoſe Originals ſhall be Similar to any given Figure, to the finding an Original Figure whoſe Image ſhall be Similar to any Figure propoſed[c]. It will therefore be unneceſſary to add more Examples of this ſort, ſeeing the preſent Problem and its Corollaries duly attended to, will render the application eaſy in any other Caſe.

PROB. XXII.

A regular Pentagon ABCDE, and the Image ab of any one of its Sides AB being given; thence to find the Image of the whole.

Fig. 44. N°. 1, 2.

Through A the neareſt angular Point of the Pentagon, draw nr, ſo that AB may incline to nr in the ſame Angle as it is ſuppoſed to do to the Interſecting Line of the Original Plane, and compleat the ſeparate Vaniſhing Plane $efnr$, A being taken as the Place of the Eye[d].

Then having found the Vaniſhing Points of any three of the required Sides of the Pentagon, as of AE, ED, and BC, by producing AE, and drawing AC parallel to ED, and AD parallel to BC till they cut ef reſpectively in v, y, and x; transfer the Points v, y, and x to EF the Vaniſhing Line of the Picture, by ſetting them in the ſame Order, and at the ſame reſpective Diſtances from o in that Line, as they ſtand from o in the Line ef: and by the help of theſe three Vaniſhing Points, and the given Image ab, the intire Image of the Pentagon may be found.

For by ab and the Points x and y, the Triangle abc is found, and by the ſame Line ab and the Points x and v, the Triangle abd is had, and by ad and the Points v and y, the Triangle aed is determined, and the Points d and c give the Side dc, which compleats the intire Image $abcde$ of the Pentagon ABCDE.

Dem.

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[a] Lem. 2. B. I.

[b] Def. 3.

[c] Gen. Cor. Prob. 4.

[d] Cor. 2. Method 3. Prob. 19.

Dem. For by reaſon of the ſeveral Vaniſhing Points uſed, the Originals of the ſeveral Triangles in the Image *abcde* are Similar to the correſponding Triangles in the Pentagon ABCDE, and *ab* being the determinate Image of AB, *abc* is the Image of ABC[a]; and for the ſame reaſon, each Triangle in *abcde* is the Image of the correſponding Triangle in ABCDE; wherefore the Figure *abcde* is the intire Image of the given Pentagon ABCDE. *Q. E. I.*

After the ſame manner, if *ae* the Image of the Side AE were given, the intire Image may be from thence compleated by the help of the Vaniſhing Points *x*, *y*, and *z* of the Sides BC, ED, and AB; and ſo, from the Image of any one Side, and the Vaniſhing Points of any three of the other Sides given, the intire Image may be found.

COR. 1.

If one of the Sides required be parallel to the Interſecting Line, that will be equivalent to a Vaniſhing Point, ſo that the Vaniſhing Points of any two others of the required Sides, will, together with the given Side, be ſufficient for compleating the Figure.

Thus if the Side DC, and conſequently EB were parallel to the Interſecting Line, and *ae* the Image of AE were given, any two of the three Points *x*, *y*, and *z* will ſuffice.

If *x* and *y* be uſed, the given Side *ae* and the Points *x* and *y* give the Triangle *aed*, *dc* drawn parallel to EF cuts *ya* in *c*, whence the Triangle *adc* is found, and *eb* parallel to *dc* cutting *xc* in *b*, determines the Triangle *abc*.

The ſame may be done in like manner, by uſing any other two of the Points *x*, *y*, and *z*, the Side *ae* being given, as is evident from the Figure.

COR. 2.

Fig. 44. N°. 1. The Angles EAD, DAC, and CAB are each equal to one fifth Part of two Rights.

For the Angles AED and DE*v* are together equal to two Rights, but the Angle AED is compoſed of the Angles AEB, BEC, and CED; and the Angle DE*v*, or its equal CAE is compoſed of the Angles EAD and DAC; wherefore the five Angles AEB, BEC, CED, EAD, and DAC, are together equal to two Rights, and being equal between themſelves, each of them is one fifth Part of two Rights.

PROB. XXIII.

Fig. 45. N°. 1, 2.

A Regular Hexagon ABCDEF, and the Image of either of its Sides being given; thence to compleat the whole Image.

In a regular Hexagon, the oppoſite Sides being parallel, all the ſix Sides can have but three Vaniſhing Points, and the Diagonals as many; of theſe it is requiſite always to have two Vaniſhing Points of the Sides, and two Vaniſhing Points of the Diagonals, when one of the given Vaniſhing Points belongs to the given Side; but if not, then one Vaniſhing Point of the Diagonals, with the two Vaniſhing Points of the required Sides will ſuffice for compleating the intire Image of the Hexagon upon the given Side.

Having therefore through A drawn *nr*, and compleated the ſeparate Vaniſhing Plane *nref*, as in the laſt Problem; produce AF, AE, AD, and AC to their Vaniſhing Points *v*, *x*, *y*, and *z*, and transfer thoſe Points to the Vaniſhing Line of the Picture as before directed; then *v* and *y* will be the Vaniſhing Points of the Sides AF and FE and their Parallels, and *x* and *z* the Vaniſhing Points of the Diagonals AE and AC and their Parallels.

If then *af* the Image of the Side AF, whoſe Vaniſhing Point is *v*, be given, all the four Points *v*, *y*, *x*, and *z*, muſt be uſed.

Thus the given Side *af* and the Points *y* and *x*, give the Triangle *afe*, and the ſame Side *af* with the Points *y* and *z*, determine the Triangle *afd*, whence *ed* is found; the Diameter *ad* with the Points *z* and *v*, give the Triangle *acd*, and the Side *ed* with the Points *v* and *x* form the Triangle *edb*, which determines the angular Point *b*, whence the remaining Sides *ab* and *bc* are found, and thereby the intire Image *abcdef* compleated.

But if *ab* the Image of the Side AB be given, then the intire Image may be had by the Vaniſhing Points *v* and *y* of the Sides, and either of the Vaniſhing Points *x* or *z* of the Diagonals.

Thus by the help of *v*, *y*, and *x*: the Side *ab* with the Points *x* and *y*, give the Triangle

[a] Prob. 19.

angle abd; the ſame ab and the Points x and v, give the Triangle abe, whence ed is found; the Diagonal ae with the Points y and v, form the Triangle aef; and the Diagonal db with the ſame Points y and v, give the Triangle bdc, which compleats the whole.

Or by v, y, and z; the Side ab with the Points y and z, give the Triangle abc; the Diagonal ac with the Points y and v, give the Triangle acd; the Diameter ad with the Points z and v, determine the Triangle adf; and the Diagonal fb being drawn, the Points v and y form thereon the Triangle fbe, whence ed is found.

The Demonſtration of this is the ſame with that of the preceeding Problem. *Q.E.I.*

COR.

If any of the required Sides of the given Hexagon be parallel to the Interſecting Line, that will ſupply the place of one Vaniſhing Point of the Sides; ſo that in this Caſe, one Vaniſhing Point of the Sides, and another Vaniſhing Point of the Diagonals will ſuffice: or if either of the Diagonals be parallel to the Interſecting Line, that will be equivalent to a Vaniſhing Point of Diagonals, and then two Vaniſhing Points of the Sides required will be ſufficient, with the given Side, to compleat the Image.

Thus if the Side AB be ſuppoſed parallel to the Interſecting Line, and af be given; the Point y, with either of the Points x or z, will ſuffice to perfect the Figure.

For af with the Points x and y, give the Triangle afe; ed drawn parallel to EF with the ſame two Points, give the Triangle ead; ab drawn parallel to ed, is cut by xd in b, whereby the Triangle abd is found; and fc drawn parallel to ab, is cut by yb in c, which determines the Triangle bcd.

The ſame may be done with the Points y and z, as is ſufficiently evident.

Or if the Diagonal AC were ſuppoſed parallel to the Interſecting Line, the Points v and y will be ſufficient, the Side ab being given.

For yb cutting ac, drawn parallel to EF, in c, gives the Triangle abc; and ac with the Points v and y, give the Triangle acd; fd drawn parallel to ac, is cut by va in f, whence the Triangle adf is found; and vb and yf, by their Interſection, give the remaining angular Point e, whereby the intire Image $abcdef$ may be finiſhed.

PROB. XXIV.

The Image of any Diameter ac of a Circle, and of its Center s being given; thence to deſcribe the intire Image of the Circle.

Fig. 46. N°. 1.

METHOD I.

Any where a-part draw a Circle ABCD, and having drawn two Diameters AC and BD perpendicular to each other, on the Extremities of thoſe Diameters draw the Square LMNR circumſcribing the Circle: Divide each of the Quadrants AB, BC, CD, and DA into three equal Parts, and through the ſeveral Diviſions, draw Lines parallel to the Sides of the Square, as in the Figure. And thus the Circumference ABCD will be divided into twelve equal Parts, each of which is marked by the Interſection of two ſtraight Lines. And the Figure thus drawn will ſerve as a Model for drawing the Image of any Circle propoſed. Fig. 46. N°. 2.

This being done, produce the given Diameter ac to its Vaniſhing Point x, and draw the Radial Ix, and perpendicular to it draw another Radial Iz; and having biſected the Angle xIz by the Line Iy[a], draw za, zc, and from y through s draw ys, cutting za and zc in l and r, and draw xl, xr; then through r draw gr parallel to the Interſecting Line, cutting za in g; divide gr in the ſame Proportion as the Side NR of the Model LMNR is divided[b], and from z draw Lines to the Diviſions of gr, and through the Interſections of theſe with the Diagonal lr, draw Lines to x, and the Figure $lmnr$ will then repreſent a Figure Similar to the Model LMNR. Laſtly drawing a Curve Line through the ſeveral Interſections of the Lines in the Figure $lmnr$, correſponding to thoſe through which the Circle paſſes in the Model LMNR, thereby the intire Image of the Circle will be compleated. Fig. 46. N°. 1.

Dem. For the Original of $lmnr$ and its Subdiviſions, is every way Similar to the Model LMNR, and alike ſituated with reſpect to the Original of the given Diameter ac, as LMNR is with reſpect to the Diameter AC of the Circle ABCD[c]: and therefore the Original of $lmnr$ is a Square circumſcribing the Circle, whoſe given Diameter is ac; the Image of which Circle muſt therefore paſs through the Diviſions of

[a] Lem. 3.

[b] Lem. 2.

[c] Prob. 15 and 20. and Cor.

$lmnr$, correſponding to thoſe in the Model LMNR, through which the Circle ABCD paſſes. *Q. E. I.*

If the given Diameter ac *be parallel to the Interſecting Line, the Figure* $lmnr$ *with its Subdiviſions may be found as in Caſe* 2. *Prob.* XX.

COR. 1.

Fig. 46. N°. 3. If the propoſed Circle be ſmall, fewer Points in its Circumference may be ſufficient to determine its Image, and a convenient Model may be made by the help of the croſs Diameters and Diagonals, as in the Figure; whereby eight Points in the Circumference will be marked: and the Image of the Circle may be found by this Model, after the ſame manner as was done by the other.

COR. 2.

Fig. 46. N°. 4. If the Image of a Semicircle alone be wanted, a very convenient Model may be made with ALMC, the half of the Square which circumſcribes the intire Circle; by drawing in it two Diagonals AM, CL, Interſecting in s, and drawing sB perpendicular to the Diameter AC, and PQ parallel to AC, through the Interſections a and b of the Diagonals with the Semicircle, as in the Figure; whereby three Points in the Semicircle, beſides the Extremities of its Diameter, are marked, whence its Image may be found as before[a].

COR. 3.

If it be required to deſcribe the Images of two concentrical Circles, the outward Circle may be deſcribed by the firſt Model, and the inner by the ſecond; and that the Lines which divide the outward Model, may not incumber the inner, thoſe which biſect each Moiety of the Sides of the outward Square need not be drawn through, but ſtop where they meet the firſt Diviſion inwards, and the Diameters and Diagonals being drawn through, the inner Model will thence eaſily be formed, by which the Images propoſed may be deſcribed, as in the Figures. Fig. 46. N°. 5, 6.

SCHOL.

Fig. 46. N°. 2. The Diviſions of the Side NR in the firſt Model, may be found without drawing the Circle; by dividing it into four equal Parts in X, D, and P, and making PQ to PR as 732 to 1000, or as 183 to 250, or as 73 to 100, or as 11 to 15, as in the Figure[b], and taking XT equal to PQ.

For the Quadrant of the Circle DgbC, being divided into three equal Parts in g and b, the Angle DSg is of 30 Degrees, the Sine of which fg, is equal to half the *Radius*, wherefore DP $= fg$ is the half of DR; and for the ſame reaſon DX is the half of DN. Again, the Angle DSb being of 60 Degrees, its Sine $eb =$ DQ is 1732 of ſuch Parts, of which the *Radius* contains 2000, and DP being the half of DR, it contains 1000 of thoſe Parts, therefore PQ contains 732 of the ſame Parts, and conſequently PQ is to PR, as 732 to 1000, the ſame Proportion as is between XT and XN.

Fig. 46. N°. 1. And as theſe Proportions between the *Radius* and the Sines, are the ſame in all Circles, any Line gr being taken and divided in the Proportion here directed, will ſerve for finding the Diviſions of the Figure $lmnr$; whence the Image of the Circle may be found without the trouble of drawing any ſeparate Model.

Fig. 46. N°. 3. The Diviſions of the Side NR of the ſecond Model are found, by biſecting it in D, and making DQ to DR, as 707 to 1000, or as 71 to 100, or nearly, as 7 to 10, and taking DT equal to DQ.

For DQ, or its equal fg, which is the Sine of 45 Degrees, is to the *Radius* DR very nearly in that Proportion.

If the Circles required be not very large, the Proportions expreſſed in the ſmaller Numbers, will ſerve to determine the Diviſions of the Model to a ſufficient Degree of Exactneſs.

Fig. 46. N°. 4. As to the third Model; the Rectangular Parallelogram ALMC being drawn, with its Side AL equal to one half of AC, the Diagonals AM, CL, and the Perpendicular sB are found of courſe; and the Point P is had by taking PL equal to one fifth of AL, whence PQ parallel to AC is found, without drawing the Semicircle ABC.

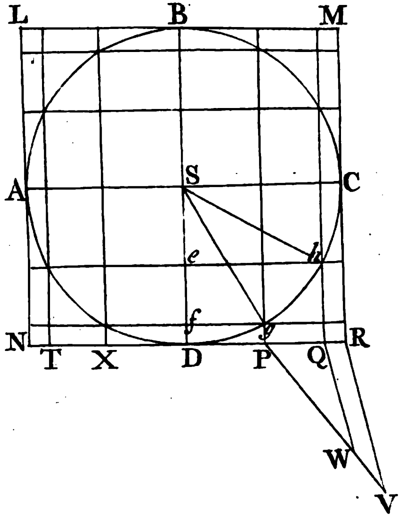
For in the Similar Triangles CAL, CDa, $CA : AL :: CD : Da$
But by Conſtruction $CA : AL :: 2 : 1$
Therefore $CD : Da :: 2 : 1$

And

[a] Prob. 20.

[b] Lem. 2.

Fig. 46. No. 2.



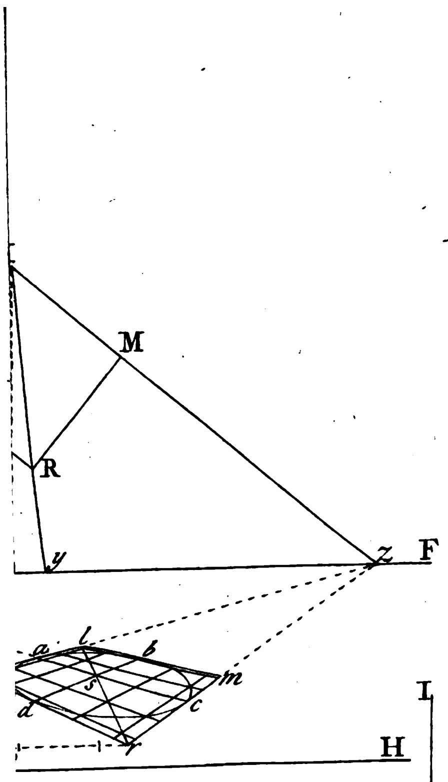
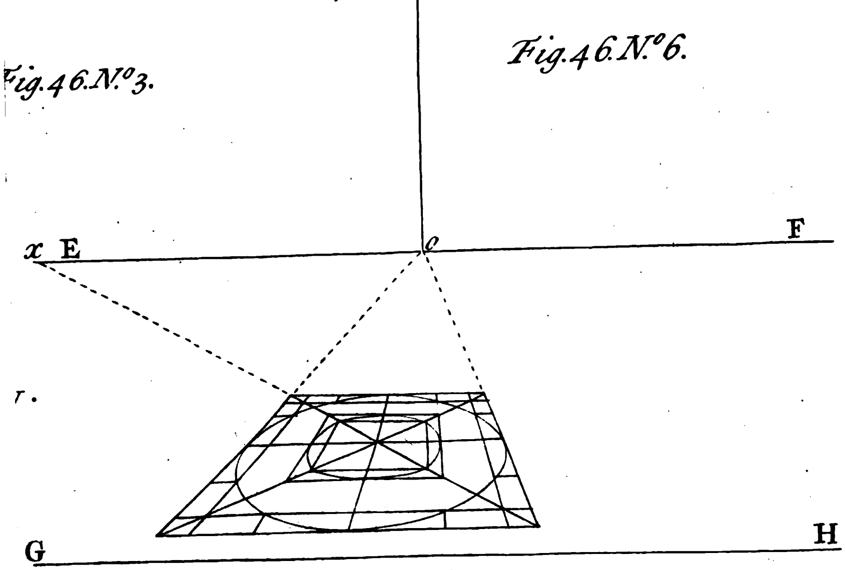


Fig. 46. No. 3.

Fig. 46. No. 6.



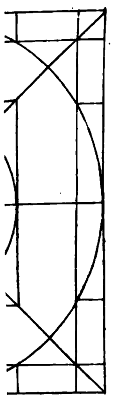
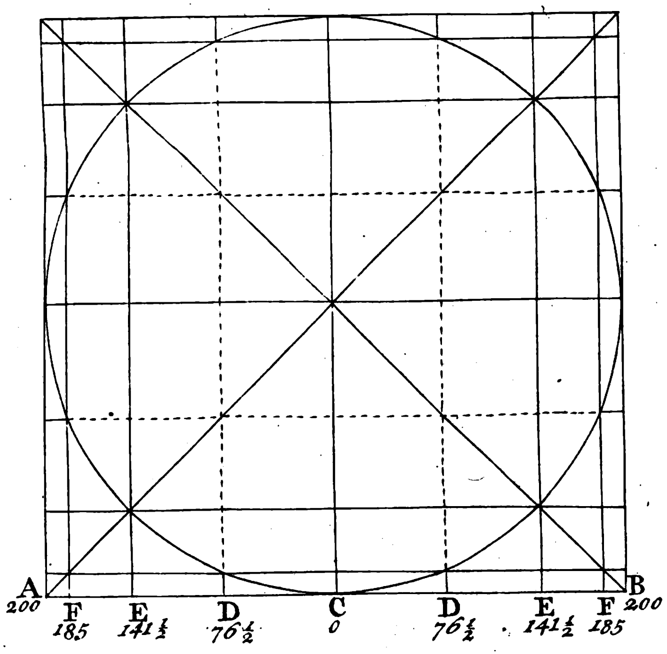


Fig. 46. No. 7.



J. Mynde sc.

And becauſe of the Semicircle ABC   CD : D*a* :: D*a* : DA
Therefore   D*a* : DA :: 2 : 1
And conſequently   2D*a*=CD : DA :: 4 : 1
But in the Similar Triangles CD*a*, *a*PL,   CD : *a*P=DA :: D*a*=AP : PL
Wherefore   AP : PL :: 4 : 1
And conſequently PL is one fifth of AL.

Or if AC be divided into five equal Parts, and from the Points of Diviſion D and F next adjoining to A and C, two Lines D*a*, F*b* be drawn perpendicular to AC, they will cut the Diagonals in the Points *a* and *b*, through which the Semicircle paſſes.

For by the above Demonſtration CD : DA :: 4 : 1. and it is evident that DA and FC are equal.

Fig. 46. N°. 7.

For large Circles, the Circumference may be divided into more Parts: if each Quadrant be divided into four, which gives ſixteen Points in the Circumference, the Model may be made on any Line AB, by biſecting it in C, and taking on each Side, three Diſtances CD, CE, CF, in the ſame Proportion to CA or CB, as 76½, 141½, and 185 are to 200 reſpectively, as in the Figure: the Angles and their Sines being nearly in thoſe Proportions. Or if the ſame Diviſion for twelve Points be retained as before, and there be added to the nearer half of the Model, the Diviſions through which the Diagonals paſs, theſe two additional Points will be a great help for drawing the Image of the nearer half of the Circle, which will need them more than the farther Semicircle, the Points through which this laſt paſſes falling cloſer together, as in the Figure.

Fig. 46. N°. 8.

As this Model will be found exact enough for the Deſcription of almoſt any Circle, it may be convenient to have a thin Braſs Ruler divided in the ſame manner as the Line AB of this Figure, by ſmall Notches in its Edge fit to receive the Point of a Compaſs, by the help of which any given Line may be eaſily and readily divided in the Proportion here required[a], without the trouble of taking the Numbers from a Diagonal Scale.

METHOD 2.

Fig. 47.

Having the Image *ab* of any Diameter of a Circle given; thence to find the Images of as many more Diameters as may be neceſſary to determine the intire Image of the Circle.

From I draw any two Radials I*y* and I*v*, making together a right Angle, and cutting the Vaniſhing Line EF in *y* and *v*; from *v* and *y* through the Extremities *a* and *b* of the given Diameter, draw *va*, *vb*, *ya*, and *yb*, Interſecting in *c* and *d*, and draw *cd*, which will be the Image of another Diameter; and the Point *s*, where *ab* and *cd* Interſect, will be the Image of the Center of the Circle. And after the ſame manner the Images of as many more Diameters may be obtained as are deſired, and a Curve Line drawn through their Extremities will be the Image of the Circle propoſed.

Dem. Becauſe the Angle *v*I*y* is Right, the Originals of the Angles *acb* and *adb* are right Angles[b], and *ab* being the Image of a Diameter, the Originals of the Points *c* and *d* are therefore Points in the Circumference of the Circle[c]. And becauſe the Originals of the Angles *vay* and *vby* are Right, the Originals of the Angles *cad* and *cbd* are alſo Right[d], wherefore the Original of *cd* which ſubtends theſe Angles, is a Diameter of the Circle, and the Interſection *s* of the Diameters *ab* and *cd* is therefore the Image of the Center. *Q. E. I.*

COR. 1.

If the Diameter *cd* be produced to its Vaniſhing Point *z*, then the Angles *z*I*y* and *y*I*x* will be equal.

For the Original of *adbc* being a Rectangular Parallelogram, the Originals of the Triangles *bca* and *bcd* are Similar; wherefore the Originals of the Angles *cab* and *cdb* are equal, and conſequently the Angles *y*I*x* and *z*I*y* made by their Radials are alſo equal[e].

COR. 2.

If the determinate Diameter *ab* be given, and any other Indefinite Diameter *cd* be drawn, having *z* for its Vaniſhing Point; the Extremities *c* and *d* of this Diameter may be found, either by biſecting the Angle *z*I*x* by the Line I*y*, or by biſecting *z*IC, the Complement to two Rights of the Angle *z*I*x* by the Line I*v*.

For the Line I*y* which biſects the Angle *z*I*x*, being perpendicular to I*v* which biſects the Angle *z*IC[f], Lines drawn from *y* and *v*, through the Extremities *a* and *b* of the given Diameter *ab*, cut each other in *c* and *d*, the Extremities of another Diameter

[a] Lem. 2.

[b] Cor. 3. Theor. 11. B. I.

[c] 31 El. 3.

[d] 13 El. 1.

[e] Cor. 3. Theor. 11. B. I.

[f] Cor. 1. Caſe 2. Prob. 15.

meter cd[a], the Radial of which Diameter makes the Angle zIy equal to the Angle yIx[b]; if therefore from z, the Indefinite Diameter cd be drawn, it is evident, that either ya and yb, or va and vb, will cut cd in the ſame Points c and d where they cut each other.

COR. 3.

Hence, if the Image of any other Diameter were required, whoſe Vaniſhing Point is out of reach; if the Angle, which the Original of the propoſed Diameter makes with the given Diameter, be known, its Image may be found.

Thus if dc be the given Diameter, and another Diameter ef be required, the Original of which may make an Angle with the Original of cd, equal to the Angle A; from I draw the Indefinite Radial Iw, making the Angle zIw equal to A, and biſect zIw by the Line Ir, to which draw It perpendicular; then td and tc, by their Interſections with rc and rd, will give e and f, the Extremities of the propoſed Diameter, and conſequently ef the Image of the Diameter required.

Or if the Indefinite Diameter ef be given, its Extremities e and f may be determined, either by the Point r or t, the Angle which the Original of ef makes with the Original of cd being known; by making the Angle zIw equal to that Angle, and thence finding the Points r and t as before directed[c].

COR. 4.

If the Indefinite Diameter ef be parallel to the Interſecting Line; then Iw and IR coinciding, the Angle zIR muſt be biſected by Ir, and It being drawn perpendicular to Ir, it will biſect zIN the Complement to two Rights of the Angle zIR[d], and t will be the true Point cf Diſtance of the Vaniſhing Point z[e], and r will be the ſame Diſtance ſet off on the other Side of z; the Angles RIr, rIz, and Irz, being by Conſtruction equal, and conſequently zr equal to Iz: and then either of the Points r or t will, by the help of c and d, determine the Extremities e and f of the Diameter ef; as the ſame Points r or t would determine the Extremities c and d of the Diameter cd, if ef, a Diameter parallel to the Interſecting Line, were given.

COR. 5.

Hence, if it be propoſed to draw through a given Point s, the Image of a Line tending to an inacceſſible Vaniſhing Point, the Angle which the propoſed Line makes with the Interſecting Line being known: from I draw an Indefinite Radial Iw, making with NR an Angle RIw equal to the propoſed Angle[f], and having through s drawn any Line cd, having its Vaniſhing Point z within reach, biſect the Angle zIw by the Line Ir, and draw It perpendicular to it, and having made sc and sd to repreſent equal Lines[g], draw rc, rd, and tc, td Interſecting in e and f; and a Line drawn through e and f, which will likewiſe paſs through s, will be the Indefinite Image ſought: and the Point of Diſtance of the Line ef being found, by biſecting the Angle NIw[h], any Parts of ef may then be determined by the uſual Methods[i], although its Vaniſhing Point is out of reach. Or if cs be made to repreſent a Line equal to the Diſtance between the Original of s and any propoſed Point in ef; tc, or rc will determine the Image of that Point, according as it lies either beyond or on this Side of s: sc, sf, and se conſtantly repreſenting equal Lines[k].

METHOD 3.

Fig. 48. The Images of any three Points a, b, and c in the Circumference of a Circle being given; thence to compleat the intire Image of the Circle.

Join the given Points a, b, and c, thereby forming a Triangle abc, and produce its Sides to their Vaniſhing Points x, y, and z, and draw their Radials Ix, Iy, and Iz; then to Ix the Radial of either of the Sides ab, whoſe Vaniſhing Point is one of the Extremes, draw Iv perpendicular, and draw a Radial Iw, making with Iv an Angle wIv equal to the Angle yIz, made by the Radials of the other two Sides ac and bc of the Triangle abc; draw wb and va Interſecting in d, on the ſame Side of ab with the Point c, then draw xd and vb Interſecting in e, laſtly draw ae cutting bd in s, and the Originals of ae and bd will be Diameters of the Circle; and their Interſection s will be the Image of the Center: by the help of which the Image of the Circle may be compleated, as in the preceeding Method.

Dem. For the Original of abc is a Triangle inſcribed in the Circle, and each of its Sides is a Chord of that Circle; if then ab be taken as the Chord of the Arch ahb, the Angle acb, whoſe Original is equal to the Angle yIz, is the Angle in the

Circle

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[a] Prob.

[b] Cor. 1.

[c] Cor. 2.

[d] Cor. 1. Caſe 2. Prob. 15.

[e] Meth. 2. Prob. 8.

[f] Theor. 11. B. I.

[g] Prob. 8.

[h] Meth. 2. Prob. 8.

[i] Prob. 8.

[k] Cor. 1. Caſe 2. Prob. 15.

Circle ſubtended by that Arch, to which all other Angles in the Circle ſubtended by the ſame Arch are equal[c], and the Angle *vIw* being made equal to the Angle *yIz*;[a] the Originals of the Angles *adb* and *acb* are equal, and conſequently *d* is the Image of a Point in the Circumference of the Circle: And becauſe *xIv* is a Right Angle, the Original of the Angle *dab* is a Right Angle, and conſequently *db* is the Image of a Diameter[c]; and becauſe the Original of the Angle *deb* is Right, the Point *e* is[b] alſo in the Image of the Circumference; laſtly, becauſe the Original of the Angle *ade* is Right, *ae* is the Image of another Diameter, and conſequently *s* is the Image of the Center. *Q. E. I.*

[a] 21 El. 3.

[b] 31 El. 3.

COR. 1.

If inſtead of the Point *w*, a Point *u* had been taken on the other Side of *v*, making the Angle *uIv* equal to the Angle *yIz* or *wIv*, the Point *u* would have equally ſerved the purpoſe; for *vb* and *ua* would have determined the Point *e*, and *xe* and *va* would have given the Point *d*.

COR. 2.

If any two other Points be taken in the Vaniſhing Line, ſo that their Radials may make together an Angle equal to *yIz*, then if thoſe Points be both on the ſame Side of *x*, they will determine a Point in the Image of the Circumference, on the ſame Side of *ab* with the Point *c*; but if the two Points be taken one on each Side of *x*, the Point determined in the Circumference by their Interſection, muſt lie on the contrary Side of *ab* from *c*.

Thus *w* and *v* determine *d*, and *v* and *u* give *e*, both on the ſame Side of *ab* with the Point *c*: but if I*r* and I*t*, making together an Angle equal to *yIz*, be drawn one on each Side of *x*, then the Lines from *r* and *t* muſt be ſo drawn to the Extremities of *ab*, as to interſect in *h* on the contrary Side of that Line.

For if inſtead of drawing *ra* and *tb*, the Lines *rb* and *ta* ſhould be drawn Interſecting in *g*, the Point *g* will not be in the Circumference of the Circle, becauſe the Angle *agb* being an outward Angle, its Original is not equal to the Angle *tIr*, but to the Complement to two Rights of that Angle[c]; but the Original of the Angle *ahb* being equal to this Complement, it is therefore the Angle ſubtended by the Arch *acb* of the Circle, ſeeing the Originals of the Angles *ahb* and *acb* are together equal to two Rights[d], and conſequently *h* is a Point in the Image of the Circumference.

[c] Cor. Prob. 3.

[d] 22 El. 3.

Note, *The Angle w*I*v is eaſily made equal to the Angle y*I*z, by drawing from* I *as a Center, any Arch* B*m*C *cutting* I*y*, I*v*, *and* I*z in l, n, and m, and making np equal to lm, and drawing* I*p: for the Arches pn and lm being equal, the Angles w*I*v and y*I*z are alſo equal*[e].

[e] 27 El. 3.

P R O B. XXV.

The Image of any Diameter *ab*, and of the Center *s* of a Circle being given; thence to find the Images of a regular Hexagon, and of two equilateral Triangles inſcribed in the Circle, having one of their Angles in one of the Extremities of the given Diameter; and alſo to find the Image of the Circle itſelf.

Fig. 49. N°. 1.

Having drawn I*x* the Radial of the given Diameter *ab*, from I as a Center, with any *Radius*, deſcribe on I*x* a Semicircle B*m*C, and divide it into three equal Parts in *l* and *m* (which is done by ſetting off the *Radius* IB, from B to *l*, and from *l* to *m*) and draw I*m* and I*l*, cutting EF in *z* and *y*, and from *z* and *y* through *s* draw *cd* and *ef*; then *cd* and *ef* will repreſent two Indefinite Diameters of the Circle, making with each other and with the given Diameter *ab*, Angles of 60 Degrees, and conſequently the Extremities of theſe Diameters *c*, *d*, *e*, and *f* being found, thoſe with the Points *a* and *b* will repreſent the ſix angular Points of the Hexagon required; which Points being joined by Lines, and likewiſe the alternate Angles as in the Figure, thereby *acfbde* the Image of the Hexagon, with two equilateral Triangles *afd* and *ceb* inſcribed in the Circle, will be obtained, and a Curve Line drawn through the ſix angular Points of the Figure will be the Image of the Circle deſired.

The Extremities *e* and *f* of the Diameter *ef* are determined by *za* and *zb*, the Line I*z* biſecting the Angle *y*IC, the Complement to two Rights of the Angle *y*I*x*, made by the Radials of the Diameters *ab* and *ef*; and the Extremities *c* and *d* of the Diameter *cd* are found by *ya* and *yb*, I*y* biſecting the Angle *x*I*z*[f]; which Points *c*

[f] Cor. 2. Method 2. and Prob. 24.

and d may alſo be had by xe and xf, the Angle xIy being equal to zIC, and therefore equal to half the Complement to two Rights of the Angle yIz made by the Radials of the Diameters ef and cd. *Q. E. I.*

COR. 1.

If the given Diameter ab be parallel to the Interſecting Line, that will be equivalent to the having the Vaniſhing Point x, the Lines which were before directed to be drawn from x, being here to be drawn parallel to E F, but the Practice in every other reſpect is the ſame as before.

COR. 2.

If either of the Vaniſhing Points as z be out of reach, ſo that neither the Indefinite Diameter cd, nor the Extremities e and f of the Diameter ef could be found by it; biſect the Angle xIy by the Line It (which here is done by making It perpendicular to Im) and ta and tb will give e and f[a], and ya and yb will cut xf and xe in c and d, by which cd is found.

For acf and edb being outward Angles, their Originals are equal to the Complement to two Rights of the Angle xIy, that is the Angle yIC, which is the Angle made by every two adjoining Sides of a regular Hexagon.

The Diameter cd and its Extremities c and d may be alſo found by biſecting the Angle yIm by the Line Ir, which will be perpendicular to Ix; for then rf and re will cut xf and xe in c and d, whereby cd is determined[b].

COR. 3.

Here t is the Vaniſhing Point of the Diagonals af and eb, and r is the Vaniſhing Point of the Diagonals ce and fd.

CASE 2.

If inſtead of the Diameter ab, the Side cf of the Hexagon, which is parallel to it, were given, the ſame Vaniſhing Points will ſerve for deſcribing the intire Figure.

For fc being produced to its Vaniſhing Point x, and the Points y and z being found as before, draw cz and fy Interſecting in s, and xs will give the Indefinite Diameter ab, which is terminated in a and b by yc and zf; laſtly yb and za cut cd and fe in d and e, whereby all the Angles being found, the intire Figure may be thence compleated. *Q. E. I.*

CASE 3.

Fig. 49. N°. 2. If the Side ab of one of the inſcribed Triangles were given, the whole Figure may from thence be compleated after the like manner.

For having produced the given Side ab to its Vaniſhing Point x, the Vaniſhing Points y and z of the two other Sides of that Triangle, are had by dividing the Semicircle BmC into three equal Parts as before[c], by which and ab the Triangle abc is found; then drawing Ir and It perpendicular to Ix and Iz, rc and ta will repreſent two Indefinite Diameters of the circumſcribing Circle, perpendicular to the Sides ab and bc of the Triangle abc, and their Interſection s will repreſent the Center; then tb will cut rc in its Extremity e, and ed being drawn to z, is terminated in d by tc; and thus the Side de of the contrary Triangle being found, the other two Vaniſhing Points y and x ſerve to compleat its Image def, whence the whole Figure may be finiſhed.

For the Originals of the Sides of the Triangle def are reſpectively parallel to the oppoſite Sides of the Triangle abc, and have therefore the ſame Vaniſhing Points. *Q. E. I.*

COR.

Here t is the Vaniſhing Point of the Sides be and cd of the Hexagon, and r of the Sides ad and bf, but the Vaniſhing Point of ea and fc is out of reach.

PROB. XXVI.

Fig. 50. The Image of any Diameter ab, and of the Center s of a Circle being given; thence to find the Images of a regular Octagon, and of two Squares inſcribed in the Circle; and alſo the Image of the Circle itſelf.

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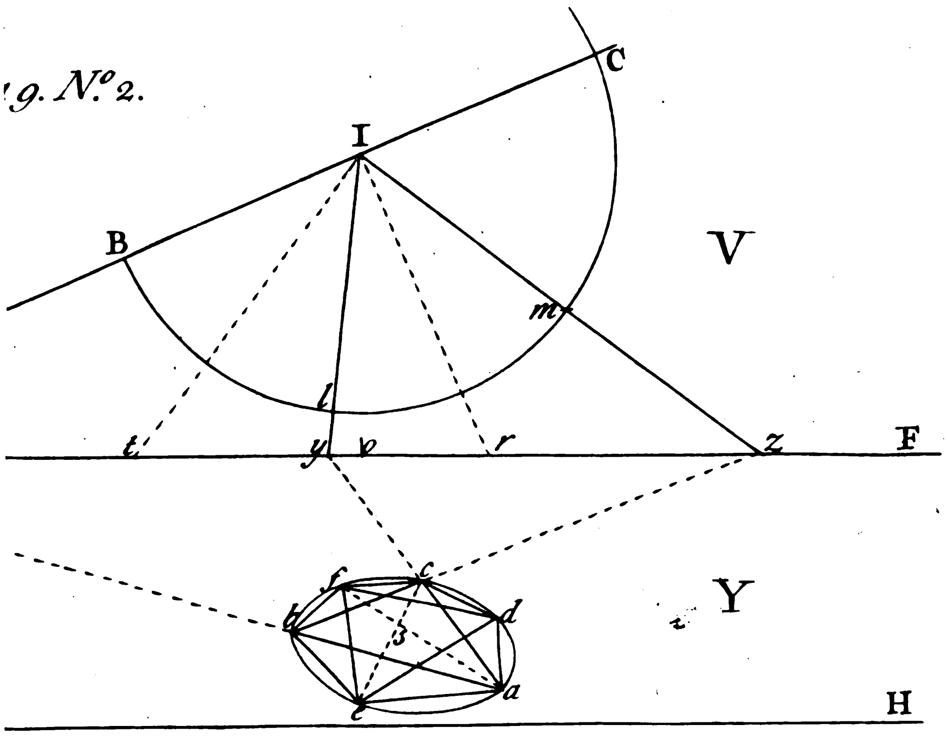
[a] Cor. 2. Method 2. Prob. 24.

[b] Cor. 3. Method 2. Prob. 24.

[c] Cor. Method 1. Prob. 19.

g. N°. 2.



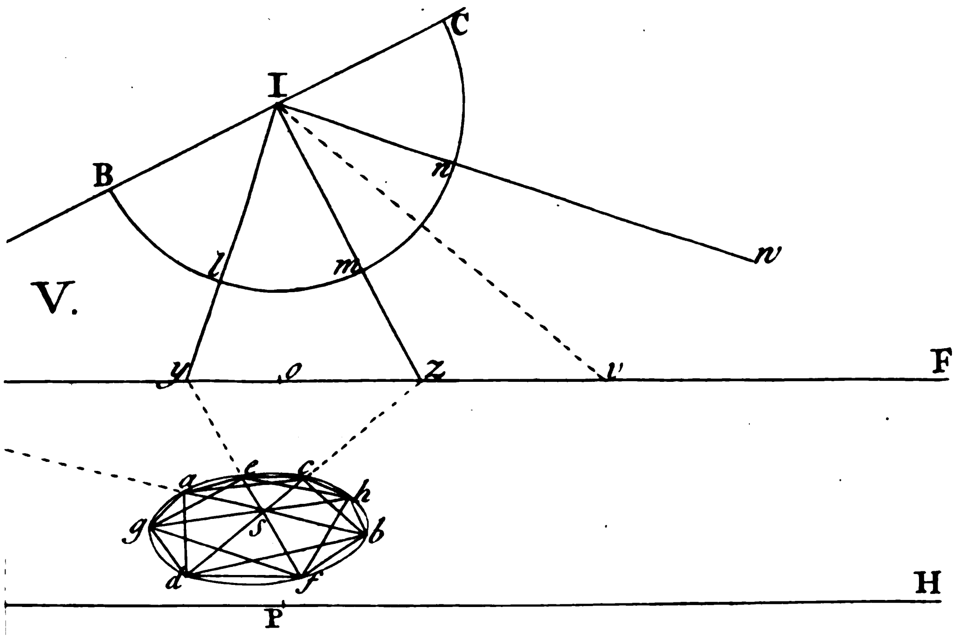
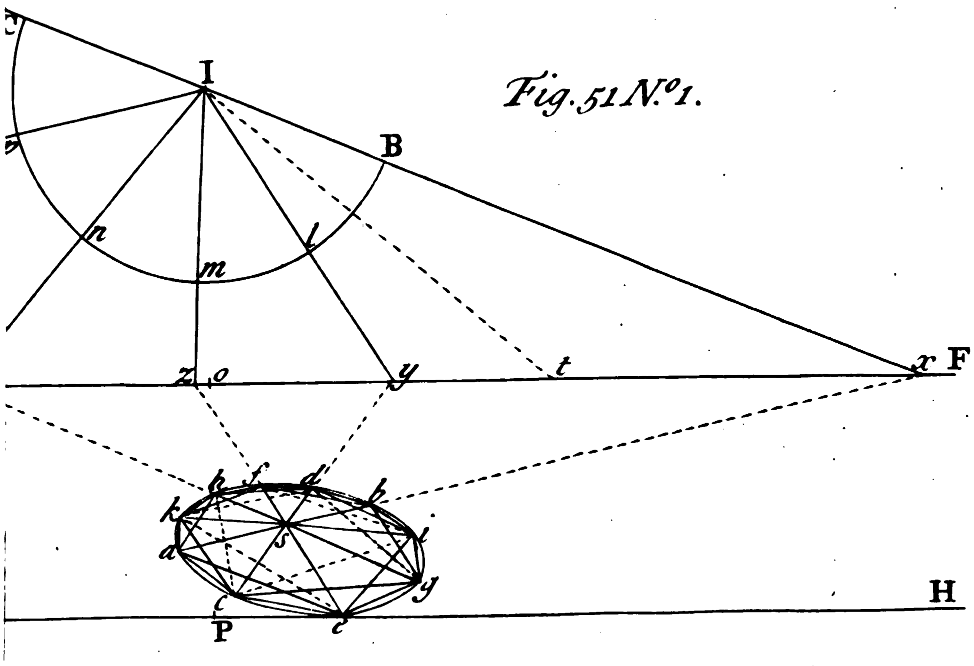


Fig. 51 N°. 1.



J. Mynde sc.

Having drawn Ix and the Semicircle BmC as before, divide BmC into four equal Parts by the Points l, m, and n, and draw the Radials Il, Im, and In, cutting EF in y, z, and w, if the laſt be within reach, otherwiſe it may be omitted: then from y and z through s, draw the Indefinite Diameters ef and cd, which, with the given Diameter ab, will be the Images of three of the four Diameters of the Circle, through the Extremities of which the propoſed Octagon muſt paſs, they making with each other Angles of 45 Degrees.

Then becauſe Iy biſects the Angle xIz, ya and yb determine d and c, the Extremities of the Diameter dc; biſect yIC, the Complement to two Rights of the Angle xIy, by the Line Iv, and va and vb will give e and f, the Extremities of the Diameter ef; laſtly, becauſe Iz biſects the Angle yIw, and Ix by Conſtruction is perpendicular to Iz, xf and xe will cut ze and zf in g and h, whereby the remaining Diameter gh will be found[a]; and the Extremities of theſe four Diameters being joined in Order by ſtraight Lines, will give $agdfbhce$ the Image of the Octagon deſired; and the alternate Angles of that Figure being alſo joined, will give $adbc$ and $egfh$ the Images of two Squares inſcribed in the Circle; and drawing a Curve Line through the angular Points of the Figure, the Image of the Circle will be thereby obtained. *Q. E. I.*

[a] Cor. 2 and 3. Method 2. Prob. 24.

Note, *What was ſaid at Cor.* 1. *of laſt Problem is equally applicable here.*

CASE 2.

If inſtead of the Diameter ab, the Side fg of one of the inſcribed Squares which is parallel to it, were given, the ſame Vaniſhing Points will ſerve for deſcribing the intire Figure.

For fg being produced to its Vaniſhing Point x, and the Points y, z, and v being found as before, zg and yf give the Triangle gef, and zf and xe compleat the Image of the Square $fgeh$, and its Diagonal gh being drawn, gives the Center s; then xs gives the Indefinite Diameter ab, whoſe Extremities a and b are found by ve and vf; laſtly ya and yb cut the Indefinite Diameter zs in its Extremities d and c, by which the reſt of the Figure may be compleated as before. *Q. E. I.*

PROB. XXVII.

The Image of any Diameter ab, and of the Center s of a Circle being given; thence to find the Images of a regular Decagon, and of two regular Pentagons inſcribed in the Circle, and alſo the Image of the Circle itſelf.

Fig. 51. N°. 1.

Having drawn Ix and the Semicircle BmC as before, divide BmC into five equal Parts by the Points l, m, n, and p; and from I through each of theſe Points draw Lines cutting EF in y, z, v, and w, if the laſt be within reach, otherwiſe it may be omitted; and from y, z, and v through s, draw the Indefinite Diameters cd, ef, and gh, which, with the given Diameter ab, will be four of the five Diameters, through the Extremities of which the Image of the propoſed Decagon muſt paſs; the Originals of thoſe Diameters dividing the Circumference of the Circle into ten equal Parts.

Then becauſe Iv biſects the Angle CIy, the Complement to two Rights of the Angle yIx, va and vb give c and d the Extremities of the Diameter cd; and becauſe Iz biſects the Angle vIy, zc and zd determine h and g, the Extremities of the Diameter gh; and becauſe the Angle xIz is equal to the Angle vIC, and therefore equal to half the Complement to two Rights of the Angle vIz, the Lines xg and xh give e and f, the Extremities of the Diameter ef; laſtly, becauſe Iv biſects the Angle wIz, draw It perpendicular to Iv, and then te and tf will cut vf and ve in i and k, the Extremities of the remaining Diameter ik, whoſe Vaniſhing Point is out of reach[b]; and thus all the angular Points of the Decagon being found, the reſt is compleated as in the Figure, and a Curve Line drawn through thoſe Points, will be the Image of the Circle propoſed. *Q. E. I.*

[b] Cor. 2 and 3. Method 2. Prob. 24.

COR.

The Indefinite Diameter ki, and its Extremities k and i may alſo be determined without the Point t, either by the Points z and y, or by v and x: for za and zb cut yh and yg in k and i; the Originals of the outward Angles akh and big being equal to the Complement to two Rights of the Angle zIy, which is the Angle made

made by every two adjoining Sides of a regular Decagon; wherefore the Points *k* and *i* are thereby rightly determined.

Or *ve* and *vf* will cut *xd* and *xc* in the ſame Points *k* and *i*. For it is evident, that in a regular Decagon, a Line drawn through the two next angular Points on the ſame Side of any Diameter, is parallel to that Diameter; therefore *ve* and *vf*, which repreſent Parallels to the Diameter *gb*, paſſing through *f* and *e*, two angular Points of the Figure next adjoining to *g* and *b*, muſt likewiſe paſs through *k* and *i* the correſponding oppoſite angular Points; and for the ſame reaſon *xc* and *xd*, which paſs through *c* and *d*, the next angular Points to the Diameter *ab*, muſt likewiſe paſs through *i* and *k*, the correſponding oppoſite angular Points; conſequently the Point *k* being both in *ve* and *xd*, and the Point *i* in *vf* and *xc*, thoſe Points are determined by the correſponding Interſections of theſe Lines.

Note, *What was ſaid at Cor.* 1. *Prob.* XXV. *is likewiſe applicable here.*

CASE 2.

Fig. 51. N°. 2. If inſtead of the Diameter of the circumſcribing Circle, any Side *ab* of either of the inſcribed Pentagons were given, the intire Figure may be thence compleated by the like Method.

For having produced the given Side *ab* to its Vaniſhing Point *x*, the Vaniſhing Points *y*, *z*, *v*, and *w* of the reſt of the Sides are found by dividing the Semicircle B*m*C into five equal Parts as before[a], by the help of which, and of the given Side *ab*, the Pentagon *abcde* may be compleated[b]. And the Vaniſhing Points thus found being alſo the Vaniſhing Points of the Sides of the contrary Pentagon, in regard that the oppoſite Sides of both Pentagons are parallel, this laſt may be alſo deſcribed, any one of its Sides being firſt determined, which may be done in this manner.

[a] Cor. 2. Prob. 22.

[b] Prob. 22.

Having drawn I*r* and I*t* perpendicular to I*x* and I*v*, the Radials of any two adjoining Sides *ab* and *ae* of the Pentagon *abcde*, draw *rd* and *tc* Interſecting in *s*; then *rd* will repreſent a Diameter of the circumſcribing Circle paſſing through the Angle *d*, perpendicular to the oppoſite Side *ab* of the Pentagon *abcde*, and *tc* will repreſent another Diameter perpendicular to the Side *ae*, and the Point *s* will repreſent the Center; and biſecting the Angle *r*I*t* by the Line I*u* (which is here perpendicular to I*w*) *ud* and *uc* will give *f* and *g* the other Extremities of thoſe two Diameters, and conſequently the Side *fg* of the contrary Pentagon, by which the intire Figure may be compleated. *Q. E. I.*

COR.

Here *r* is the Vaniſhing Point of the Sides *fe* and *bc* of the Decagon, *u* of the Sides *ek* and *bh*, and *t* of *kd* and *gb*; and if a Radial were drawn perpendicular to I*y*, it would cut EF in the Vaniſhing Point of the Sides *af* and *ci*, but the Vaniſhing Point of the Sides *ag* and *di* is out of reach.

LEM. 4.

Fig. 52. From a given Point K without a Circle ADBE, to draw two Tangents to the Circle.

From K to O the Center of the Circle draw KO, and biſect it in *s*, and from *s* as a Center with the *Radius s*O, deſcribe the Arch DOE, cutting the given Circle in D and E, then KD, KE will be Tangents to the Circle in D and E.

Dem. Draw OD, OE; then becauſe KO is the Diameter of the Circle KDOE the Angles KDO, KEO are Right[c]; and OD and OE being each a *Radius* of the Circle ADBE, KD and KE which are perpendicular to OD and OE, are therefore Tangents to this Circle in D and E[d]. *Q. E. I.*

[c] 31 El. 3.

[d] 16 El. 3.

COR.

The Chord DE, which joins the Points of Contact D and E, is perpendicular to KO, and biſected by it in C.

For OD and OE being equal, and the Angles KDO, KEO Right, the Triangles KDO, KEO, which have the Side KO common to both, are Similar and equal[e]; wherefore the Sides KE and KD, and the Angles DKO, OKE are equal, and in the Iſoſceles Triangle DKE, the Angle DKE being biſected by KC, that Line alſo biſects DE in C[f], and DE is therefore perpendicular to KC[g].

[e] 4 El. 1.

[f] 3 El. 6.

[g] 3 El. 3.

PROB. XXVIII.

Fig. 53. The Image of any Diameter *ab*, and of the Center *s* of a Circle be-

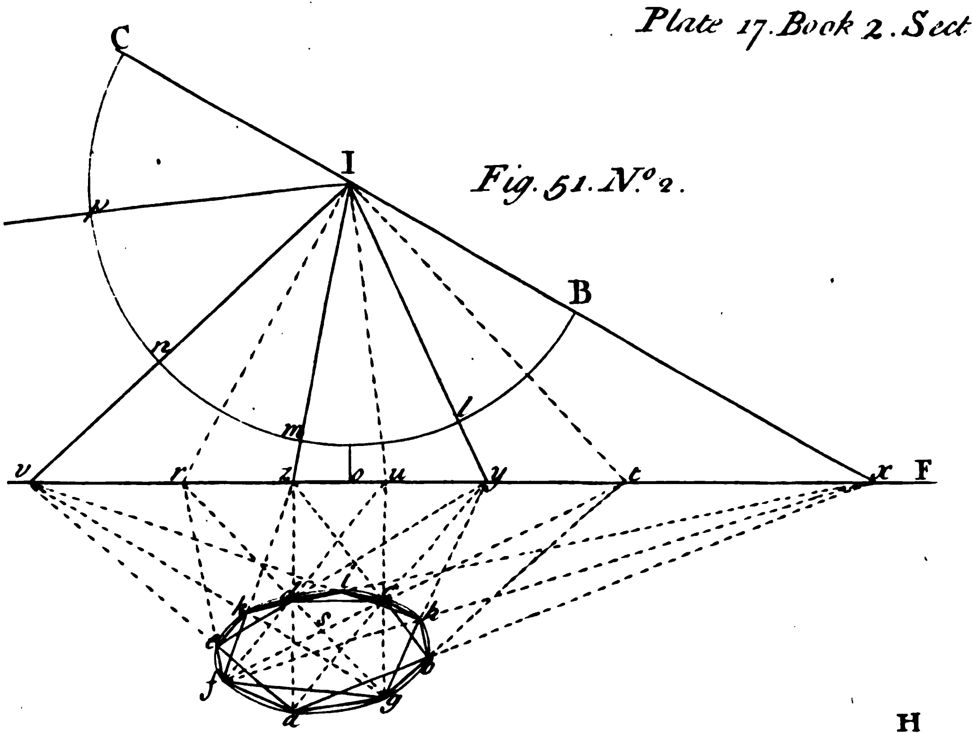


Fig. 51. No. 2.

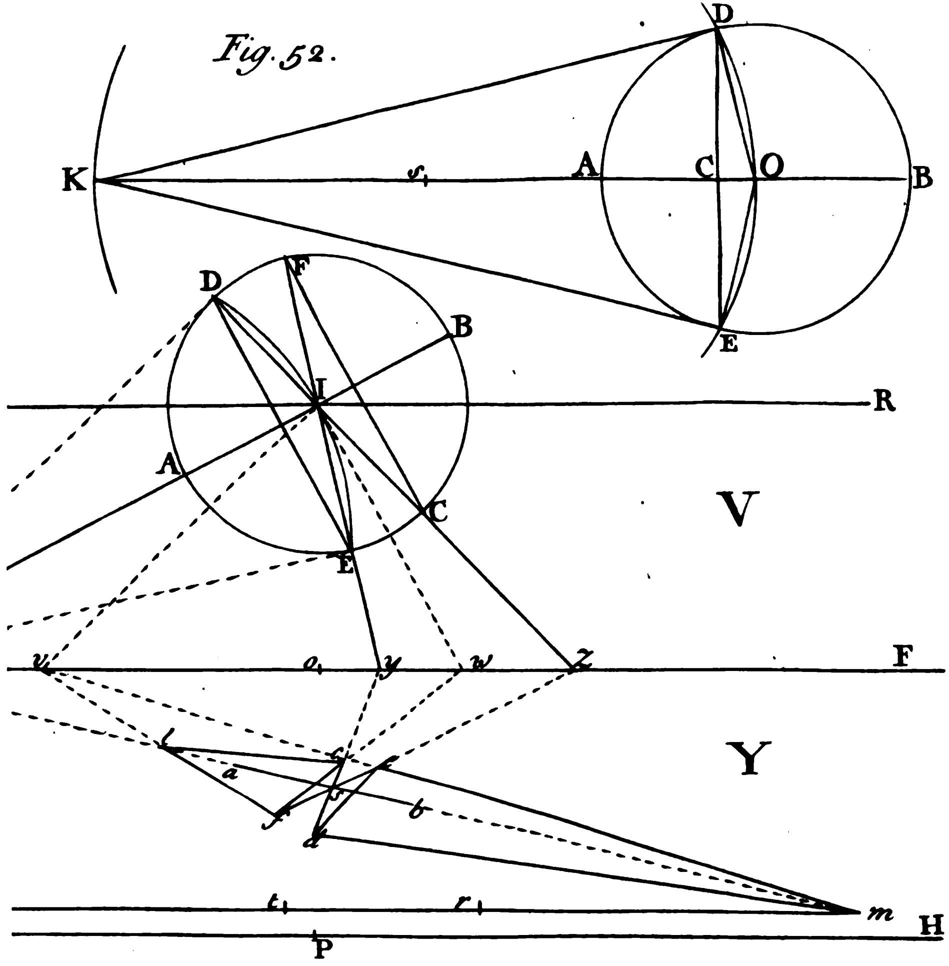


Fig. 52.

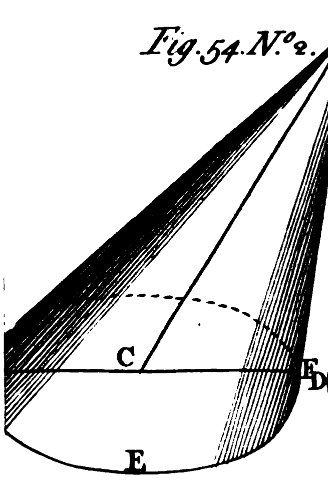


Fig. 54. No. 2.

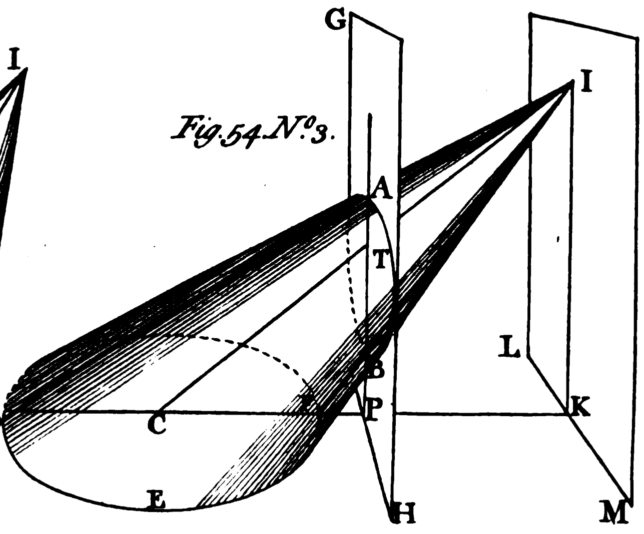


Fig. 54. No. 3.

J. Mynde sc.

ing given; from any Point m in that Diameter produced without the Circle, to draw the Images of two Tangents to the Circle.

Produce ab to its Vaniſhing Point x, and find the Proportions of the Originals of ms and bs[a]; which let be as mt to rt: and having drawn the Radial xI, make AI to xI, as rt to mt, and from I as a Center, with the *Radius* IA, deſcribe the Circle ADBE, and find D and E, the Points where Tangents from x meet the Circle[b], and draw DI and IE cutting the Vaniſhing Line in z and y: then it is evident, that z and y will be the Vaniſhing Points of the Diameters of the Circle, through the Extremities of which the Tangents from m paſs.

Therefore from y and z through s draw two Indefinite Diameters cd and ef, then draw Iv perpendicular to Iz, and Iw perpendicular to Ix, cutting the Vaniſhing Line in v and w; laſtly, draw vm cutting ef in e, and we cutting cd in d, and me and md will be the Images of the Tangents deſired.

Dem. Becauſe the Angle zIv is Right, the Originals of me and fe are perpendicular, and fe being the Indefinite Image of one of the Diameters of the Circle, through whoſe Extremity one of the Tangents paſſes, and to which that Tangent is perpendicular, me is therefore the Image of that Tangent, and cuts the Indefinite Diameter fe in its proper Extremity e; ſeeing no Line drawn from m can repreſent a Perpendicular to fe, but what hath v for its Vaniſhing Point. And becauſe Iw is perpendicular to Ix, the Originals of de and xm are perpendicular; and thus the Original of de paſſing through e one of the Points of contact, and being perpendicular to the Original of ms, it is therefore the Chord of the Tangents to the Circle from the Original of m[c], and conſequently cuts the Diameter cd, through which the other Tangent paſſes, in its proper Extremity d, and md is therefore the other Tangent required. *Q. E. I.*

COR. 1.

If the Point l were the given Point, from whence the Tangents were required to be drawn, the Originals of ls and sm being ſuppoſed equal; the ſame Points v and w would ſerve to determine the Extremities of the Diameters, through which the Tangents paſs, only with this difference, that from m, the Tangents paſs through the Extremities d and e of the Diameters cd and ef, but from the Point l they muſt paſs through c and f the contrary Extremities of thoſe Diameters; and then vl gives the Point f, and wf gives the Point c, and lf and lc are the Images of the Tangents deſired.

COR. 2.

If the Diſtance from m to s be ſo great in Proportion to the Semidiameter sb, that xI being divided in that Proportion, would make the *Radius* IA too ſmall for determining the Diameters DC and EF, and conſequently the Points z and y with ſufficient exactneſs; Ix may be produced beyond x at pleaſure to k, and kI and AI being made in the Proportion required, AI will be thereby enlarged, and then the Tangents from k will determine the Diameters DC and EF to a greater nicety.

For it is evident, that while the Proportion of the *Radius* AI to the Diſtance of x from I continues the ſame, whether thoſe Diſtances be increaſed or diminiſhed, the Triangles xID in either Caſe will be Similar, ſo that the Inclination of DI to xI will always be the ſame.

COR. 3.

If the given Diameter ab be parallel to the Interſecting Line, the only difference is, that inſtead of xI, the Line NR muſt be uſed; and as in this Caſe ms and bs would be in the ſame Proportion to each other as their Originals, the trouble of finding that Proportion is ſaved.

COR. 4.

If the Tangents were required to be drawn from the Directing Point of ab; find the Proportion of bs to ſo much of the Original of ab, as lies between s and its Directing Point[d], and proceed as before, only obſerving, that the Images of the Tangents muſt be drawn through d and e parallel to each other, and to the given Diameter ab, they all having by Suppoſition the ſame Directing Point[e], and conſequently a Line drawn from v parallel to ab will give the Point e.

COR. 5.

If the Originals of the Tangents required were to be parallel to the given Diameter ab; it is evident they muſt paſs through the Extremities of a Diameter, whoſe

[a] Cor. 2. Prob. 9.

[b] Lem. 4.

[c] Cor. Lem. 4.

[d] Cor. Prob. 12.

[e] Cor 4. Theor. 12. B. I.

Original is perpendicular to that of ab. Therefore if from w through s an Indefinite Diameter be drawn, and its Extremities determined by any of the Methods before propoſed, Lines from x to thoſe Extremities will be the Tangents ſought.

PROB. XXIX.

To find the Image of the Plan, or Ichnography of a Building, Fortification, Pavement, Garden, or any other Figure in the Original Plane, having ſome kind of regularity, when the Original Figure propoſed conſiſts of ſo many Sides, of ſuch different Inclinations, that it would be tedious to find their ſeveral Vaniſhing Points and Dimenſions by the Rules already given.

This may be done by incloſing the Original Figure in a Parallelogram, ſubdivided by Lines parallel to its Sides, paſſing through the principal or moſt remarkable Points or Places of the Original Figure.

For having found the Image of this Parallelogram and its Subdiviſions [a], thoſe Parts of the Original Figure, which are incloſed in any of the Subdiviſions of the Parallelogram, will have their Images in the correſponding Subdiviſions of the Image of that Parallelogram, to which with a little care they may be transferred. *Q. E. I.*

COR.

It is not neceſſary that either of the Sides of the Original Parallelogram ſhould be parallel to the Interſecting Line, nor that it ſhould be Rectangular, but it may be drawn in ſuch manner, as it may moſt commodiouſly agree with the form of the Original Figure to be deſcribed.

Thus if in the Original Figure there be any conſiderable Number of parallel Lines, one of the Sides of the Parallelogram may be drawn parallel to them, whereby their Vaniſhing Point will be the ſame with the Vaniſhing Point of that Side; and the Original Lines and their Images will then more naturally fall in with the Subdiviſions of the Parallelogram, and its Image.

PROB. XXX.

To find the Image of the Plan of a Town, Field, or Country, where there is no regularity in the Situation of the Houſes, Rivers, Trees, or other Things there to be deſcribed.

This may be done by deſcribing a Figure upon the Original Plane, the Image of which may be a Parallelogram ſubdivided into ſmaller ones in any Proportion as may beſt ſuit with the Deſign [b]; and the Parallelogram being accordingly drawn in the Picture, what of the Original Plane is incloſed within any of the Subdiviſions of the Figure, muſt be transferred to the correſponding Subdiviſions of the Parallelogram in the Picture, and ſo the whole Image may be compleated. *Q. E. I.*

COR.

In theſe Caſes it may be moſt convenient ſo to draw the Figure on the Original Plane, as that its Image may be a Parallelogram, having its Sides parallel to thoſe of the Picture; for then the Images of all ſuch Objects as are incloſed within the Figure in the Original Plane, will come within the Bounds of the Picture.

Thus if the Picture be rectangular, as is moſt uſual, and its Baſe agree with the Interſecting Line of the Original Plane; the Figure to be drawn on this Plane may have the Interſecting Line for one of its Sides, and conſequently the Subdiviſions parallel to that Side will alſo be parallel to the Interſecting Line [c]; and the contrary Sides and their Parallels will tend to the Foot of the Eye's Director [d].

But if the Picture were intended for the ſloping Side of a Staircaſe, ſo as to have the Angles of its Frame oblique, it may then be better that the Figure in the Original Plane be ſo drawn, as that its Image may be Similar to the Frame of the intended Picture.

SCHOL.

It is unneceſſary to add more Examples for finding the Images of Figures lying in an Original Plane, ſeeing the Rules already given for determining the Images of Points, Angles, Lines, and Parts or Diviſions of Lines (which altogether compoſe a kind of

Perſpective

[a] Prob. 20.

[b] Prob. 21.

[c] Theor. 2. B. I.

[d] Cor. 3. Theor. 12. B. I.

Perſpective Geometry) are fully ſufficient for the Deſcription of the intire Image of any Figure propoſed, the Image of any one Line in it, and the Proportions of the Sides and Angles of the Original Figure being given; for if the Original Figure be rectilinear, its Image may be found either by dividing the Original into Triangles, and finding the Images of thoſe Triangles in order one after another; or by finding the Images of all the angular Points, and joining them by ſtraight Lines; or laſtly, by incloſing the Original Figure in a Parallelogram, conveniently ſubdivided: which laſt Method is generally uſeful for the Deſcription of the Images of curvilinear Figures, or of ſuch as are ſo irregular, that the other ways would be inconvenient. But it muſt be obſerved, that the Method of finding the Images of Objects by the help of the Vaniſhing Points of their Sides, when it can be done, has this peculiar Advantage, that when the neceſſary Vaniſhing Points are once found, they ſerve not only for the Original Figure propoſed, but for all other Figures in the Original Plane which are Similar to it, and alike poſited with reſpect to the Interſecting Line, let their Sizes be ever ſo different.

Thus if a Floor, or Pavement were compoſed of Triangles, Parallelograms, Hexagons, or any other Similar Figures; the Vaniſhing Points which ſerve for any one of them, ſerve equally for all the reſt; and the ſame Points will alſo ſerve for deſcribing any given Figure, ſuch as a Pentagon, Hexagon, *&c.* within or about a Figure of the ſame kind: examples of all which any one who underſtands the Rules here taught, may eaſily form to himſelf, by which Exerciſe he may profit more than if the Figures were ready drawn for him.

GENERAL COROLLARY.

It appearing from what has been ſhewn, that the Shapes of the Images of all Figures in the Original Plane, and the Places of thoſe Images in the Picture, depend on the Angles made by the Directors of their Sides, and on the Proportions between the Radials, Directors, and the Parts of the Original Lines themſelves, all which continue the ſame whatever Inclination the Picture hath to the Original Plane, while the Interſecting and Directing Lines, and the Place of the Eye in the Directing Plane remain unaltered[a]; and the Angle of Inclination of the Picture to the Original Plane being no wiſe concerned in the Demonſtrations of any of the Propoſitions of this Book; it follows, that when once the Interſecting and Directing Lines, and the Place of the Eye in the Directing Plane are choſen, the Shape of the Image of any Figure in the Original Plane, and the Place of that Image on the Picture become fixed, and receive no Alteration by any Change made in the Inclination of the Picture to the Original Plane. And as in all the Rules hitherto laid down, the Original Plane has been conſidered abſtractedly, without any regard to its Poſition with reſpect to the Horizon, thoſe Rules are alike applicable to any Plane whatſoever, whether it be the Side, Ceiling, or Floor of any Building, or any other Plane, either in a direct, inclining, reclining, or otherwiſe oblique Situation, with reſpect to the Picture, the Ground, or the Eye; the Work on every particular Plane being to be guided by its own peculiar Vaniſhing, Interſecting, and Directing Lines, and the other Points and Lines which depend on them.

[a] Gen. Cor. after Def. 20. B. 1.

STEREOGRAPHY,
OR A
COMPLEAT BODY
OF
PERSPECTIVE,
In all its BRANCHES.

BOOK III.

IN Problem XXIV. of the preceeding Book, the Methods of finding the Image of a Circle were deduced from the Conſideration of the Properties of the Circle itſelf; but as the Image of a Circle, in whatever Poſition it be, except when its Plane paſſes through the Eye, is always a regular Curve, it may not be amiſs here to mention ſome things touching the Nature and Properties of the Curves thus generated, and of ſuch Lines and Points relating to them, as are neceſſary to determine their Figure; and to ſhew how the Originals of ſuch Lines and Points may be aſcertained in the Original Plane, and from thence to derive Methods for finding the principal Lines themſelves in the Picture, without the Aſſiſtance of the Original Plane; and laſtly to propoſe ſome eaſy ways of drawing thoſe Curves by the help of the Lines thus found.

This will naturally lead us to take notice of ſome of the firſt Principles of Conick Sections; but as that is a Science of itſelf diſtinct from what is here handled, though of great Affinity to it, we ſhall not trouble the Reader with the Demonſtrations of thoſe Principles, but refer, as to the Proof of them, to the Writers on that Subject.

SECTION I.
Of the ſeveral Curves produced by the Image of a Circle in different Poſitions.

1. THE Rays by which a Circle is ſeen, form a Cone, of which the Circle is the Baſe, and the Eye *the Vertex*, and a Line drawn from the Eye to the Center of the Circle is *the Axe of that Cone.*

Fig. 54. N°. 1. Thus let DEF repreſent a Circle, and I the Spectator's Eye; the Line IC drawn from the Eye to the Center of the Circle is *the Axe*, and the Lines ID, IF, and all others that can be drawn from the Eye to the Circumference of the Circle, compoſe the Conick Surface, and each of thoſe Lines is called *a Side of the Cone.*

2. If the Axe be perpendicular to the Plane of the Circle, then the Cone is Equilateral or a Right Cone; and all Lines drawn from the Eye to the Circumference of the Circle are then equal, as are alſo the Angles of Inclination of the Sides to the Plane of the Baſe; but if the Axe incline to the Plane of the Circle, the Sides of the Cone, as well as their Angles of Inclination to the Baſe, will be unequal, and it is then called *a Scalene Cone.* Fig. 54. N°. 2.

3. A Circle being thus conſidered as the Baſe of a Cone, of which the Eye is the Vertex, it follows, that the Image of any Circle is one or other of the Conick Sections, it being the Section of the Conick Surface by the Plane of the Picture; and every Line

drawn

drawn from the Eye to any Point of the Circular Baſe, may be ſaid to *form* the Image of that Point by its Interſection with the Picture; ſo that every poſſible Point in the Circular Baſe will have an Image, unleſs the forming Line, which ought to produce that Image, be parallel to the Picture.

4. The ſeveral Sections of the Cone, are the Triangle, the Circle, the *Ellipſis*, the *Parabola*, and the *Hyperbola*.

5. If a Plane IDF paſſing through the Vertex I of any Cone IDEF, cut its Baſe in any Line DF, the Section of the Cone by that Plane will be a Triangle DIF: but this Section can have no Place in *Stereography*, in regard that if the cutting Plane be taken as the Picture, and the Vertex of the Cone as the Eye, the Eye muſt then be ſuppoſed to be in the Plane of the Picture, which in *Stereography* cannot be[a]. Fig. 54. N°. 1, 2, 3.

6. If any Cone having a Circular Baſe, be cut by a Plane parallel to that Baſe, the Section will be a Circle.

7. If a Scalene Cone IDEF be cut by a Plane IDF paſſing through its Axe IC, perpendicular to the Plane of the Baſe DEF, thereby forming a Triangle IDF, and if the ſame Cone be cut by another Plane GH, perpendicular to the Plane of the Triangle IDF, and cutting that Plane in AB in ſuch manner, that the Angles at the Baſe of that Section may be equal to the contrary Angles at the Baſe of the Cone, that is, that the Angle IBA may be equal to the Angle IDF, or the Angle IAB equal to the Angle IFD; then the Section ATB of the Cone by the Plane GH will be a Circle, and the Cone is then ſaid to be *cut Subcontrarily* by that Plane. Fig. 54. N°. 3.

8. If a Plane ILM touch a Cone IDEF only in its Vertex I, and the Cone be cut by any other Plane GH parallel to the Plane ILM; the Section of the Cone by the Plane GH will be an *Ellipſis*, unleſs this Plane be either parallel to the Baſe of the Cone, or cut it ſubcontrarily, in either of which Caſes the Section will be a Circle, as already mentioned[b].

And here the Plane ILM being taken as the Directing Plane, the Plane GH as the Picture, and the Plane of the Baſe DEF as the Original Plane; it is evident, that the Plane GH being parallel to none of the forming Lines, every Point of the Circular Baſe has a real Image, and that therefore the intire Image or Section ATB muſt be one continued Figure returning into itſelf.

9. Hence it follows, that if a Circle DEF in an Original Plane lie wholly on one Side of the Directing Plane ILM, its Image muſt be either an *Ellipſis* or a Circle.

10. If a Plane ILM touch a Cone IDEF in either of its Sides ID; the Section EAH of that Cone, by any Plane GH parallel to the Plane ILM, is called a *Parabola*. Fig. 54. N°. 4.

And here the Planes ILM and GH being taken to repreſent the Directing Plane and Picture as before, the Part EAH of the *Parabola* thus formed, is the Perſpective of EFH, ſuch Part of the Circular Baſe as lies beyond the Picture; and the Remainder of the *Parabola* which lies below E and H, is formed by the Projections of thoſe Parts of the Circular Baſe which lie between E and D, and H and D; every Point of which has a real Projective Image, except only the Point D, whoſe forming Line ID is the only one which is parallel to the Plane of the Section: ſo that the Sides AE and AH of the *Parabola* muſt be Indefinite, and can never meet together to compoſe a Figure returning into itſelf, and the *Parabola* muſt therefore remain open at that end, for want of the Image of the Point D, which ought to cloſe it, and which is infinitely diſtant[c].

11. And as the Points in the Circular Baſe which lie neareſt to D, are projected fartheſt off, and the Lines which form the Images of thoſe Points, approach nearer and nearer to the Line ID, as the Points themſelves do to D; thoſe forming Lines may be conceived ultimately to coincide with ID, and conſequently the Indefinite Sides AE and AH of the *Parabola*, formed by the Projections of thoſe Points, may be conceived to become ultimately parallel to ID, which may be taken as the Director of the infinitely ſmall Parts of the Circular Baſe adjoining to D, to which the Images of thoſe infinitely ſmall Parts are therefore parallel; ſeeing an infinitely ſmall Part of a Curve may be conſidered as a ſtraight Line.

12. Hence it follows, that if a Circle DEH in an Original Plane, touch the Directing Line in any Point D, its Image muſt be a *Parabola*.

13. If the Sides of a Cone IDEF be produced beyond its Vertex I, ſo as to form an oppoſite Cone IPGQ, and theſe two Cones be cut by a Plane NRLM paſſing through their common Vertex I, thereby forming two oppoſite Triangles NIR, LIM; any Plane GH parallel to the Plane NRLM, muſt cut both thoſe Cones, and the Fig. 54. N°. 5.

[a] Art. 3. Sect. 3. B. I.

[b] Art. 6, 7.

[c] Cor. 1. Theor. 4. B. I.

Sections EAH and GBT thereby produced, are called *oppoſite Hyperbolas*, and are every way equal and Similar. They are alſo called *oppoſite Sections*, there being no other Section of a Cone by any Plane not paſſing through its Vertex, beſides the *Hyperbola*, which has an oppoſite, in regard that no Plane can cut both the oppoſite Cones, unleſs it be parallel to ſome Plane, as LMNR, which cuts them through their common Vertex.

And here the Planes NRLM and GH being conſidered as the Directing Plane and Picture, the Part EAH of one of the *Hyperbolas* thus formed is the Perſpective of EFH, ſuch Part of the Circular Baſe as lies beyond the Picture; and the Remainder of that *Hyperbola* which falls below E and H, is formed by the Projections of EL and HM, ſuch Parts of the Circular Baſe as lie between the Picture and Directing Plane; and the oppoſite *Hyperbola* GBT is produced by the Tranſprojection of LDM, that Part of the Circular Baſe which lies behind the Directing Plane: but in regard the Points L and M of the Circular Baſe lie in the Directing Line, thoſe two Points can have no Images, wherefore the Sides AE and AH of the *Hyperbola* AEH, as alſo the Sides BG and BT of the oppoſite *Hyperbola* GBT, are Indefinite, and can never meet to cloſe thoſe Figures.

14. The Indefinite Sides AE and BT of the oppoſite *Hyperbolas*, the Original of the infinitely diſtant Extremities of which is the Directing Point L, may be conceived to become ultimately parallel to the Line LIR, the Director of the infinitely ſmall Parts of the Circular Baſe adjoining to L; and the contrary Sides AH and BG of thoſe Sections, may be conceived to become ultimately parallel to MIN, the Director of the infinitely ſmall Parts of the Circular Baſe adjoining to the Point M, for the like reaſon as mentioned in Article 11. and the oppoſite *Hyperbolas* muſt therefore be two ſeparate and indefinite Curves, each open at one end, neither of which can ever meet or interfere with the other; ſeeing the Parts of the Circular Baſe which are infinitely near L and M on the Side of the Picture, are projected at an infinite Diſtance below E and H in the *Hyperbola* EAH; and the Parts of the Circular Baſe infinitely near the ſame Points L and M on the contrary Side of the Directing Plane, are tranſprojected at an infinite Diſtance above T and G in the oppoſite *Hyperbola* GBT.

15. Hence it follows, that if a Circle DEH in an Original Plane cut the Directing Line in any two Points L and M, the Image of that Circle will be two oppoſite *Hyperbolas*.

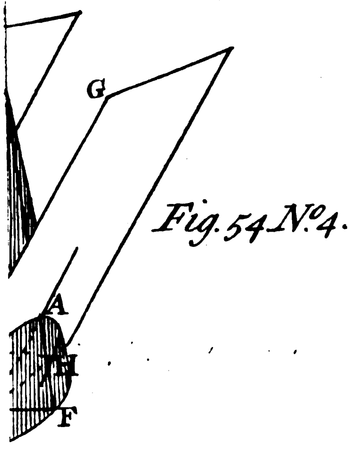
16. If a Cone be cut by any Number of parallel Planes (whereof none paſs through its Vertex) all the Sections of that Cone by thoſe Planes will be Similar, and of the ſame Denomination; and therefore the Species of the Section produced by the Image of a Circle, doth not depend on the Situation of the Picture or cutting Plane with reſpect to the Circle, but on the Poſition of the forming Circle with regard to the Directing Line of its Plane; and when once the Situation of the Directing Plane with reſpect to the forming Circle is determined, the Species of the Section to be produced is alſo determined, wherever the Picture be placed, whether before or behind, or ſo as to cut the Original Circle; the Picture in all Situations being conſtantly ſuppoſed parallel to the Directing Plane.

17. If from any Point without the Plane of a Conick Section or the oppoſite Sections, there be drawn Lines to every poſſible Point in the Section or Sections, thoſe Lines will altogether compoſe a Conick Surface, of the ſame kind with the Cone which is produced from a Circular Baſe.

Fig. 54. N°. 3. For as the Cone IDEF is produced by Lines drawn from I to the ſeveral Points of the Circular Baſe DEF, the Poſition of thoſe Lines with reſpect to each other is nowiſe altered by whatever Plane the Cone is cut; if then the Cone be cut by the Plane GH, whereby the *Ellipſis* BTA is produced, it is evident, that if this *Ellipſis* be taken as a new Baſe, Lines drawn from I to the ſeveral Points of this new Baſe, will produce the ſame Cone as before.

Fig. 54. N°. 4. Likewiſe if the Plane of the *Parabola* EAH be taken as an Original Plane, the Plane ILM as its Vaniſhing Plane, and the Plane of the Circle DEH as a Picture, and from any Point I in the Plane ILM, there be drawn Lines to the ſeveral Points of the *Parabola* indefinitely produced; it is evident, that all the Lines which form the Cone IDEF, meet the *Parabola* ſomewhere, except only the Line ID which is parallel to its Plane, and that therefore thoſe Lines will, by their Interſection with the Picture, form the Circle DEF, which will be compleat except only for want of the Point D; but in regard this Point is formed by a Point at an infinite Diſtance in the Plane of the *Parabola*, the Image of that Point becomes a Vaniſhing Point in LM the

Plate 18. Book 3 Sect. 1.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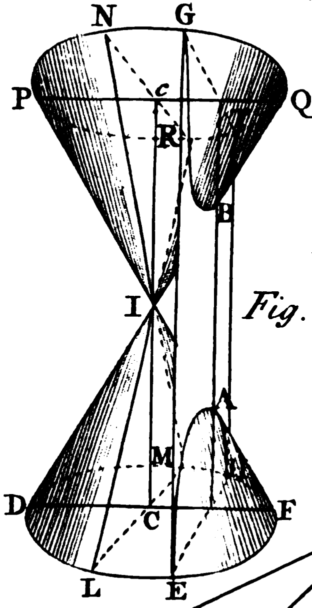
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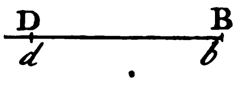


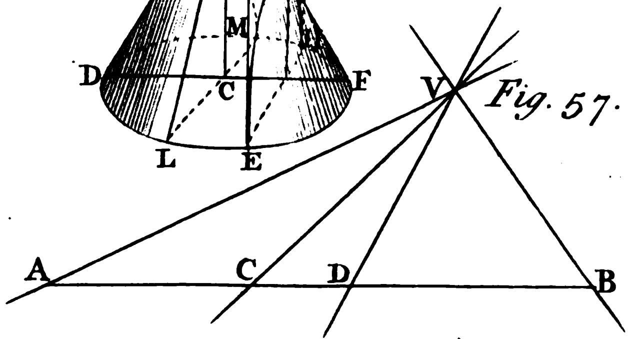
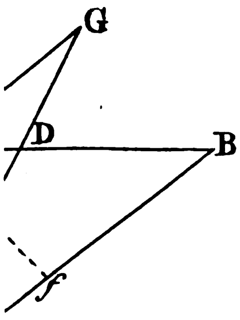


Fig. 57.



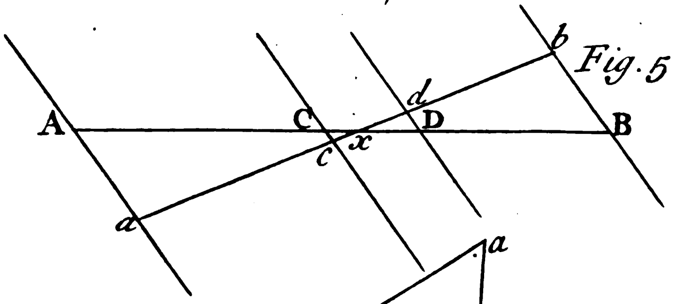
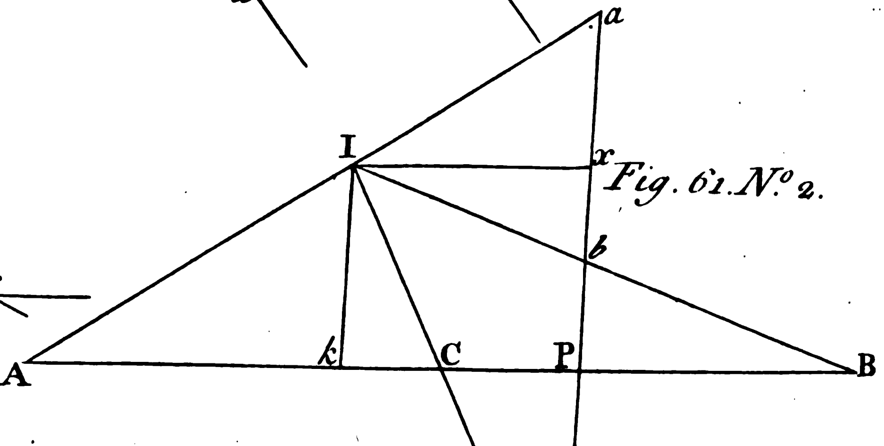


Fig. 58.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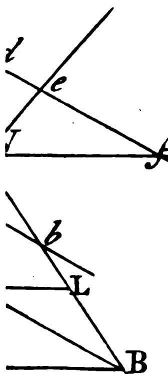
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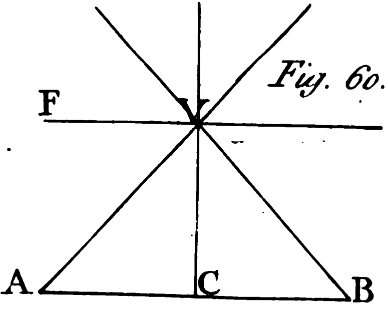
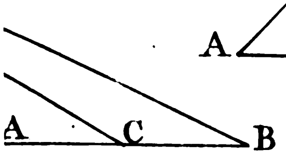
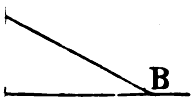


Fig. 60.



61. No 3.



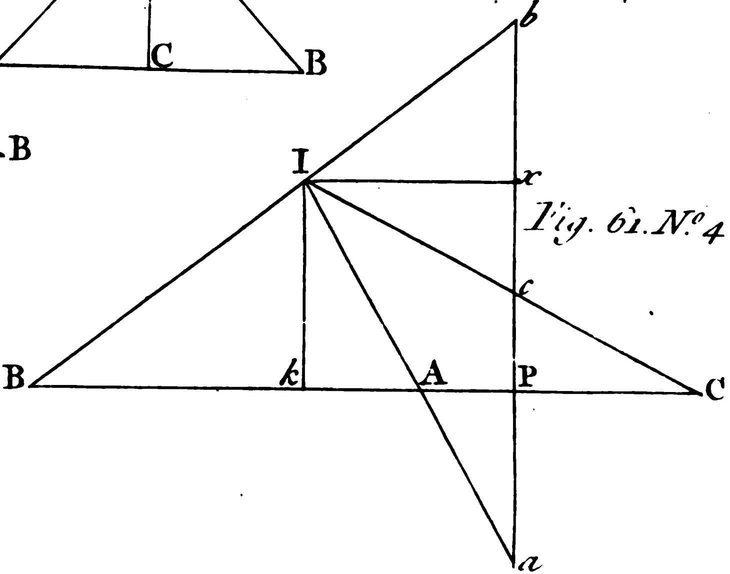


Fig. 61. No 4.

J. Mynde sc.

the Vaniſhing Line of that Plane, which Vaniſhing Point is the ſame with the Point D, where the Images of the infinitely diſtant Extremities of the Sides AE and AH of the *Parabola* unite and compleat the circular Image, and conſequently the Cone IDEF.

Laſtly, if the Plane of the oppoſite *Hyperbolas* EAH, GBT be taken as an Original Plane, the Plane NRLM as its Vaniſhing Plane, and the Plane of the Circle DEF as the Picture, and from any Point I in the Plane NRLM there be drawn Lines to the ſeveral Points of the oppoſite *Hyperbolas* indefinitely produced; it is evident, that all the Lines which form the oppoſite Cones IDEF and IPGQ, will meet the one or the other of the oppoſite *Hyperbolas* ſomewhere, except only the Lines IL and IM which are parallel to the Plane of the *Hyperbolas,* and that therefore thoſe Lines will, by their Interſections with the Picture, form the Circle DEF, which will be compleat, excepting only for want of the two Points L and M, which laſt Points having their Originals at an infinite Diſtance in the Plane of the *Hyperbolas,* do therefore become Vaniſhing Points in the Vaniſhing Line LM of that Plane, and conſequently the Vaniſhing Point L will be the Point of meeting of the Images of the infinitely diſtant Extremities of the Sides AE and BT of the oppoſite *Hyperbolas,* which are repreſented by FL and DL, and M will be the Point of Concourſe of the Images of the infinitely diſtant Extremities of the contrary Sides AH and BG of the *Hyperbolas,* which are repreſented by FM and DM, by which means the intire circular Image DLFM, and conſequently the oppoſite Cones will be compleated. Fig. 54. N°. 5.

18. And as the Cones thus produced from any Conick Section as a Baſe, can be ſo cut by a Plane as to produce a Circle, it is apparent the ſame Cones may be likewiſe cut by other Planes ſo as to produce any other of the Conick Sections.

19. If two Conick Sections agree in any five Points; or if they touch a given ſtraight Line in the ſame Point, and agree with each other in three more Points; or if they touch two given Straight Lines in the ſame two Points, and agree in one other Point; in either of theſe Caſes, thoſe two Conick Sections will agree intirely, and coincide with each other.

Of the Properties of Lines Harmonically divided.

Having thus premiſed ſome things touching the Sections of a Cone in general, and given Rules to know, according to the Situation of the Original Circle with reſpect to the Directing Line, which of the Conick Sections will be produced by the Image of the Circle, we ſhould proceed to the other Enquiries before propoſed; but in order thereto, it will be convenient firſt to lay down ſome Propoſitions touching the Properties of Lines Harmonically divided, by way of *Lemmas,* for the eaſier Demonſtration of what ſhall be advanced on this Subject, in which we ſhall employ the Remainder of this Section.

DEF. 1.

If a Line AB be ſo divided into three Parts by the Points C and D, as that the whole Line AB may have the ſame Proportion to either of the extreme Parts AC, as the other extreme Part DB hath to the middle Part CD; or which is the ſame, if the Rectangle between the whole Line AB and its middle Part CD, be equal to the Rectangle between the extreme Parts AC and DB, then the Line AB is ſaid to be *Harmonically divided in the Points* A, B, C, *and* D. Fig. 55.

LEM. 1.

To divide a given Line Harmonically.

1. Let AB be the given Line, and C one of its intermediate Points of Diviſion, and let it be required to find a fourth Point D between C and B, ſo that the given Line AB may be thereby divided Harmonically in A, C, D, and B. Fig. 56.

From any Point E without AB, draw EA and EB, and through the given intermediate Point C, draw GF parallel to EB cutting EA in F, make CG equal to CF, and draw EG, which will cut AB in D, the Point ſought.

Dem. Becauſe of the Similar Triangles AEB, AFC, AB : AC :: EB : FC=CG
And becauſe of the Similar Triangles BED, DGC, BD : DC :: EB : CG
Therefore AB : AC :: BD : DC
And conſequently AB is Harmonically divided in A, B, C, and D.

2. If the Point ſought were required to fall between C and A; through C draw *fg* parallel

parallel to E A cutting E B in *f*, and having taken C*g* equal to C*f*, draw E*g*, which will cut A B in *d* the Point deſired, by which A B will alſo be Harmonically divided in A, *d*, C, and B.

The Demonſtration of this is the ſame as before, the Triangles B A E, B C*f*, and A E *d*, *dg*C being Similar.

3. If any three adjoining Points of Diviſion be given, the fourth, which muſt be an extreme Point, is found after the ſame manner.

Thus if A, *d*, and C be given, and the Point B be deſired; through C, the intermediate Point neareſt to the Extremity required, draw *gf* parallel to E A, cutting E*d* in *g*, and C*f* being made equal to C*g*, the Line E*f* gives the Point B.

Or if B, D, and C be given, and the Extremity A required; through C the neareſt Point to A, draw F G parallel to E B cutting E D in G, and thereby the Point F, and thence A, may be found. *Q.E.I.*

COR. 1.

The middle Part C D muſt always be leſs than either of the Extremes A C or D B. For as already ſhewn $\mathrm{DB} : \mathrm{DC} :: \mathrm{BE} : \mathrm{CG} = \mathrm{CF}$.

If then D C and D B be equal, C F will be equal to B E, and conſequently C B and F E will be parallel, their Interſection A will therefore be infinitely diſtant, ſo that no extreme Point can then be found to compleat the Harmonical Diviſion of that Line. And if D C be larger than D B, C F being then alſo larger than B E, C B and F E will converge beyond B, and the leſſer Part D B will then become the middle Part.

COR. 2.

If in a determinate Line A B, any Point whatever be taken as C, a fourth Point D or *d*, on either Side of C within that Line may be found, which will compleat its Harmonical Diviſion; for wherever the Point C is taken between A and B, a Line C F or C*f* may be drawn through it, parallel to B E or A E, cutting A E or B E in ſome Point F or *f*, whence the Point D or *d* may be found as in this *Lemma*.

LEM. 2.

If two Lines Harmonically divided, being laid upon each other, agree in any three Points of Diviſion, whereby one Part in the one, muſt neceſſarily agree with a Part in the other; the fourth Point of each will alſo agree, provided the agreeing Part be either an extreme Part, or the middle Part of both.

Fig. 55. Let AB and *ab* be the two given Lines laid upon each other; and firſt, let the Points A, B, and C of the one, agree with the Points *a*, *b*, and *c* of the other, whereby the Parts A C and *ac*, which are both extreme Parts, as alſo the whole Lines AB and *ab* do agree; it muſt be proved, that the Points D and *d* alſo agree.

Dem. Becauſe of the Harmonical Diviſion of AB, $\mathrm{AB} : \mathrm{AC} :: \mathrm{DB} : \mathrm{DC}$
And for the ſame reaſon in the Line *ab* $ab : ac :: db : dc$
But $\mathrm{AB} = ab$ and $\mathrm{AC} = ac$ as before, therefore $\mathrm{DB} : \mathrm{DC} :: db : dc$
And by Compoſition $\mathrm{DB} + \mathrm{DC} = \mathrm{CB} : \mathrm{DC} :: db + dc = cb : dc$
But C B is equal to *cb*, therefore $cb : \mathrm{DC} :: cb : dc$
Conſequently $\mathrm{DC} = dc$
And therefore the Points D and *d* coincide.

Again, let the Points A, C, and D agree with the Points *a*, *c*, and *d*, by which means the extreme Parts A C and *ac*, and the mean Parts C D and *cd* agree; it muſt be ſhewn, that the Points B and *b* alſo agree.

Becauſe of the Harmonical Diviſion of A B, $\mathrm{AC} : \mathrm{CD} :: \mathrm{AB} : \mathrm{DB}$
And for the ſame reaſon in the Line *ab* $ac : cd :: ab : db$
But $\mathrm{AC} = ac$ and $\mathrm{CD} = cd$, therefore $\mathrm{AB} : \mathrm{DB} :: ab : db$
And by Diviſion $\mathrm{AB} - \mathrm{DB} = \mathrm{AD} : \mathrm{DB} :: ab - db = ad : db$
But AD is equal to *ad*, therefore $ad : \mathrm{DB} :: ad : db$
Conſequently $\mathrm{DB} = db$
And therefore the Points B and *b* agree.

After the ſame manner it may be ſhewn, that if the Points C, D, and B agree with the Points *c*, *d*, and *b*, the Points A and *a* will alſo agree. *Q.E.D.*

DEF. 2.

Fig. 57. If a Line A B be Harmonically divided in A, B, C, and D, and from any Point V without that Line, there be drawn four Lines V A, V C, V D, and V B through

the Points of Diviſion of A B, theſe four Lines produced both ways from V, are called *Harmonical Lines.*

D E F. 3.

And if through the ſame Points A, C, D, and B, four Lines be drawn parallel to each other, and making any Angle whatſoever with A B, thoſe four Lines are called *Harmonical Parallels.* Fig. 58.

L E M. 3.

If four Harmonical Parallels A*a*, C*c*, D*d*, B*b*, formed by a Line A B Harmonically divided in A, B, C, and D, be cut by any other Line *ab*; the Line *ab* will be Harmonically divided in its Interſections with thoſe Parallels. Fig. 58.

Dem. If *ab* and A B were parallel, it is evident they would be divided in the ſame Proportion by the Harmonical Parallels, ſeeing the correſponding Parts in each would be equal; and if the Lines A B and *ab* croſs each other in any Point *x*, either within or without the Harmonicals, the Triangles *x*A*a*, *x*C*c*, *x*D*d*, *x*B*b*, will ſtill be Similar, and conſequently the Segments *ca*, *cx*, *xd*, *db*, will have the ſame Proportion to CA, C*x*, *x*D, and D B, as *xb* hath to *x*B, and will therefore be reſpectively proportional; and the Line AB being by Suppoſition Harmonically divided in A, B, C, and D, the Line *ab* will therefore be divided Harmonically in the correſponding Points *a*, *b*, *c*, and *d*. *Q. E. D.*

L E M. 4.

If a Line A B be biſected in C, and from any Point V without that Line, there be drawn three Lines VA, VC, and VB cutting AB in A, C, and B, and through the ſame Point V another Line VF be drawn parallel to AB; then the four Lines VA, VC, VB, and VF, produced on both Sides of the Point V, will be Harmonical Lines. Fig. 59.

Dem. Having drawn any Line BF cutting all the four Lines VA, VC, VB, and VF in E, D, B, and F; through D draw HL parallel to AB, which will therefore be biſected in D, AC and CB being equal by Suppoſition.

Now in the Similar Triangles FVB, DLB, FB : DB :: FV : DL = HD
And in the Similar Triangles FVE, EHD, FV : HD :: FE : ED
Wherefore FB : DB :: FE : ED

That is, the Line FB is Harmonically divided in the Points F, B, E, and D [a], and conſequently the Lines VA, VC, VB, and VF are Harmonical Lines [b]. *Q. E. D.*

L E M. 5.

If the Angle AVB made by any two Lines VA and VB, be biſected by a Line VC, then if another Line VF be drawn through V perpendicular to VC, the four Lines VA, VC, VB, and VF will be Harmonical Lines. Fig. 60.

Dem. Through C draw AB perpendicular to VC, then the Triangles VCA, VCB, being every way Similar and equal, the Line AB will be biſected in C; but AB being perpendicular to VC, is therefore parallel to VF; conſequently the Lines VA, VC, VB, and VF are Harmonical Lines [c]. *Q. E. D.*

L E M. 6.

If four Harmonical Lines VF, VE, VD, and VB, formed by the Line FB Harmonically divided in the Points F, B, D, and E, be cut by any other Line *fb* parallel to FB, the Line *fb* will alſo be divided Harmonically in the correſponding Points *f*, *b*, *d*, and *e*. Fig. 59.

Dem. Becauſe the Parts *fe*, *ed*, *db* will be reſpectively proportional to the Parts FE, ED, and DB [d]. *Q. E. D.*

L E M. 7.

If four Harmonical Lines VF, VA, VC, and VB formed by the Line *fb* Harmonically divided in *f*, *b*, *d*, and *e*, meet in the Point V; then any Line HL, drawn parallel to any one of the Harmonicals, as VF, will cut the other three, and be biſected by them in the Point D. Fig. 59.

Dem. Firſt it is plain, the Line HL muſt cut the three Harmonicals VA, VC, and VB, ſeeing none of them are parallel to VF, to which HL is parallel by Suppoſition.

Through D, the middlemoſt Point of HL, draw FB parallel to *fb*, then FB will be Harmonically divided in the Points F, E, D, and B [e].

[a] Def. 1.

[b] Def. 2.

[c] Lem. 4.

[d] Lem. 2. B. I.

[e] Lem. 6.

Now in the Similar Triangles F V E, E H D, FV : HD :: FE : ED
And in the Similar Triangles F V B, DLB FV : DL :: FB : DB
[a] Lem. 6. But becauſe FB is Harmonically divided as already ſhewn[a], FE : ED :: FB : DB
Wherefore FV : HD :: FV : DL
And conſequently HD = DL

Therefore H L cuts three of the Harmonicals, and is biſected by them in the Point D. *Q. E. D.*

C O R.

If the Angle made by any two of the four Harmonical Lines, not adjoining together, be Right, the Angle comprehended between the other two will be biſected by the intermediate Line.

For if V F and V D be perpendicular, H L drawn parallel to V F will alſo be perpendicular to V D, and H L being biſected in D[b], the Rectangular Triangles V D H, V D L are Similar, and conſequently the Angles D V H, D V L are equal, that is, the Angle E V B is biſected by V D. ([b] Lem. 7.)

L E M. 8.

Fig. 59. If four Harmonical Lines VA, VC, VB, and VF meeting in V, be cut any where by a Line F B, that Line will be Harmonically divided by them in the Points F, E, D, and B.

Dem. Through any of the Diviſions of FB, as D, draw H L parallel to one of the Harmonicals V F, ſo that the Point D may be between H and L, then H L will be biſected in D[c]. ([c] Lem. 7.)

Now in the Similar Triangles FVE, EHD, FE : ED :: FV : HD=DL
And in the Similar Triangles F V B, DLB, FB : DB :: FV : DL
Wherefore FE : ED :: FB : DB

Conſequently F B is Harmonically divided in the Points F, E, D, and B. *Q. E. D.*

C O R. 1.

Fig. 61. N°. 1, 2. If in an Original Line, any Part AB be taken and biſected in C, not its Directing Point; whether the Part taken, lie all on the ſame ſide, or part on one ſide and part on the other of its Directing Point; the Indefinite Image of that Line will be Harmonically divided by the Images of A, B, and C, and its Vaniſhing Point.

Let I$x$$k$P repreſent the Radial Plane of an Original Line kB, in which the Part A B is taken and biſected in C: then becauſe A C and C B are equal, the Lines IA, IC, IB, and Ix, which laſt is always parallel to A B, are Harmonical Lines[d]; therefore the Indefinite Image Px, which cuts all the four Harmonicals (it being parallel to none of them) is Harmonically divided by them in a, b, c, and x[e]. ([d] Lem. 4. [e] Lem. 8.)

C O R. 2.

Fig. 61. N°. 3, 4. If in the Indefinite Image of a Line, any Part ab be taken and biſected in c, not its Vaniſhing Point; whether ab lie wholly on one ſide, or part on one ſide and part on the other of its Vaniſhing Point; the Indefinite Original of that Line will be Harmonically divided by the Originals of a, b, and c, and its Directing Point.

For ab being biſected in c, and Ik being parallel to it, the Lines Ia, Ic, Ib, and Ik are Harmonical Lines, therefore the Indefinite Original kP, which cuts all theſe four Harmonicals (it being parallel to none of them) is Harmonically divided by them in A, B, C, and k.

C O R. 3.

Fig. 61. N°. 5, 6. If either of the Points A, B, or C of the Original Line be its Directing Point, the Indefinite Image will be biſected by the Images of the two other Points and its Vaniſhing Point; and *vice verſa*, if either of the Points a, b, or c of the Indefinite Image be its Vaniſhing Point, the Indefinite Original will be biſected by the Originals of the two other Points and its Directing Point.

For in the firſt Caſe Ik is one of the Harmonicals, to which the Indefinite Image is parallel, and therefore cuts the other three, and is biſected by them; In the other Caſe Ix is one of the Harmonicals, to which the Original Line is parallel, and therefore cuts the other three, and is biſected by them[f]. ([f] Lem. 7.)

C O R. 4.

Fig. 61. N°. 7, 8. If an Original Line be Harmonically divided in the Points A, B, C, and D, neither of which is its Directing Point; whether that Line lie wholly on one ſide, or part on one

ig. 61. N.o 5.

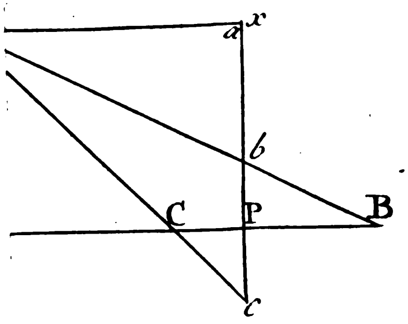
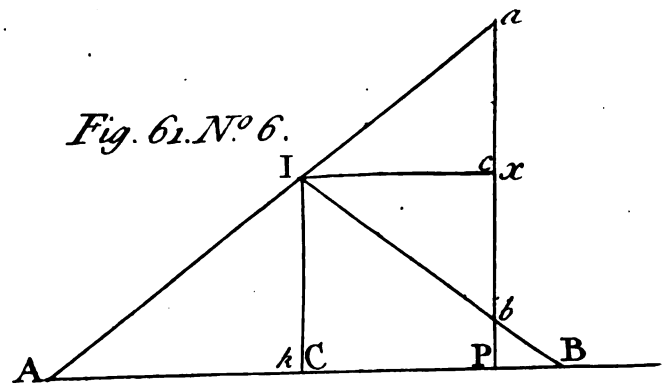


Fig. 61. N.o 6.



ig. 61. N.o 7.

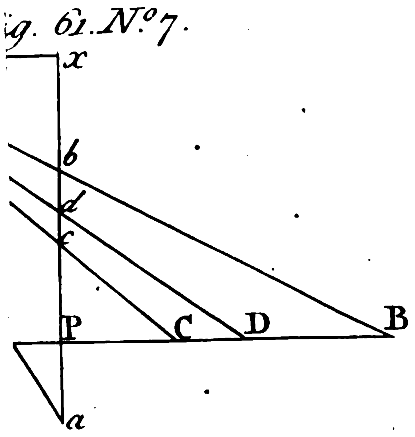


Fig. 61. N.o 8.

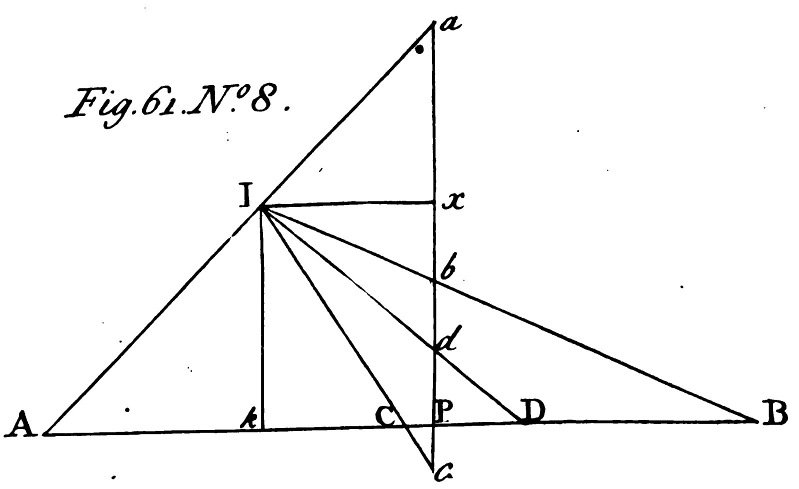


Fig. 62. N.o 1.

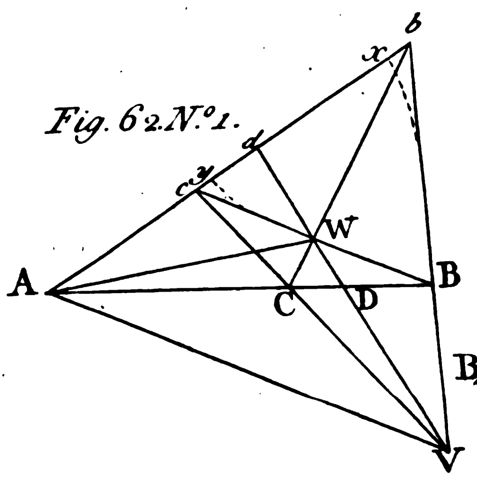


Fig. 62. N.o 2.

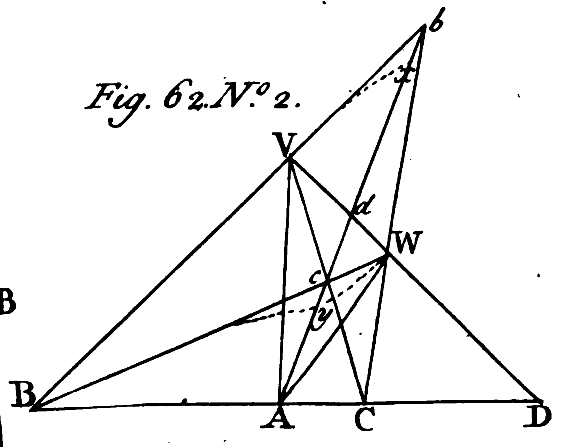


Fig. 62. N.o 3.

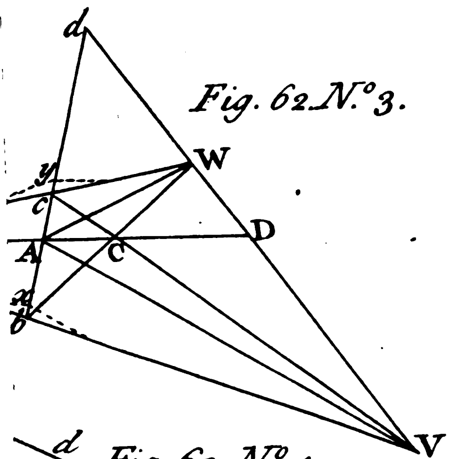


Fig. 62. N.o 5.

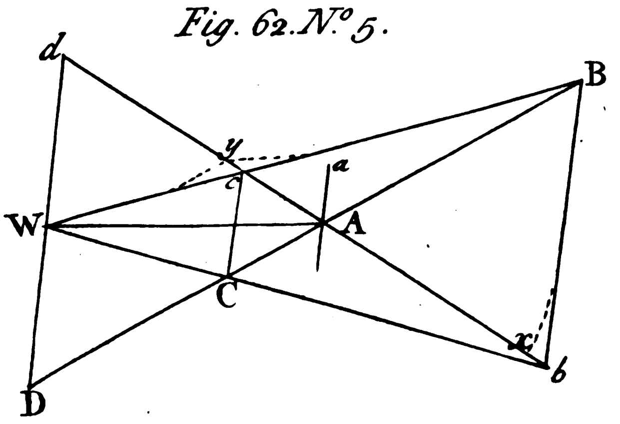
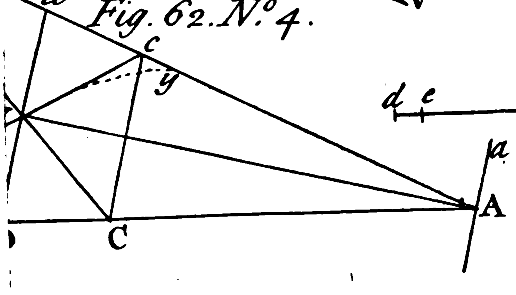


Fig. 62. N.o 4.



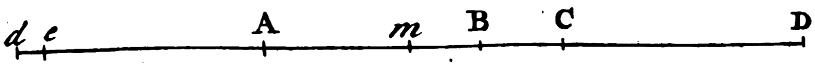


Fig. 63.

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one ſide and part on the other of its Directing Point; its Indefinite Image will alſo be Harmonically divided by the Images of A, B, C, and D; and *vice verſa*, if the Indefinite Image of a Line be Harmonically divided in a, b, c, and d, neither of which is its Vaniſhing Point, its Original will alſo be Harmonically divided by the Originals of a, b, c, and d.

For I A, I B, I C, and I D being Harmonical Lines[a], the Indefinite Image Px cutting them all four (it being parallel to none of them) is therefore Harmonically divided by them in a, b, c, and d[b]. In like manner, if Ia, Ib, Ic, and Id be Harmonical Lines, the Indefinite Original Line kB, which cuts them all four, is Harmonically divided by them in A, B, C, and D.

COR. 5.

If either of the Points of Harmonical Diviſion of the Original Line be its Directing Point, the Indefinite Image of that Line will be biſected by the Images of the other three Points; and *vice verſa*, if either of the Points of Harmonical Diviſion of the Indefinite Image be its Vaniſhing Point, the Original Line will be biſected by the Originals of the other three Points.

Fig. 61. N°. 3, 4.
Fig. 61. N°. 1, 2.

This is the Converſe of the firſt and ſecond Corollaries, and is demonſtrated in the ſame manner.

COR. 6.

If either Extremity of a Line Harmonically divided be taken as the Vaniſhing Point of that Line, the two Parts which lie fartheſt from that Extremity, will repreſent equal Lines.

This plainly follows from the latter part of the laſt Corollary.

LEM. 9.

If two Lines Harmonically divided, cut each other in any one common Point of Diviſion; then if a Line be drawn from the ſecond Point of Diviſion from the common Point in the one Line, to the ſecond Point of Diviſion from the common Point in the other Line, and the two remaining Points in each Line be alſo joined by two Lines in any Order; all theſe joining Lines, produced if neceſſary, will either meet in one common Point, or elſe will be parallel to each other.

Firſt, let A C D B and Acdb be two Lines Harmonically divided by their intermediate Points, and cutting each other in A, a Point of Diviſion common to both of them. Having joined the Points D and d (the ſecond Points of Diviſion in each from their common Point A) by the Line Dd, join alſo the Points C and c (which are the firſt Points in each from their common Point A) by the Line Cc, and produce Dd and Cc till they Interſect in V, which they muſt do ſomewhere if they be not parallel; then from V to B, the remaining Point of Diviſion of A B, draw V B; it muſt be proved, that V B, produced if neceſſary, will paſs through b the remaining Point of Ab.

Fig. 62. N°. 1, 2, 3.

Dem. If V B does not paſs through b, let us imagine it to cut Ab in another Point x.

Now becauſe A C D B is Harmonically divided by its Intermediate Points, the Lines VA, V C, V D, and V Bx are Harmonical Lines, wherefore the Line Acdx is Harmonically divided by thoſe Lines[c]; but the Line Acdb is, by Suppoſition, Harmonically divided by its Intermediate Points, therefore the Lines Acdx, and Acdb, which have three Points A, c, and d in common, have their fourth Points x and b different, which cannot be[d]; and therefore the Points x and b muſt coincide, and conſequently a Line drawn from b through B muſt paſs through the ſame Point V. *Q. E. D.*

Again, having drawn Dd as before, join the contrary Points C and b by the Line Cb, which muſt neceſſarily cut Dd in ſome Point W, and from B through W draw BW; it muſt be proved, that BW, produced if neceſſary, will paſs through c the remaining Point of Ab.

Fig. 62. N°. 1, 2, 3, 4, 5.

Dem. If BW do not paſs through c, let it cut Ab in any other Point y.

Now becauſe A C D B is Harmonically divided, the Lines WA, W C, W D, and W By are Harmonical Lines, wherefore the Line Aydb is Harmonically divided by them; but the Line Acdb is, by Suppoſition, Harmonically divided; and this Line having the Points A, d, and b in common with the Line Aydb, their fourth Points y and c muſt alſo be the ſame. *Q. E. D.*

Laſtly, the ſame things being ſuppoſed, and the Points D and d being joined as before; if a Line Cc, drawn through the firſt Points of Diviſion of the two Lines A B

Fig. 62. N°. 4, 5.

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[a] Def. 2.

[b] Lem. 8.

[c] Lem. 8.

[d] Lem. 2.

and A*b* from their common Point A, be parallel to D*d*, then a Line B*b*, drawn through the remaining Points of thoſe two Lines will alſo be parallel to D*d*.

Dem. If B*b* be not parallel to D*d*, let B*x* be parallel to it, and through A draw A*a* parallel to D*d*.

Now becauſe ACDB is Harmonically divided, the Parallels D*d*, C*c*, A*a*, and B*x*, drawn through the Diviſions of that Line, are Harmonical Parallels, and therefore the Line A*cdx* is Harmonically divided by them[a]; but by the Suppoſition A*cdb* is Harmonically divided, and theſe two Lines agreeing in the Points A, *c*, and *d*, they muſt alſo agree in the fourth Point, and conſequently *b* and *x* coincide. *Q. E. D.*

L E M. 10.

Fig. 63. If a Line AD be Harmonically divided in A, B, C, and D, and any two adjoining Parts AB and BC taken together, be biſected in *m*; then *m*B, *m*C, and *m*D will be in continual Proportion, that is, $mB : mC :: mC : mD$.

Dem. Take A*d* equal to CD,
Then by the Suppoſition $AB : BC :: AD : CD$
And by Compoſition $AB + BC = AC : BC :: AD + CD = dD : CD$
Therefore $\frac{1}{2}AC = mC : BC :: \frac{1}{2}dD = mD : CD$
And by Diviſion $mC - BC = mB : mC :: mD - CD = mC : mD$.
Q. E. D.

C O R. 1.

The ſame things being ſuppoſed as before, BC will be to BD, as B*m* to BA.
For by the third ſtep of the *Lemma*, alternate $mC : mD :: BC : CD$
And by the *Lemma* $mC : mD :: mB : mC = mA$
Therefore $BC : CD :: mB : mA$
Conſequently by Compoſition $BC : BC + CD = BD :: mB : mB + mA = BA$.

C O R. 2.

The ſame things continuing, CD will be to BD, as *m*D to AD.
For by the firſt ſtep of the laſt Cor. inverting, $mD : mC :: CD : BC$
And becauſe of the Harmonical Diviſion of AD, $CD : BC :: AD : AB$
Therefore $mD : mC :: AD : AB$
And by Diviſion $mD - mC = CD : mD :: AD - AB = BD : AD$
Therefore by alternation $CD : BD :: mD : AD$.

C O R. 3.

On the ſame Suppoſition, CD will be to BD, as *m*A to BA,
For by Cor. 1. inverting $BD : BC :: BA : mB$
Therefore by Diviſion $BD - BC = CD : BD :: BA - mB = mA : BA$.

C O R. 4.

The ſame Things being ſuppoſed as before, if A*e* be taken equal to AB; then *m*D, AD, and *e*D will be continually proportional, that is, $mD : AD :: AD : eD$.
For by the third Step of Cor. 2. $mD : mC = mA :: AD : AB = Ae$
Therefore by Compoſition $mD : mD + mA = AD :: AD : AD + Ae = eD$.

L E M. 11.

Fig. 64. If from a Point K without a Circle ADBE, there be drawn two Tangents KD and KE, touching the Circle in D and E, and thoſe Points be joined by the Chord DE; then if any Line K*b* be drawn from K, cutting the Circle in *a* and *b*, and the Chord of the Tangents DE in *c*, the Line K*b* will be Harmonically divided in the Points K, *a*, *c*, and *b*.

Dem. From the Center of the Circle O, to the Point *o*, where K*b* cuts the Circle KDOE by which the Points D and E were found[b], draw O*o*; then becauſe of the Semicircle KEO, the Angle K*o*O is Right[c], and therefore *ao* and *ob* are equal[d].
Now becauſe of the Circle KDOE[e] $Kc : cD :: cE : co$
And likewiſe becauſe of the Circle ADBE $ac : cD :: cE : cb$
Wherefore $ac : Kc :: co : cb$.

But the Parts *ac* and *cb* of the Line K*b* being biſected in *o*, as already ſhewn, the Line K*b* is therefore Harmonically divided in K, *a*, *c*, and *b*[f]. *Q. E. D.*

[a] Lem. 3.

[b] Lem. 4. B. II.

[c] 31 El. 3.

[d] 3 El. 3.

[e] 35 El. 3.

[f] Cor. 1 Lem. 10.

Plate 20. Book 3.
Sect. 1.

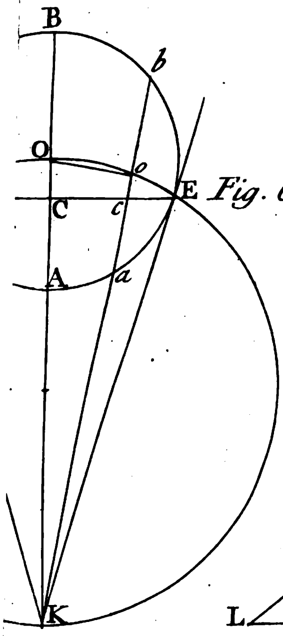


Fig. 64.

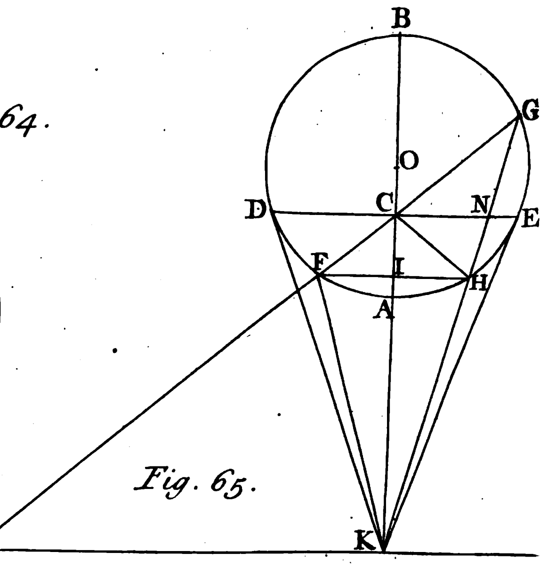
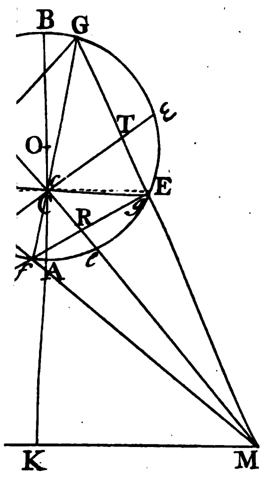


Fig. 65.



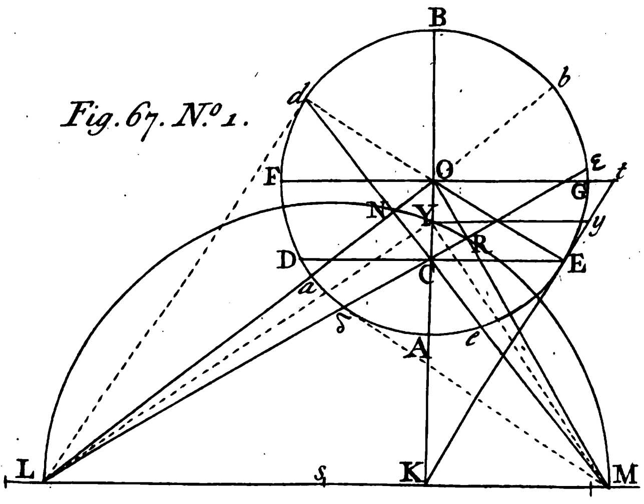


Fig. 67. N.° 1.

Fig. 67. N.° 3.

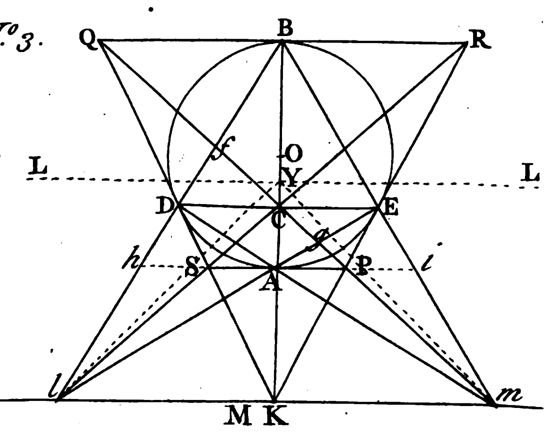
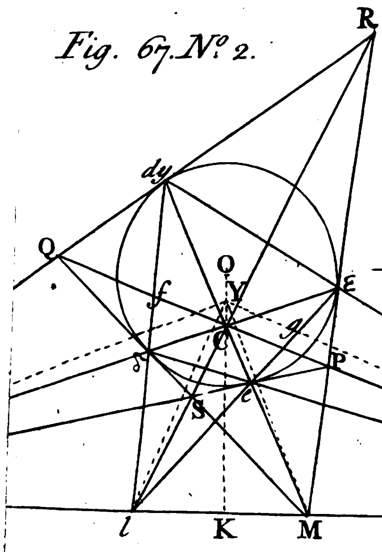


Fig. 67. N.° 2.



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If the Line KB *paſs through* O *the Center of the Circle, the Demonſtration that it is Harmonically divided in* K, A, C, *and* B, *will be the ſame, the Parts* AC *and* CB *of that Line being biſected in* O.

C O R.

Hence, if a Line K*b* drawn from K through a Circle ADBE cutting it in *a* and *b*, be Harmonically divided in the Points K, *a*, *c*, and *b*; the Point *c*, if within the Circle, will be a Point in the Chord of the Tangents from K.

Becauſe in the Line K*b*, three Points K, *a*, and *b*, being determined, and the Part K*a* being taken as an extreme Part, there can be no Point between *a* and *b* but one, which can divide that Line Harmonically [a], and therefore it muſt be the Point *c* where it is cut by DE, the Chord of the Tangents from K.

L E M. 12.

Fig. 65.

If from a Point K without a Circle ADBE, a Line KB be drawn through O the Center of the Circle, cutting DE the Chord of the Tangents from K in C, and another Line LK be drawn through K perpendicular to KB, and conſequently parallel to DE [b]; then if from any Point L in LK, a Line LG be drawn through C, cutting the Circle in F and G, the Line LG will be Harmonically divided in the Points L, F, C, and G.

Dem. From K to G draw KG, cutting the Circle in H and G, and DE the Chord of the Tangents from K, in N, and draw CH; then KG being Harmonically divided in K, H, N, and G [c], CG, CN, CH, and CK are Harmonical Lines [d]: from H draw HF parallel to DE one of the Harmonicals, cutting the other three CG, CK, and CH in F, I, and H, then FH will be biſected by them in I [e]; but FH being perpendicular to AO, which paſſes through the Center of the Circle ADBE, and H being a Point in the Circumference, HF is therefore biſected by AO in I [f], and conſequently F is a Point in the Circumference of the Circle, as well as in the Harmonical CG. Laſtly draw KF, and becauſe FH is biſected in I, and FH and LK are parallel, KL, KF, KI, and KH are Harmonical Lines [g], and conſequently the Line LG, which cuts theſe four Harmonicals, is Harmonically divided by them in the Points L, F, C, and G [h]. *Q. E. D.*

C O R.

Hence, the Chord of the Tangents from any Point L in the Line LK, muſt paſs through the ſame Point C: and no Lines in the Circle, except ſuch as paſs through C, can be Chords of Tangents which will meet in any Point of LK.

Becauſe from any Point L in LK a Line may be drawn through C, which will be Harmonically divided by that Point and the Circle, and therefore the Chord of the Tangents from L muſt paſs through C [i].

But if the propoſed Chord do not paſs through C, and the Tangent at either of its Extremities be produced till it meet LK in any Point L, the Tangent at its other Extremity cannot paſs through the ſame Point L, in regard that the Chord of the Tangents from L muſt paſs through C, and no more than two Tangents can be drawn to a Circle, which ſhall meet each other in one and the ſame Point [k].

L E M. 13.

Fig. 66.

If the Line *de* paſſing through C, be the Chord of the Tangents to the Circle ADBE from the Point L, and any two other Lines LG and L*g* be drawn from L, cutting the Circle in F, G and *f*, *g*, and the Chord *de* in N and R, and the Points F, *f*, and G, *g*, be joined by ſtraight Lines; theſe Lines produced, will either meet *de* in ſome one Point M without the Circle, or elſe will be parallel to it: and if the contrary Points of Diviſion G, *f*, and *g*, F, be joined by ſtraight Lines, theſe will interſect in a Point C in the Line *de* within the Circle.

Dem. Becauſe the Lines LG and L*g* are Harmonically divided in L, G, F, N, and L, *g*, *f*, R [l], and have one Point of Diviſion L in common, and the Line *de* joins N and R, the ſecond Point of Diviſion in each from the common Point L, therefore F*f* and G*g* which join the other Points of Diviſion, will either cut *de* in ſome one Point M, or will be parallel to it [m]; and for the ſame reaſon F*g* and *f*G which join the contrary Points (and which cannot be parallel) will interſect in ſome Point C in the Line *de*; and it is evident from the Nature of a Circle, that the Point M muſt fall without the Circle, and the Point C within it. *Q. E. D.*

[a] Lem. 2.

[b] Lem. 4. B. II.

[c] Lem. 11.

[d] Def. 2.

[e] Lem. 7.

[f] 3 El. 3.

[g] Lem. 4.

[h] Lem. 8.

[i] Cor. Lem11.

[k] 16 and 18 El. 3.

[l] Lem. 11.

[m] Lem. 9.

L E M. 14.

Fig. 66. The ſame things being ſuppoſed as before, if the Lines F*f* and G*g* meet *de* in any Point M, then a Line $\delta\varepsilon$ drawn from L through C, will be the Chord of the Tangents to the Circle from the Point M.

Dem. Becauſe *de* is by Suppoſition the Chord of the Tangents from L, the Line LG is Harmonically divided in L, G, F, and N[a], and therefore CG, CN, CF, and CL, produced both ways from C, are Harmonical Lines[b], conſequently MF and MG, which cut all theſe four Harmonicals, are Harmonically divided by them in the Points M, *f*, S, F, and M, *g*, T, G[c], wherefore the Points S and T are Points in the Chord of the Tangents from M[d]; but thoſe Points are in the Line $\delta\varepsilon$ drawn from L through C, that being one of the four Harmonicals, and therefore $\delta\varepsilon$ is the Chord of the Tangents from the Point M. *Q. E. D.*

C O R. 1.

Hence, as all Lines drawn from L cutting the Circle, are Harmonically divided by the Circle and the Line *de*, ſo all Lines drawn from M and cutting the Circle, are Harmonically divided by the Circle and the Line $\delta\varepsilon$[e].

C O R. 2.

Hence alſo, if any Point M be taken in *de*, the Chord of the Tangents from L, produced without the Circle; then the Chord of the Tangents from the aſſumed Point M being produced, will paſs through the Point L.

From M draw any Line MG cutting the Circle in G and *g*, and from L to G and *g*, draw LG and L*g* cutting the Circle in F and *f*, then a Line drawn through F and *f* will meet *de* in the ſame Point M with the Line G*g*[f]; but the Lines FG, *fg* drawn through the Interſections F, G, *f*, *g* of the Lines MF, MG with the Circle, will alſo meet the Chord of the Tangents from M in ſome one Point[g], and theſe by Conſtruction meeting in L, the Chord of the Tangents from M muſt therefore alſo paſs through L.

L E M. 15.

The ſame things remaining as before, if through L and M a Line LM be drawn, and from O the Center of the Circle a Line OK be drawn perpendicular to LM cutting it in K; the Line OK will paſs through the ſame Point C, and a Line DE drawn through C perpendicular to OK will be the Chord of the Tangents from K.

Dem. Find DE the Chord of the Tangents from K, which muſt be perpendicular to OK, and will cut it in ſome Point *c*[h].

Now LM being by Conſtruction perpendicular to OK, the Chord of the Tangents from any Point L or M in LM, paſſes through the ſame Point *c*[i]; but *de* is by Suppoſition the Chord of the Tangents from L, and $\delta\varepsilon$ is the Chord of the Tangents from M[k]; wherefore the Point C, where theſe two Chords interſect, is the ſame with the Point *c*, where the Chord DE cuts OK. *Q. E. D.*

L E M. 16.

Fig. 67. N°. 1. If from any Point K without a Circle ADBE, a Line KB be drawn paſſing through O the Center of the Circle, and the Point C where that Line is cut by the Chord of the Tangents from K be determined, and through K a Line LM be drawn perpendicular to KB; then if any Point L be taken in that Line, and *de* the Chord of the Tangents from that Point be produced till it cut LM in another Point M, and upon LM as a Diameter, a Semicircle LYM be deſcribed, that Semicircle will cut KB in a Point Y between C and O, which Point Y will conſtantly be the ſame, wherever the Point L is taken in the Line LM.

Dem. Draw LO and MO; then in the Triangles LKO, CNO, *de* being perpendicular to LO[l], the Angles CNO and LKO are Right, and the Angle NOC common to both Triangles, therefore theſe two Triangles are Similar; and in the Triangles CNO, CKM, the Angles at N and K are Right, and the Vertical Angles at C equal, therefore theſe two Triangles are Similar, and conſequently the Triangle LKO is Similar to the Triangle CKM, and therefore LK : KO :: KC : KM

And becauſe of the Semicircle LYM[m] LK : KY :: KY : KM

Therefore KC : KY :: KY : KO

But the Point C in the Line KB being conſtantly the ſame, as well as the Point O, wherever the Point L is taken in the Line LM[n]; therefore the Lines KO and KC

muſt

[a] Lem. 11.

[b] Def. 2.

[c] Lem. 8.

[d] Cor. Lem. 11.

[e] Lem. 11.

[f] Lem. 13.

[g] Lem. 13.

[h] Lem. 4. B. II. and Cor.

[i] Cor. Lem. 12.

[k] Lem. 14.

[l] Cor. Lem. 4. B. II.

[m] 35 El. 3.

[n] Cor. Lem. 12.

muſt always continue the ſame, and conſequently ſo will the Line KY, which is a mean proportional between them, and for the ſame reaſon the Point Y will always fall between C and O. *Q. E. D.*

COR. 1.

'Tis evident the Semicircle LYM alſo paſſes through the Points N and R, where LO and MO cut *de* and *δε* the Chords of the Tangents from L and M, the Angles LNM and LRM being both Right[a].

[a] 31 El. 3. and Cor. Lem. 4. B. II.

COR. 2.

If LY and MY be drawn, the Angle LYM will conſtantly be a Right Angle, wherever the Point L is taken in LM[b].

[b] 31 El. 3.

LEM. 17.

The ſame things being ſuppoſed as before, the Line KY is equal to KE the Tangent to the Circle from the Point K. Fig. 67. N°. 1.

Dem. Becauſe of the Similar Triangles KCE, KEO, $KC : KE :: KE : KO$

But $KC : KY :: KY : KO$[c]

Therefore $KY = KE$. *Q. E. D.*

[c] Lem. 16.

COR.

Hence, if KY be made equal to KE, and from any Point *s* in the Line ML as a Center with the *Radius* *s*Y, a Semicircle LYM be deſcribed, cutting LM in any Points L and M; then a Line *de* drawn from M through C, and terminated by the Circle in *d* and *e*, will be the Chord of the Tangents from L; and a Line *δε* drawn from L through C, and terminated by the Circle, will be the Chord of the Tangents from M.

LEM. 18.

If from any Point L in LM a Line LY be drawn, it will be equal to L*d*, the Tangent to the Circle from the Point L. Fig. 67. N°. 1.

Dem. Draw O*d*; then in the Similar Triangles LN*d*, L*d*O, $LN : Ld :: Ld : LO$

And in the Similar Triangles LNM, LKO, $LN : LM :: LK : LO$

Wherefore $LM \times LK = Ld^2$.

But[d] $LM \times LK = LK \times KM + LK^2$

And becauſe of the Semicircle LYM, $LK \times KM = KY^2$

Therefore $LM \times LK = KY^2 + LK^2 = Ld^2$

But[e] $KY^2 + LK^2 = LY^2$.

Conſequently $LY = Ld$. *Q. E. D.*

[d] 3 El. 2.

[e] 47 El. 1.

COR.

Hence, if from any Point L in LM as a Center, with a *Radius* equal to LY, an Arch be drawn, it will cut the Circle in *d* and *e* the Extremities of the Chord of the Tangents from L.

LEM. 19.

The ſame things remaining, if the Diameter FG of the Circle, which is parallel to LM, be produced till it cut the Tangent KE in *t*; then the *Radius* OG will be a mean proportional between CE the Semichord of the Tangents from K and the Line O*t*. Fig. 67. N°. 1.

Dem. Draw the *Radius* OE, then becauſe OE is perpendicular to K*t*[f], the Triangles KOE, O*t*E are Similar; and becauſe CE is perpendicular to KO, the Triangles KOE, OEC are Similar[g], wherefore the Triangle O*t*E is Similar to the Triangle OEC.

And conſequently $Ot : OE = OG :: OG : CE$. *Q. E. D.*

[f] 18 El. 3.

[g] 8 El. 6.

COR.

If through Y a Line Y*y* be drawn parallel to FG cutting K*t* in *y*, then Y*y* will be equal to OG.

For becauſe of the Similar Triangles KCE, KY*y*, KO*t*; CE, Y*y* and O*t* are in the ſame Proportion to each other as KC, KY, and KO; but KY is a mean Proportional between KC and KO[h], therefore Y*y* is a mean Proportional between CE and O*t*, and conſequently equal to OG.

[h] Lem. 16.

LEM. 20.

The ſame things being ſuppoſed as before, if the Angle LYM be biſected by a Line Y*l* cutting LM in *l*; then two Lines *l*δ, *l*ε drawn from *l* through the Extremities of Fig. 67. N°. 2.

$\delta\varepsilon$, the Chord of the Tangents from M, will alſo paſs through d and e, the Extremities of the Chord of the Tangents from L.

Dem. From Y draw Ym perpendicular to Yl cutting LM in m, the Point in LM through which the Chord of the Tangents from l paſſes[a]; and draw mC, lC the Chords of the Tangents from l and m[b].

Then becauſe the Angle LYM is biſected by Yl, to which Ym is perpendicular, the Line LM is Harmonically divided by YL, Yl, YM, and Ym, in L, l, M, and m[c], wherefore CL, Cl, CM, and Cm are Harmonical Lines[d], and $l\delta$ which cuts them all four, is Harmonically divided by them in l, δ, f, and y[e]; but becauſe mC is the Chord of the Tangents from l, $l\delta$ is alſo Harmonically divided in l, f, and its Interſections δ and d with the Circle[f], and the Points l, δ, and f in both theſe Diviſions being the ſame, the fourth Point y is the ſame with d the Interſection of MC with the Circle[g].

In the ſame manner it may be proved, that $l\varepsilon$ cuts the Circle in e the other Extremity of the Chord de; ſeeing $l\varepsilon$ is Harmonically divided in l, e, g, and ε, as well by the the four Harmonicals CL, Cl, CM, and Cm, as by the Circle and the Chord Cm. *Q. E. D.*

COR. 1.

If from m two Lines be drawn through δ and ε, they will likewiſe paſs through e and d.

For Lm and Lε which meet in L, being Harmonically divided in L, l, M, m and L, δ, C, ε, and MC paſſing through the ſecond Point of Diviſion in each from their common Point L, $l\delta$ and $m\varepsilon$ meet MC in the ſame Point d, and $l\varepsilon$ and $m\delta$ meet MC in the ſame Point e[h].

COR. 2.

It is evident, that ed and $\delta\varepsilon$ the Chords of the Tangents from L and M are the Diagonals of the Trapezium $e\delta d\varepsilon$ inſcribed in the Circle, and that the Points l and m where the oppoſite Sides of that Trapezium meet, are in the ſame ſtraight Line LM.

COR. 3.

Fig. 67. N°. 3.

If on each Side of K in the Line lm, a Diſtance equal to KY be ſet off at l and m, and DE the Chord of the Tangents from K be drawn; then lD and lE, or mD and mE, drawn from l or m through the Extremities D and E of that Chord, will alſo paſs through A and B, the Extremities of the Diameter AB, which paſſes through K perpendicular to lm.

For by this Conſtruction, the Angle lYm being Right, mC and lC are the Chords of the Tangents from l and m[i]; and becauſe the Angle lCm is biſected by CK, to which DE is perpendicular, Cl, CK, Cm, and DE are Harmonical Lines, and cut lD and lE Harmonically in l, D, f, B and l, A, g, E[k], but lD and lE are likewiſe Harmonically divided by their Interſections with the Circle and the Chord mC[l], and D and E being Points of the Circle, and the Points l, D, f, and l, E, g, being the ſame in both Diviſions, the fourth Points B and A in both Diviſions are alſo the ſame[m].

After the like manner it may be ſhewn, that mD and mE paſs through A and B.

COR. 4.

Here YK being perpendicular to lm, YL is parallel to it, and KY and Kl being equal, lY biſects the Angle LYK, ſo that K and M coincide, and the Point L is infinitely diſtant; and the Tangents QR and SP from that infinitely diſtant Point being therefore parallel to YL, the Diameter AB becomes the Chord of thoſe Tangents.

LEM. 21.

Fig. 67. N°. 2.

The ſame things being ſuppoſed as in the preceeding *Lemma*, if there be drawn from L two Tangents Ld, Le, to the Circle, and from M two other Tangents Mδ, Mε, forming by their mutual Interſections a Trapezium SQRP; the Chords of the Tangents from l and m will be Diagonals of that Trapezium.

Dem. Becauſe Md is Harmonically divided in M, e, C, and d[n], LM, Le, LC, and Ld are Harmonical Lines, and divide the Tangent Mε Harmonically in M, P, ε, and R. And becauſe Lε is Harmonically divided in L, δ, C, and ε, ML, Mδ, MC, and Mε are Harmonical Lines, and divide the Tangent Ld Harmonically in L, Q, d, and R[o].

Join Q and P the ſecond Points of Diviſion of LR and MR from their common

[a] Cor. 2. Lem. 16.

[b] Cor. Lem. 17.

[c] Lem. 5 and 8.

[d] Def. 2.

[e] Lem. 8.

[f] Lem. 11.

[g] Lem. 2.

[h] Lem. 9.

[i] Cor. Lem. 17.

[k] Lem. 5 and 8.

[l] Lem. 11.

[m] Lem. 2.

[n] Lem. 11.

[o] Lem. 5, and 8.

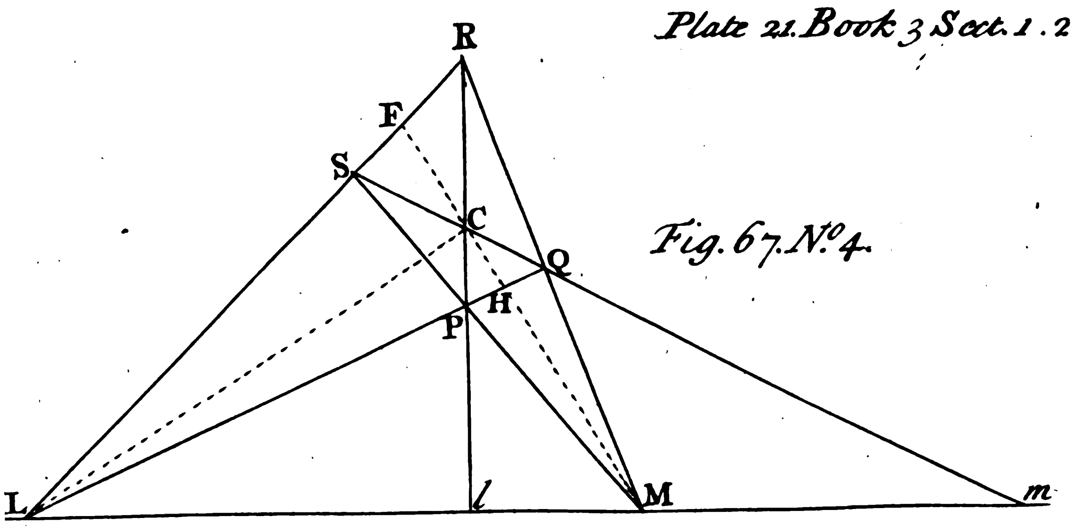


Fig. 67. No. 4.

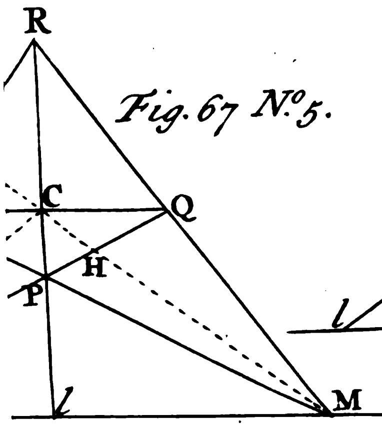
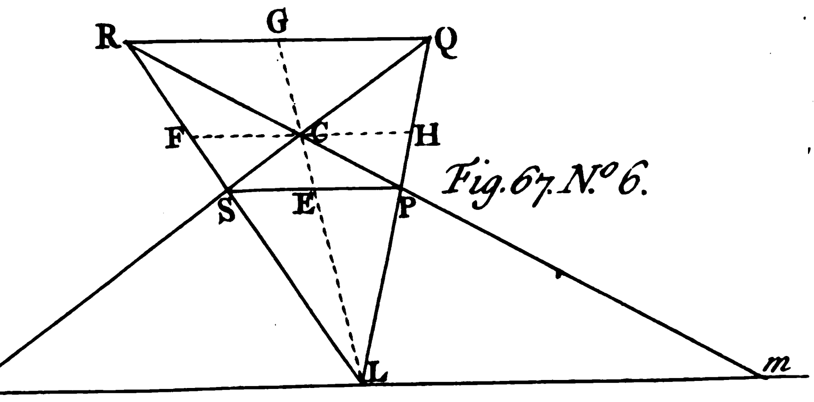


Fig. 67 No. 5.



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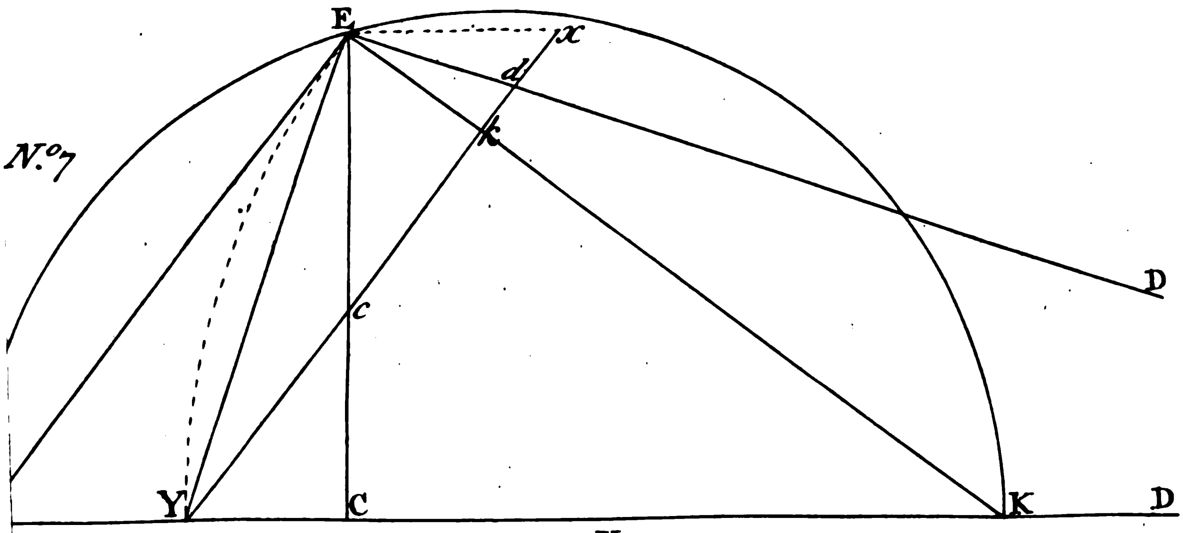

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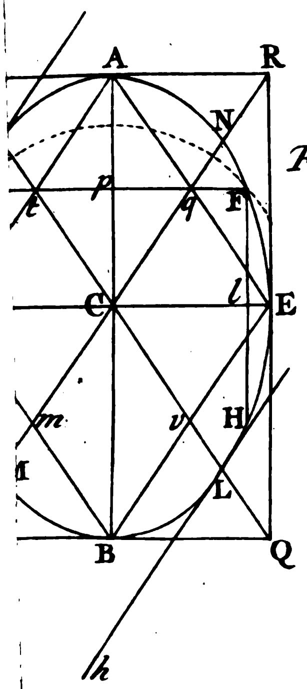


Fig. 68. No. 1.

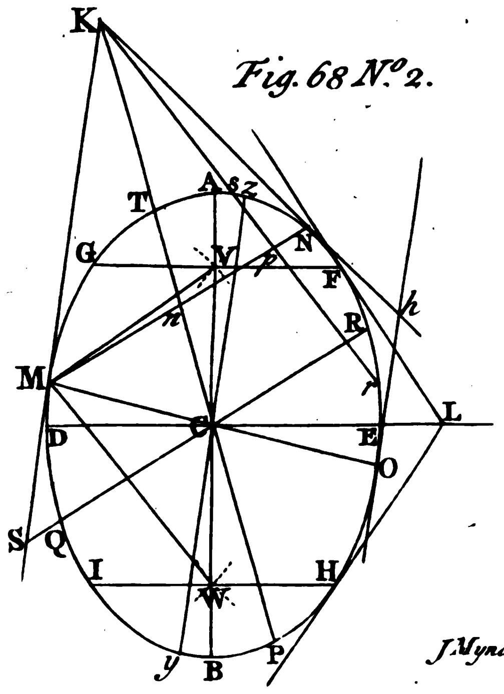


Fig. 68 No. 2.

J Mynde sc.

Point R; then L*ε* and M*d* muſt Interſect in ſome Point of QP[a], but they Interſect in C[b], wherefore C is a Point in QP; alſo *dε* and LM Interſect in ſome Point of QP, but they Interſect in *m*[c], wherefore QP alſo paſſes through *m*, and conſequently the Diagonal QP is the ſame with *m*C, the Chord of the Tangents from *l*.

Again, becauſe of the Harmonicals ML, M*δ*, MC, and M*ε*, the Line Q*m* is Harmonically divided in Q, C, P, and *m*, and *m*L being alſo Harmonically divided in *m*, M, *l*, and L[d], join their ſecond Points of Diviſion from *m* by the Line C*l*; then LP and MQ muſt Interſect in ſome Point of C*l*, but they Interſect in S, wherefore C*l* paſſes through S; alſo LQ and MP Interſect in ſome Point of C*l*, but they Interſect in R, wherefore C*l* alſo paſſes through R, and conſequently the Diagonal SR is the ſame with *l*C, the Chord of the Tangents from *m*. *Q. E. D.*

COR.

The ſame things being ſuppoſed as in the ſecond Corollary of the preceeding *Lemma*; if from K two Tangents KD, KE be drawn, cutting the Tangents QR and SP, and thereby forming a Trapezium SQRP; the Chords of the Tangents from *l* and *m* will be Diagonals of that Trapezium. Fig. 67. N°. 3.

Becauſe KB is Harmonically divided in K, A, C, and B, *lm*, SP, DE, and QR are Harmonical Parallels, which therefore divide KD and KE Harmonically in K, S, D, Q, and K, P, E, R; and SR and QP therefore Interſect in ſome Point of DE[e]; but SP and QR being biſected in A and B, SR and QP alſo Interſect in ſome Point of AB, and conſequently in C the Interſection of AB with DE.

Produce SP till it cut *l*B and *m*B in *h* and *i*; then *l*B and *m*B will be Harmonically divided in *l*, *h*, D, B, and *m*, *i*, E, B; and theſe meeting KB in B, and *hi* joining the ſecond Points of Diviſion of each of theſe Lines from their common Point B, *l*C and KD muſt Interſect in ſome Point of *hi*, but KD cuts *hi* in S, wherefore *l*C paſſes through S; likewiſe KE and *m*C Interſect in ſome Point of *hi*, and KE cutting it in P, *m*C alſo paſſes through P. And conſequently the Diagonals SR and QP are the ſame with *l*C and *m*C, the Chords of the Tangents from *m* and *l*.

LEM. 22.

If in a given Line LM any two Points L and M be taken, and from each of thoſe Points two Lines LR, LQ and MS, MR be drawn at pleaſure, forming by their mutual Interſections a Trapezium SPQR; then if the Diagonals SQ and RP which croſs in C, be produced till they cut LM in *l* and *m*, the Line LM will be Harmonically divided in L, *l*, M, and *m*; and the Diagonals *l*R and *m*S will likewiſe be Harmonically divided in *l*, P, C, R, and *m*, Q, C, S. Fig. 67. N°. 4.

Dem. Find in the Sides SR and PQ of the Trapezium, two Points F and H, which may divide them Harmonically in L, S, F, R, and L, P, H, Q, and draw FH.

Then becauſe LR and LQ are Harmonically divided, and have one common Point L, and FH joins their ſecond Points of Diviſion from L, the Lines SP and RQ muſt meet in ſome Point of FH, if they be not parallel to it[f], but by Conſtruction SP and RQ meet in M, therefore FH alſo paſſes through M; likewiſe RP and SQ which join the contrary Points of Diviſion, muſt croſs in ſome Point of FH[g], but they croſs in C, the Line FH therefore alſo paſſes through C: now becauſe of the Harmonical Diviſion of LQ, the Lines CL, CP, CH, and CQ are Harmonical Lines, wherefore LM (if it cut them all four) will be Harmonically divided by them in L, *l*, M and *m*[h]; and for the ſame reaſon ML, MP, MH, and MQ being Harmonical Lines, the Diagonals *l*R and *m*S which cut them all four, are Harmonically divided by them in *l*, P, C, R, and *m*, Q, C, S reſpectively. *Q. E. D.*

COR. 1.

If either of the Diagonals SQ be parallel to LM, the other Diagonal RP will biſect SQ in C, and the Line LM in *l*. Fig. 67. N°. 5.

For CL, CP, CH, and CQ being Harmonical Lines as before, LM parallel to CQ one of theſe Harmonicals, is biſected by the other three[i], and in the Triangle RLM, LM being biſected in *l*, SQ parallel to LM is biſected by R*l* in C.

COR. 2,

If the Point M be infinitely diſtant, that is, if the Sides SP and RQ of the Trapezium be parallel to L*m*, then thoſe Sides will be biſected by CL in E and G, and the Line *lm* will be biſected in L. Fig. 67. N°. 6.

[a] Lem. 9.

[b] Cor. Lem. 17.

[c] Cor. 1. Lem. 20.

[d] Lem. 20.

[e] Lem, 9.

[f] Lem. 9.

[g] Lem. 9.

[h] Lem. 8.

[i] Lem. 7.

For CL, CP, CH, and CQ, being ſtill Harmonical Lines, *lm*, SP, and RQ, which are parallel to CH one of theſe Harmonicals, are therefore biſected in L, E, and G, by the other three.

LEM. 23.

Fig. 67. N°. 7. To divide a Line KO in K, C, Y, and O, in ſuch manner that KC, KY, and KO may be in continual Proportion.

1. The whole Line KO and the Point C being given, thence to find Y.

On KO as a Diameter deſcribe a Semicircle OEK, from C erect CE perpendicular to KO cutting the Semicircle in E, and having drawn KE, make KY equal to it, and Y will be the Point ſought.

[a] 8 El. 6. For the Triangles KCE, KEO being Similar[a], $KC : KE = KY :: KY : KO$.

2. The whole Line KO and the Point Y being given, thence to find C.

Having drawn the Semicircle OEK as before, from K as a Center with the *Radius* KY deſcribe an Arch cutting the Semicircle in E, from whence EC drawn perpendicular to OK will cut it in C the Point deſired.

3. The Points K, C, and Y being given, thence to find the Extremity O.

Draw CE perpendicular to KY, and from the Center K with the *Radius* KY deſcribe an Arch cutting CE in E, and having drawn KE, draw EO perpendicular to it, which will cut KY in O the Point required.

4. The Points O, Y, and C being given, thence to find the Extremity K.

From any Point E without the given Line draw EO and EC, and from Y draw Y*x* parallel to EO cutting EC in *c*, and a Line E*x* parallel to OC in *x*; and having in Y*x* taken *cd* equal to *c*Y, find a Point *k* between *c* and *d*, whereby Y*x* may
[b] Lem. 1. be Harmonically divided in Y, *k*, *d*, and *x*[b]; then E*k* being drawn, it will cut OC in K the Point ſought.

Dem. Becauſe by the Suppoſition $KC : KY :: KY : KO$

Therefore by Diviſion $KC : KY - KC = CY :: KY : KO - KY = YO$

And KC being leſs than KY, CY is therefore leſs than YO, or its equal E*x*, wherefore alſo Y*c*, or its equal *cd*, is leſs than *cx*, and conſequently E*d* produced will meet OC in ſome Point D.

Now becauſe Y*d* parallel to EO is biſected in *c*, EY, E*c*, E*d*, and EO are Har-
[c] Lem. 4. monical Lines[c], and the Line OD, which is parallel to none of them, is therefore Har-
[d] Lem. 8. monically divided by them in O, Y, C, and D[d]; and becauſe Y*x* is Harmonically di-
[e] Def. 2. vided in Y, *k*, *d*, and *x*, EY, E*k*, E*d*, and E*x*, are Harmonical Lines[e], wherefore
YD which is parallel to E*x*, one of theſe Harmonicals, is biſected by the other three
[f] Lem. 7. in Y, K, and D[f]; but the whole Line OD being Harmonically divided in O, Y, C,
and D, and its two adjoining Parts YC and CD taken together, being biſected in K,
[g] Lem. 10. as already ſhewn, therefore KC : KY : : KY : KO[g], and conſequently the Point
K is rightly determined. *Q. E. I.*

SECTION II.

Of the Ellipſis.

Fig. 68. N°. 1. LET ADBE be an *Ellipſis*.

1. Draw any two parallel Lines AD, BE in the *Ellipſis*, terminated by it in A, D, B, and E; biſect AD and BE in *t* and *v*, and draw *tv* terminated by the Curve in T and L, and biſect TL in C.

Then C is the Center, and TL a Diameter of the *Ellipſis*, and DA, BE, and all other Lines parallel to them, drawn within and terminated by the Curve, are biſected by the Diameter TL, and each Moiety D*t*, *t*A, B*v*, *v*E, &c. of ſuch Lines, is an *Ordinate* to that Diameter; and if through the Extremities T and L of the Diameter TL, two Lines T*g*, L*b* be drawn parallel to any Ordinate D*t*, they will be *Tangents* to the *Ellipſis* in T and L.

2. If through the Center C, a Line MN be drawn parallel to an Ordinate D*t* of any Diameter TL, and terminated by the Curve in M and N, MN will alſo be a Diameter, and theſe two with reſpect to each other are called *Conjugate Diameters*, and

all Lines as AE, DB, drawn parallel to TL within the Curve, and terminated by it, are biſected by the Diameter MN, and each Moiety D*m*, *m*B, A*q*, *q*E of ſuch Lines is an Ordinate to the Diameter MN.

3. From C as a Center, with any *Radius* large enough to cut the Section in two Points, draw an Arch of a Circle cutting it in G and F, and having drawn GF, biſect it in P, and draw PC cutting the Section in A and B, and through C draw DE perpendicular to AB, and terminated by the Curve in D and E.

Then AB and DE will be two Conjugate Diameters, GF and all other Lines parallel to DE, and terminated by the *Ellipſis*, are biſected by AB, to which they are double Ordinates, and FH and all other Lines parallel to AB, and terminated by the Section, are biſected by DE, to which they are double Ordinates; PR and SQ drawn through A and B parallel to DE, are Tangents to the *Ellipſis* in A and B, and SP and QR drawn through D and E parallel to AB, are Tangents in D and E: theſe two Diameters AB and DE are called *the Axes*, of which AB the longer, is called *the firſt* or *Tranſverſe Axe*, and DE the ſhorter, is *the ſecond Axe*, and with reſpect to each other, they are called *Conjugate Axes*, and are the only two Conjugate Diameters of the *Ellipſis*, which are perpendicular to each other, and to their reſpective Ordinates. The Tranſverſe Axe AB is the longeſt of all the Diameters of the *Ellipſis*, and the ſecond Axe DE is the ſhorteſt, and the Extremities A and B of the Tranſverſe Axe are alſo called *the Vertices of the Ellipſis.*

4. All Lines drawn through the Center C, and terminated by the *Ellipſis*, are called *Diameters*, and biſect each other in C; and every Diameter hath another Diameter Conjugate to it; and either of any two Conjugate Diameters is parallel to the Ordinates, and to the Tangents at the Extremities of the other.

5. If any two Lines MN and TL, drawn in and terminated by the *Ellipſis*, biſect each other in a Point C; theſe two Lines are *Diameters of the Ellipſis*, and the Point C where they Interſect is *the Center.* And if any Line AD not a Diameter, drawn in and terminated by the *Ellipſis*, be biſected in *t* by any Diameter TL, it will be *a double Ordinate to that Diameter*; and no Lines drawn in the *Ellipſis*, beſides ſuch as are parallel to AD, can be biſected by the Diameter TL.

6. If from the Extremities A and B of the Tranſverſe Axe, to the Extremities D and E of the ſecond Axe, there be drawn AD, AE, and BD, BE; ADBE will be an Equilateral Parallelogram: and if through C two Lines TL, MN be drawn parallel to AE and AD, and terminated by the *Ellipſis* in T, L, M, and N, the Lines TL and MN will be two Conjugate Diameters equal to each other, and are the only two Conjugate Diameters of the *Ellipſis* which are equal; AD and BE will be biſected in *t* and *v* by the Diameter TL, to which they are double Ordinates, and AE and BD will be biſected in *q* and *m* by the Diameter MN, to which they are double Ordinates; and either Moiety D*t* or *t*A of any of theſe double Ordinates, as AD, is equal to C*t*, the Segment of the Diameter TL through which it paſſes, intercepted between the Center C and that Ordinate; and theſe two Diameters TL and MN will likewiſe paſs through P, Q, S, and R, the Angles of the Rectangular Parallelogram PRSQ formed by the Tangents at the Extremities of the Axes: and all Parallelograms formed by Tangents at the Extremities of any two Conjugate Diameters, are equal to each other, and to the Parallelogram PRSQ.

7. The Acute Angle TCN made by the Diameters TL, MN, is biſected by the Tranſverſe Axe AB, and the Obtuſe Angle TCM made by the ſame Diameters is biſected by the ſecond Axe; but the Diameters TL and MN in the *Ellipſis* cannot be perpendicular, for if they were, they would then be the Axes[a], and being equal, the Curve would be a Circle.

8. If any Diameter of the *Ellipſis* be drawn within the Acute Angle TCN, its Conjugate will fall within the Obtuſe Angle TCM; and every Diameter of the *Ellipſis* which falls within the Acute Angle TCN, is larger than its Conjugate.

9. If from either Extremity D or E of the ſecond Axe as a Center, with a *Radius* equal to CA the Semitranſverſe Axe, an Arch of a Circle be drawn cutting that Axe in V and W, each of the Points V and W is called *a Focus of the Ellipſis*; and if through V and W two double Ordinates GF and IH to the Axe AB be drawn, theſe two will be equal, and either of them is the Meaſure of the *Parameter* of the Axe AB. And if from any Point M in the *Ellipſis* two Lines MV, MW be drawn to the *Foci*, theſe two Lines together will be equal to the Tranſverſe Axe AB; and if through M a Tangent SK be drawn, the Angles VMK, WMS will be equal. Fig. 68. N°. 2.

10. If from the Center C there be taken on the ſecond Axe DE, a Diſtance CL

equal

[a] Art. 3.

equal to the Semitranſverſe Axe CA, two Tangents LH, LF drawn from L to the *Ellipſis* will touch it in H and F, one of the Extremities of the double Ordinates IH and GF drawn through the *Foci* W and V; and if the like Diſtance were ſet off on the other Side of C, Tangents from thence would touch the *Ellipſis* in I and G, the other Extremities of thoſe Ordinates.

11. If through any Point M in the *Ellipſis*, a Tangent MK be drawn till it cut any Diameter TP produced in K, and from the ſame Point M an Ordinate Mn be drawn to that Diameter cutting it in n, the Semidiameter CT will be a mean Proportional between Cn and CK, the Parts of that Diameter intercepted between the Center and its Interſections with the Ordinate and Tangent; and the whole Diameter TP produced to K, will be Harmonically divided in the Points K, T, n, and P, and Mn produced to N, will be the Chord of the Tangents from K. And if from K any Line Kr be drawn cutting the *Ellipſis* in s and r, and the Line MN in p, the Line Kr will be Harmonically divided in K, s, p, and r.

12. The *Parameter* of any Diameter, is a third Proportional to that Diameter and its Conjugate, putting the Diameter, whoſe Parameter it is as the firſt Term; thus if GF be the Parameter of the Diameter AB, to which DE is the Conjugate, then AB is to DE, as DE to GF. or $AB^2 : DE^2 :: AB : GF$.

13. If an Ordinate Mn be drawn to any Diameter TP, cutting it in any Point n, then a mean Proportional between Tn and nP the Segments of that Diameter, will be to the Ordinate Mn, as the Diameter TP is to its Conjugate QR, that is,

$$\sqrt[2]{Tn \times nP} : Mn :: TP : QR :: TC : QC.$$

14. If through the Vertex M of any Diameter MO, a Tangent be drawn cutting any two Conjugate Diameters TP and QR in K and S, the Rectangle between MK and MS the Segments of that Tangent, will be equal to the Square of Cz, the half of the Diameter zy, conjugate to the Diameter MO.

15. If two Tangents MK and Ob at the Extremities of any Diameter MO, be cut by any other Tangent Kb in K and b, the Rectangle between MK and Ob will be equal to the Square of Cz the Semidiameter Conjugate to the Diameter MO.

16. A Circle may be conſidered as an *Ellipſis* whoſe Conjugate Axes are equal, and whoſe *Foci* coincide.

PROB. I.

An Original Circle, which doth not cut or touch the Directing Line, being given; therein to determine the Originals of the Axes, or any other Conjugate Diameters of the *Ellipſis* formed by the Image of the Circle, and other the Lines and Points in the *Ellipſis* above deſcribed.

Fig. 69. N°. 1. Let Z be the Original Plane, LM the Directing Line, and IK the Eye's Director, and let ADBE be a Circle in that Plane, and O its Center.

1. To find the Originals of the Conjugate Axes and their Ordinates, and of the Center of the *Ellipſis*, and alſo of the Tangents at the Extremities of the Axes.

Through the Center O draw a Diameter ab perpendicular to LM cutting it in k, and find de the Chord of the Tangents to the Circle from k[a], cutting ab in C; take kY in ka equal to the Tangent ke and draw IY, and biſect it by the Perpendicular To, cutting LM in o; from o as a Center with the *Radius* oI or oY, draw the Semicircle LIYM, cutting LM in L and M: laſtly, from M and L through C draw MA, LE terminated by the Circle in A, B, D, and E.

Then AB and DE will be the Originals of the Conjugate Axes of the *Ellipſis*, and C the Original of its Center, and all Lines drawn from L through the Circle and terminated by it, will be Originals of double Ordinates to the Axe whoſe Original is AB, and all Lines drawn from M, terminated in like manner by the Circle, will be the Originals of Ordinates to the Axe whoſe Original is DE, and L and M will be the Directing Points of thoſe Ordinates reſpectively.

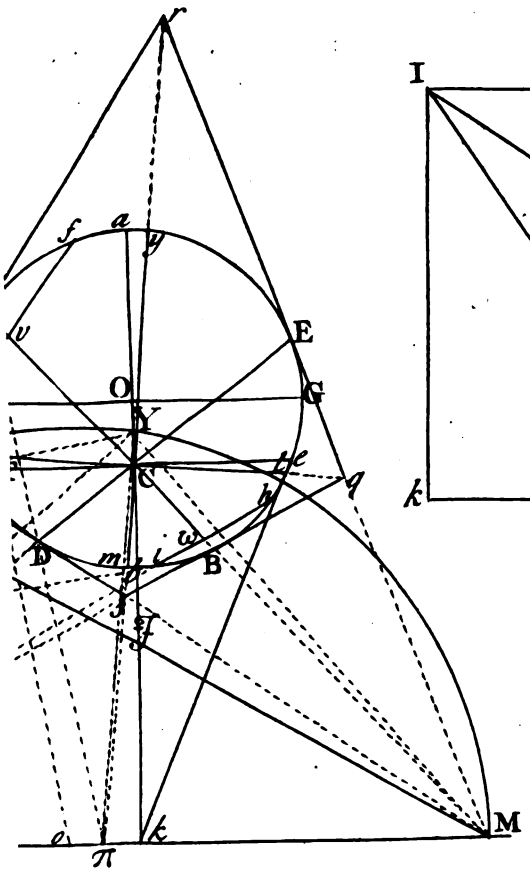
Dem. Becauſe AB is the Chord of the Tangents to the Circle from L, and DE is the Chord of the Tangents from M[b], therefore LE and MA are Harmonically divided by the Circle and the Point C[c]; and L and M being Directing Points, the Images of AB and DE are therefore biſected by the Image of C[d]; wherefore AB and

[a] Lem. 4. B. II.

[b] Cor. Lem. 17.

[c] Lem. 11.

[b] Cor. 5. Lem. 8.



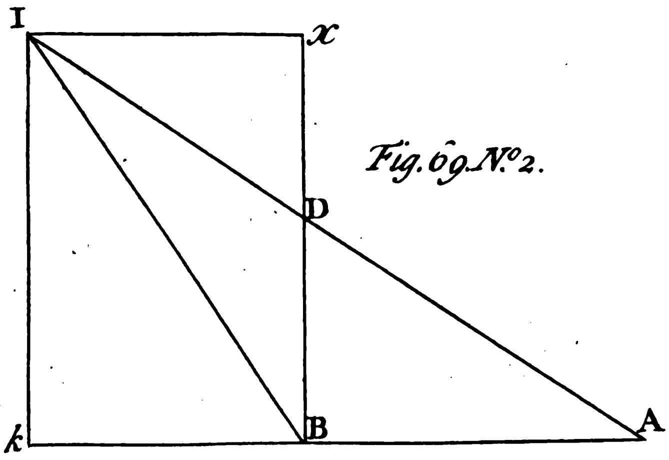


Fig. 69. No. 2.

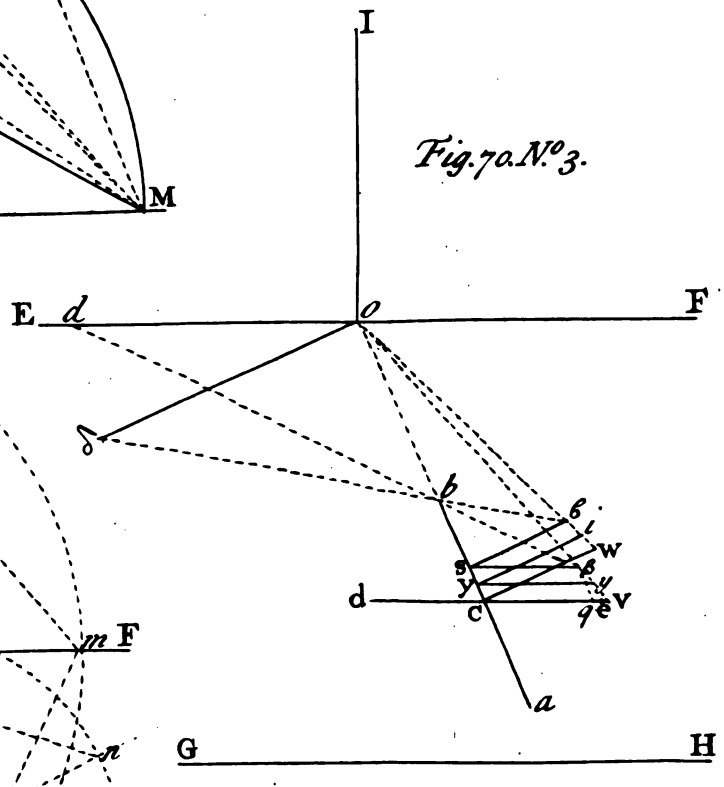
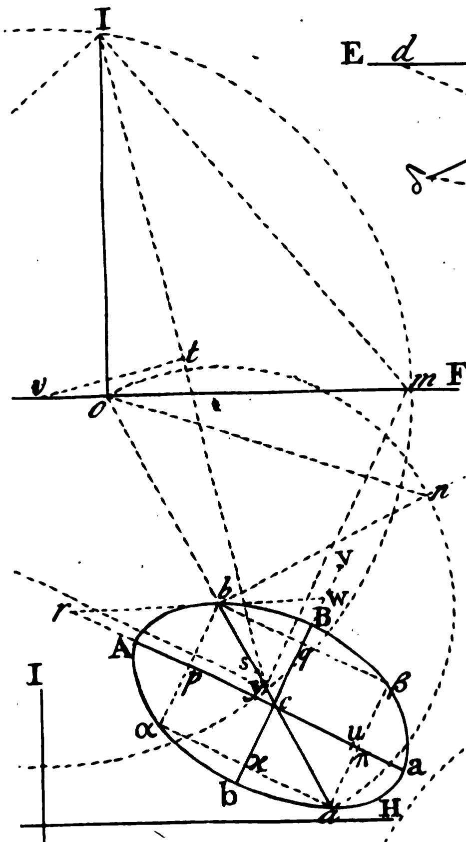
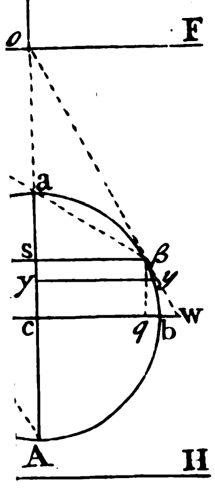


Fig. 70. No. 3.





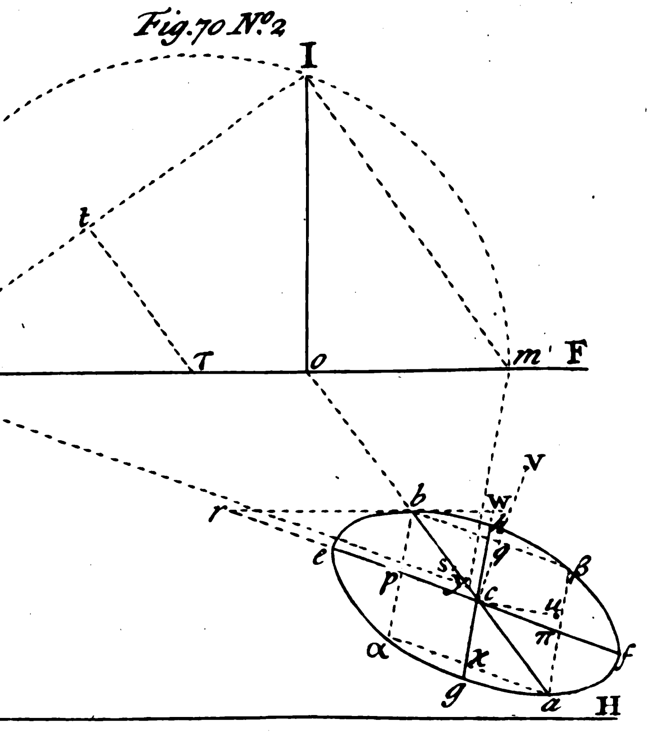


Fig. 70. No. 2.

J. Mynde sc.

and D E are the Originals of two Diameters, and C the Original of the Center of the *Ellipſis*[a]; and becauſe all Lines drawn from L cutting the Circle and the Line AB, are Harmonically divided by the Circle and that Line[b], the Images of the Parts of thoſe Lines which lie within the Circle, are biſected by the Image of AB[c], and are therefore double Ordinates to the Diameter repreſented by AB[d]; and becauſe of the Directing Point L, the Images of all thoſe Lines being parallel to each other, and to the Image of D E[e], D E is therefore the Original of a Diameter of the *Ellipſis*, Conjugate to the Diameter repreſented by A B[f]; and as the Images of all Lines drawn from M and terminated by the Circle, are parallel to the Image of A B, and biſected by the Image of D E, they are therefore double Ordinates to the Diameter repreſented by D E. Laſtly becauſe of the Semicircle L I Y M, the Angle L I M is Right[g], and I M and I L being the Directors of A B and D E, their Images are therefore perpendicular[h], conſequently A B and D E being the Originals of two Conjugate Diameters, which are perpendicular to each other and to their reſpective Ordinates, A B and D E are the Originals of the Axes of the *Ellipſis*[i], and M and L are the Directing Points of their reſpective Ordinates; and if through M and L, the Tangents to the Circle M D, M E, L A, L B be drawn, their Images will be Tangents to the *Ellipſis* in the Extremities of the Axes repreſented by D E and AB. *Q. E. I.*

C O R. 1.

The Originals A B and D E of the Axes being found; thence to determine which of them repreſents the Tranſverſe Axe.

Biſect the Angle L I M made by the Directors of the Axes, by the Line I *x* cutting L M in *x*, and from *x* through the Extremity A of either of the Axes A B, draw *x* A till it cut D E the Original of the other Axe in *n*; then if the Point *n* fall without the Circle, A B will be the Original of the Tranſverſe Axe, but if *n* fall within the Circle, A B will be the Original of the ſecond Axe.

For the Angle L I M being biſected by I *x*, the Image of the Triangle *n* C A is an Iſoſceles Triangle, having its Sides correſponding to C *n* and C A equal[k]; wherefore if C D be ſhorter than C *n*, its Image will be ſhorter than the Image of C *n*, and conſequently ſhorter than that of C A, wherefore C A is the Original of the longer or Tranſverſe Semiaxe. On the contrary, if *n* did fall within the Circle, the Image of C D would be greater than the Image of C *n* or A C, and D E would then be the Original of the Tranſverſe Axe.

If inſtead of drawing *x* A, a Line *x* D were drawn, it would cut A C within the Circle, which would ſtill ſhew C A to be the Original of the longer Semiaxe.

C O R. 2.

The Chord of the Tangents to the Circle from any Point L in the Line L M, is always the Original of a Diameter of the *Ellipſis*, and if a Line be drawn through the ſame Point L and the Point C, it will be the Original of a Diameter Conjugate to the other.

For the Chord of the Tangents from any Point L in the Line L M, always paſſes through C[l], which is the Original of the Center of the *Ellipſis*; and L being the Directing Point of the Ordinates to that Diameter, a Line drawn through that Point and the Point C, muſt be the Original of another Diameter of the *Ellipſis*, parallel to the Ordinates of the firſt, and conſequently Conjugate to it[m].

C O R. 3.

The Diameter *a b* of the Circle, which is perpendicular to the Directing Line, is always the Original of a Diameter of the *Ellipſis*; and *d e* the Chord of the Tangents to the Circle from *k*, where the perpendicular Diameter *a b* meets the Directing Line, is always the Original of a Diameter of the *Ellipſis* Conjugate to the Diameter repreſented by *a b*; and *ab* is the only Diameter of the Circle, the Image of which can be a Diameter of the *Ellipſis*.

For *a b* paſſing always through C, is therefore the Original of a Diameter of the *Ellipſis*, and *d e* which paſſes through C, is the Original of another Diameter[n]; and in regard the Image of *d e*, and of all other Lines drawn in the Circle parallel to *d e* and terminated by the Circle, are parallel to each other, and biſected by the Image of *a b*[o], all ſuch Lines are the Originals of double Ordinates to the Diameter repreſented by *a b*; wherefore the Diameter repreſented by *d e*, which is parallel to thoſe Ordinates, is Con-

E jugate

[a] Ellip. Art. 5.

[b] Lem. 11.

[c] Cor. 5. Lem. 8.

[d] Ellip. Art. 5.

[e] Cor. 4. Theor. 12. B. I.

[f] Ellip. Art. 4.

[g] 31 El. 3.

[h] Cor. 4. Theor. 12. B. I.

[i] Ellip. Art. 3.

[k] Cor. 1. Prob. 15. and Schol. Prob. 21. B. II.

[l] Cor. Lem 12.

[m] Ellip. Art. 4.

[n] Ellip Art. 4.

[o] Cor. 1. Theor. 23. B. I.

jugate to the Diameter repreſented by ab; and it is evident, that no other Diameter of the Circle beſides ab can paſs through C, and therefore that no other Diameter of the Circle can be the Original of a Diameter of the *Ellipſis*.

2. To determine the Originals of any two Conjugate Diameters of the *Ellipſis*.

From any Point o in the Line LM with the *Radius* oY, deſcribe a Semicircle LYM, cutting LM in any two Points L and M; then two Lines drawn from L and M through C will be the Originals of two Conjugate Diameters of the *Ellipſis*[a], and L and M will be the Directing Points of their reſpective Ordinates, and of the Tangents at their Extremities. *Q. E. I.*

3. To determine the Originals of the two Conjugate Diameters of the *Ellipſis* which are equal.

Biſect the Angle LYM by the Line Yπ cutting LM in π, and find tl the Chord of the Tangents to the Circle from π, and from π through C draw my; then tl and my will be the Originals of the Diameters required.

From π draw πD, which will likewiſe paſs through A[b]; then becauſe the Images of tl and my are two Conjugate Diameters of the *Ellipſis*[c], and the Image of tl biſecting the Image of AD[d], to which the Image of my is parallel[e], therefore tl and my are the Originals of the Diameters ſought[f].

The Lines tl and my may likewiſe be found, by drawing tl through p and q, and my through s and r[g], the Figure $sprq$ repreſenting the Parallelogram SPRQ in Fig. 68. No. 1.[h] *Q. E. I.*

4. To determine the Originals of the *Foci*.

Biſect the Angle LIM made by the Directors of the Axes, by the Line Ix cutting LM in x, and draw xA cutting DE produced in n; from n draw two Tangents to the Circle touching it in g and i, and from L through g and i draw gf and ib cutting AB in v and w; then v and w will be the Originals of the *Foci*.

For the Image of CA the Semitranſverſe Axe, being equal to the Image of Cn, which is Part of the Conjugate Axe[i], and the Images of the Tangents ng and ni being Tangents to the *Ellipſis* in the Points repreſented by g and i, and the Images of gf and ib being Ordinates to the Tranſverſe Axe paſſing through thoſe Points, the Points v and w, where thoſe Ordinates cut that Axe, are therefore the Originals of the *Foci*[k]. *Q. E. I.*

COR.

The Lines gf and ib are each the Original of the Meaſure of the Parameter of the Tranſverſe Axe[l].

CASE 2.

If the Center of the Circle be in the Line of Station, that is, if k were the Foot of the Eye's Director, then ab and de would be the Originals of the Conjugate Axes.

Dem. For ab and de are the Originals of two Conjugate Diameters[m], and their Images in this Poſition of the Circle being perpendicular, they are therefore the Originals of the Axes[n]. *Q. E. D.*

COR.

If J be the Place of the Eye, and it be required to determine which of the Lines ab or de is the Original of the Tranſverſe Axe; Take kx on the Directing Line, equal to kJ the Height of the Eye, and from x through d draw a Line, which if it cut ab within the Circle, will ſhew ab to be the Original of the Tranſverſe Axe, but if it cut ab without the Circle, ab will be the Original of the ſhorter Axe[o].

For here de the Original of one of the Axes, being parallel to the Directing Line, its imaginary Director is a Line drawn parallel to it through J, and Jk is the Director of the other Axe; if then the Right Angle made by theſe two Directors be biſected by a Line from J, it is evident, that Line muſt cut LM in x, ſo that Jk and kx will be equal.

CASE. 3.

If the Center of the Circle be in the Line of Station, and the Height of the Eye be equal to kY, the Image of the Circle will be a Circle, that is, the Section of the Viſual Cone by the Picture will be ſubcontrary.

It has been already ſhewn, that when the Center of the forming Circle is in the Line of Station, the Lines ab and de are the Originals of the Conjugate Axes[p]; it

muſt

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[a] Lem. 16. and Part firſt of this Prob.

[b] Lem. 20.

[c] Cor. 2. Part firſt of this Prob.

[d] Lem. 11. and Cor. 5. Lem. 8.

[e] Cor. 4. Theor. 12. B. I.

[f] Ellip. Art. 6.

[g] Lem. 20.

[h] Ellip. Art. 6.

[i] Cor. 1. Part firſt of this Prob.

[k] Ellip. Art. 10.

[l] Ellip. Art. 9.

[m] Cor. 3. Part firſt of this Prob.

[n] Ellip. Art. 3.

[o] Cor 1. Part firſt of this Prob.

[p] Caſe 2.

muſt be now ſhewn, that at the Height of the Eye kY, the Images of ab and de are equal, and conſequently that the Curve produced is a Circle[a].

Dem, Becauſe of the Circle ADBE[b] $Ca : Ce :: Ce : Cb$ Fig. 69. N°. 1.
And alſo $ka : ke :: ke : kb$
And becauſe ka is Harmonically divided in k, b, C, and a[c], $Ca : Cb :: ka : kb$
Conſequently $Ca : ka :: Ce : ke$
But by the Suppoſition ke is equal to kY the Director of the Line ka, therefore } $Ce : kY :: Ca : ka$

And conſequently the Images of Ca and Ce are equal[d]; but the Images of Ca and Cb being equal, as alſo the Images of Ce and Cd[e], the Image of ab is therefore equal to the Image of de, and conſequently the Curve produced by the Image of the Circle, is a Circle. *Q. E. D.*

COR. 1.

When the Height of the Eye is equal to kY, the Images of all thoſe Lines in the forming Circle, which ſhould be Conjugate Diameters of the *Ellipſis*, will be perpendicular to each other.

It has been ſhewn, that all Semicircles deſcribed from any Point in the Directing Line as a Center, and paſſing through Y, will cut the Directing Line in the Directing Points of the Originals of two Conjugate Diameters of the *Ellipſis*[f]; but if Y be the Place of the Eye, the Directors drawn from thence to the Extremities of the Diameter of any ſuch Semicircle will be perpendicular to each other[g]; therefore the Images of all Lines drawn from the two Directing Points, which are the Extremities of that Diameter, and conſequently of thoſe Lines which ſhould produce two Conjugate Diameters of the *Ellipſis*, will be perpendicular to each other[h].

COR. 2.

If the Height of the Eye be greater than kY, ab will be the Original of the Tranſverſe Axe; if the Height of the Eye be leſs than kY, ab will be the Original of the ſecond Axe, but in either Caſe the Image of the Circle muſt be an *Ellipſis*.

It has been proved, that when the Height of the Eye is equal to kY, the Images of ab and de are equal[i]. Now if the Height of the Eye be increaſed or diminiſhed, the Image of ab will be increaſed or diminiſhed in the ſame Proportion[k], whilſt the Image of de continues of the ſame Length at all Heights of the Eye in the ſame Directing Plane[l]; wherefore if the Height of the Eye be greater than kY, the Image of ab will be longer than the Image of de; and if the Height of the Eye be leſs than kY, the Image of ab will be ſhorter than that of de; and thus the two Axes being unequal, the Figure produced muſt be an *Ellipſis*.

SCHOL.

Although it be a ſufficient Proof that the Section is ſubcontrary when the Height of the Eye is equal to kY, by ſhewing the Image produced to be a Circle; yet it may be otherwiſe ſhewn, that in this Caſe the Viſual Cone is cut ſubcontrarily by the Picture.

Let I$x$$k$B repreſent the Vertical Plane, wherein kA is the Line of Station, AB the Diameter of the Original Circle, and kI the Height of the Eye taken a mean Proportional between kB and kA; then IBA will be the Triangle formed by the Section of the Cone with the Vertical Plane. It muſt be proved, that DB the Section of the Picture with that Triangle cuts it ſubcontrarily. Fig. 69. N°. 2.

In the Similar Triangles IkA, DBA, $BA : BD :: kA : kI$
But by the Suppoſition $kA : kI :: kI : kB$
Therefore $BA : BD :: kI : kB$

And conſequently the Triangles IkB and ABD are Similar, the Angles IkB, DBA being equal[m], and the Sides ſubtending thoſe Angles, proportional[n]; wherefore the Angles BAD and kIB are equal; but the alternate Angles kIB, IBD are equal[o], the Angle IBD is therefore equal to the Angle BAD, and conſequently the Triangle IBA is cut ſubcontrarily by BD: and ſo it will be wherever the Picture is placed, ſo long as it remains parallel to the ſame Directing Plane; for ſtill the Section of the Picture with the Triangle IBA will be parallel to Ik, and conſequently to BD.

Hence it is evident no ſubcontrary Section can be produced at any different Height of the Eye, the Diameter of the Circle BA, and the Diſtance kB remaining the ſame.

For

[a] Ellip. Art. 7.

[b] 35 and 36 El. 3.

[c] Lem. 11.

[d] Theor. 32. B. I.

[e] Cor. 5. Lem. 8.

[f] Part ſecond of this Prob.

[g] 31 El. 3.

[h] Cor. 4. Theor. 12. B. I.

[i] Caſe 3.

[k] Theor. 27. B. I.

[l] Cor. 4. Theor. 23. B. I.

[m] 29 El. 1.

[n] 6 El. 6.

[o] 29 El. 1.

For although B A will ſtill be to BD, as *k* A to *k*I; yet if *k*I be either bigger or leſs than *k* Y, *k* A will not then be to *k*I, as *k* I to *k*B; and conſequently the Triangles I*k*B and ABD will not be Similar, on the Similitude of which, the above Demonſtration is founded.

PROB. II.

The Image of that Diameter of a Circle which is perpendicular to the Directing Line of its Plane, being given; thence to determine the Axes, or any two other Conjugate Diameters of the *Ellipſis* formed by the Image of the Circle.

1. To determine the Axes.

Fig. 70. N°. 1. Let E F be the Vaniſhing Line of the Plane of the Circle, *o* the Center of that Vaniſhing Line, and I *o* its Diſtance, and let *a b* be the given Image of the Diameter of the Circle, its Vaniſhing Point being *o*, and s the Image of the Center of the Circle.

Biſect *a b* in *c*, which will be the Center of the *Ellipſis*, *a b* being one of its Diameters[a]; then take *o* y in the Line *ab*, a mean Proportional between *ob* and *o a*[b], and draw Iy: biſect I y in *t* by the Perpendicular *t v*, cutting E F in *v*, and from *v* as a Center with the *Radius* *v* y or *v* I, deſcribe the Circle I *l* y *m* cutting E F in *l* and *m*, and draw *l* y and *m* y: laſtly, through *c* draw A a parallel to *l* y, and B b parallel to *m* y, and A a and B b will be the Indefinite Axes ſought.

Fig. 69, 70. *Dem.* Here *a b* repreſents the Diameter *a b* of the Circle A D B E in Fig. 69, N°. 1. and *c* repreſents C[c]; and becauſe *k*Y in that Figure, is a mean Proportional between *k a* and *kb*[d], the Image of Y will fall in ſuch manner in *a b*, as that *o* y will be a mean Proportional between *o b* and *o a*[e], wherefore y, found as before directed, is the Repreſentation of Y in the other Figure.

Again, the Situation of L and M with reſpect to Y in the Original Plane is ſuch, that Lines drawn from L and M to Y, are not only perpendicular, but have perpendicular Images: now becauſe of the Circle I *l* y *m*, whoſe Diameter is *lm*, the Angles *l* I *m* and *l* y *m* are both Right[f], therefore *l* y and *m* y are perpendicular, as well as their Originals[g], and conſequently repreſent L Y and M Y in the Original Plane, there being no two other Lines which can paſs through y with theſe Conditions, in regard that *v* is the only Point in E F, from whence as a Center a Circle can be deſcribed which ſhall paſs through I and y: and becauſe of the Directing Points L and M, the Images of L C and M C which paſs through C, are parallel to the Images of L Y and M Y[h], wherefore the Indefinite Lines A a and B b, drawn in the Picture through *c* parallel to *l* y and *m* y, repreſent L C and M C in the Original Plane, and are therefore the Indefinite Axes deſired[i]. *Q. E. I.*

Now to determine the Length of the Axes thus found:

Through either of the Extremities *b* of the given Diameter *ab*, draw *r* w parallel to E F, cutting A a and B b in *r* and w, and through *b* and *a* draw *b β* and *a α* parallel to A a, and *b α* and *a β* parallel to B b, cutting A a in *p* and *π*, and B b in *q* and *κ*; on *π r* as a Diameter deſcribe a Semicircle cutting B b in v, and make *c* A and *c* a each equal to *c* v; alſo on *κ* w as a Diameter deſcribe a Semicircle cutting A a in *u*, and make *c* B and *c* b each equal to *c u*, then A a and B b will be the determinate Axes ſought.

Dem. For the Original of *r* w being perpendicular to the Original of *a b*, which is a Diameter of the forming Circle, it is therefore a Tangent to the Circle in the Point repreſented by *b*[k], *r* w is therefore a Tangent to the *Ellipſis* formed by the Image of the Circle, in *b*, and *b p* being an Ordinate to the Axe A a, and *r* the Point where the Tangent from *b* cuts that Axe, the half of that Axe is a mean Proportional between *c p* and *c r*[l]; but by the Conſtruction *c π* and *c p* are equal, therefore *c* v, which is a mean Proportional between *c π* and *c r*[m], is alſo a mean Proportional between *c p* and *c r*, conſequently *c* A and *c* a being taken each equal to *c* v, the Axe A a is thereby rightly found: after the ſame manner *c q* and *c κ* being by Conſtruction equal, *c u* which is a mean Proportional between *c κ* and *c* w, is alſo a mean Proportional between *c q* and *c* w; wherefore *c* B and *c* b being each taken equal to *c u*, the Axe B b is thereby truly determined. *Q. E. I.*

2. To determine any two Conjugate Diameters of the *Ellipſis*, either of the Indefinite Diameters being given.

Fig. 70. N°. 2. The Image *a b* of the perpendicular Diameter of the forming Circle being given, and the

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[a] Cor. 3. Part firſt of Prob. 1.

[b] Lem. 23.

[c] Prob. 1.

[d] Lem. 17.

[e] Theor. 30. B. I.

[f] 31 El. 3.

[g] Cor. 3. Theor. 11. B. I.

[h] Cor. 4. Theor. 12. B. I.

[i] Prob. 1.

[k] 18 El. 3.

[l] Ellip. Art. 11.

[m] 13 El. 6.

the Points c, y, and s found in it as before; let ef be an Indefinite Diameter of the *Ellipſis* propoſed, to which it is required to find the proper Conjugate, and to determine the Extremities of both.

Through y draw yl parallel to ef cutting EF in l, and having drawn lI, draw Im perpendicular to it, cutting EF in m, and draw my, and parallel to it through c draw gh, then ef and gh will be two Indefinite Conjugate Diameters of the *Ellipſis*.

Dem. Becauſe of the Vaniſhing Points l and m, the Originals of ly and my are perpendicular, and they paſſing through Y in the Original Plane, if they be produced to their Directing Points, thoſe Points will be the Extremities of the Diameter of a Circle in that Plane, which will paſs through Y[a], wherefore thoſe Points will be the Directing Points of two Conjugate Diameters of the *Ellipſis*[b], conſequently ef and gh which paſs through c parallel to ly and my, and which have therefore the ſame Directing Points with them[c], are Indefinite Conjugate Diameters of the *Ellipſis*. *Q. E. I.*

Fig. 69. N°. 1.

[a] 31 El. 3.

[b] Part ſecond of Prob. 1.

[c] Cor. 5. Theor. 12. B. I.

Now to determine the Extremities of theſe Conjugate Diameters; through b draw rw parallel to EF cutting ef and gh in r and w, and through b and a draw $b\beta$, $a\alpha$ parallel to ef, and $b\alpha$, $a\beta$ parallel to gh, cutting ef and gh in p, π, q, and $\varkappa$, and make ce and cf each a mean Proportional between cp and cr, and alſo cg and ch each a mean Proportional between cq and cw, and thereby the determinate Conjugate Diameters ef and gh will be found.

Dem. This is demonſtrated in the ſame manner as the Preceeding[d]; but in the Practice for finding the Length of the mean Proportionals, the Lines ef and gh not being perpendicular (as the Axes are) there muſt be two Lines cv and cu drawn through c perpendicular to ef and gh, on which thoſe Proportionals may be marked at v and u, by Semicircles drawn on the Diameters πr and $\varkappa$w as before. *Q. E. I.*

[d] Ellip. Art. 11.

COR.

Hence if from any Point τ in EF as a Center, a Semicircle be deſcribed paſſing through I, and cutting EF in any two Points l and m, and ly and my be drawn; two Lines drawn through c parallel to ly and my, will be Indefinite Conjugate Diameters of the *Ellipſis*.

For the Radials Il and Im will be perpendicular to each other.

3. To determine the Diameter of the *Ellipſis* Conjugate to the given Diameter ab.

The Points c, y, and s being found in ab as before, through c and s draw de and sβ parallel to EF, and having made sβ to repreſent a *Radius* of the forming Circle[e], draw $o\beta$ cutting de in v, and from β draw βq parallel to ab, cutting cv in q, then make cd and ce each a mean Proportional between cq and cv, and the Conjugate Diameter de will be thereby determined.

Fig. 70. N°. 3.

[e] Cor. 4. Method 2. Prob. 24. B. II.

Dem. In the firſt Place, ab is a Diameter of the *Ellipſis*, as well as the Image of the perpendicular Diameter of the forming Circle, and de drawn through c parallel to EF, is the Indefinite Diameter of the *Ellipſis* conjugate to ab[f]. Now the Originals of $o\beta$ and sβ being perpendicular, and sβ repreſenting a *Radius* of the forming Circle, the Original of $o\beta$ is a Tangent to the Circle in the Point repreſented by β, $o\beta$ is therefore a Tangent to the *Ellipſis* in β, and βq is an Ordinate to the Diameter de; wherefore ce and cd being each made a mean Proportional between cq and cv, the Conjugate Diameter de is thereby rightly determined[g]. *Q. E. I.*

[f] Cor. 3. Part firſt of Prob. 1.

[g] Ellip. Art. 11.

COR. 1.

If through y a Line yy be drawn parallel to sβ, till it cut $o\beta$ in y, yy will be equal to ce or cd, whence theſe laſt may be readily found.

For oy being a mean Proportional between oc and os[h], yy is a mean Proportional between cv and sβ[i], which laſt is equal to cq.

[h] Lem. 16.

[i] Cor. Lem. 19.

COR. 2.

If the Diſtance Io be equal to oy, the Conjugate Diameters ab and de will be equal.

Through s, y, and c draw sς, yi, and cw perpendicular to oa, and draw $o\delta$ parallel to them, and equal to Io or oy, and draw δb cutting sς in ς.

Then becauſe oa is Harmonically divided in o, b, s, and a, if on ab as a Diameter, a Circle were deſcribed, s would be the Point in that Diameter, through which the Chord of the Tangents to that Circle from o would paſs[k]; therefore sς is the Indefinite Semichord of the Tangents from o, and δb determines its Extremity ς[l], and

[k] Lem. 11.

[l] Cor. 3. Lem. 20.

*o*ϐ cutting y*i* and *c*w in *i* and w, will determine y*i* the mean Proportional between sϐ and *c*w, and y*i* will be equal to *c b* the *Radius* of this Circle[a].

Now in the Similar Triangles δ*ob*, *b*sϐ, *ob* : *b*s :: δ*o* : sϐ
And in the Similar Triangles *d o*ϐ, *b*sβ, *ob* : *b*s :: *do* : sβ
But *do* = δ*o*, therefore sϐ = sβ.
Again, in the Similar Triangles *o*sϐ, *o*y*i*, *o*s : *o*y :: sϐ : y*i*
And in the Similar Triangles *o*sβ, *o*y*y*, *o*s : *o*y :: sβ : y*y*
But sϐ = sβ as before, therefore y*i* = y*y*

Conſequently *c*e which is equal to y*y*[b], is alſo equal to y*i*, which laſt was before ſhewn to be equal to *c b*; wherefore the Conjugate Diameters *ab* and de are equal.

C O R. 3.

Fig. 70. N°. 4. If the Center of the Original Circle be in the Line of Station, then the given Diameter A a coinciding with the Vertical Line, will be one of the Axes, and B b drawn through *c* parallel to E F will be the other Indefinite Axe[c], the Extremities of which may be determined as before; and if I*o* and *o*y be equal, the Axes A a and B b will be equal[d], and the Image of the Circle will therefore be a Circle[e].

S C H O L.

That A a and B b are the Axes, may be alſo demonſtrated from the Principles in the firſt Part of this Problem.

Fig. 70. N°. 1. For here A a and I*o* making one continued ſtraight Line, a Line from I to y coincides with it, and if I y be biſected by a Perpendicular, that Perpendicular muſt either be parallel to, or coincide with E F, and therefore can never cut E F, to determine the Center of the Circle which is to paſs through I and y, in order for the finding the Points *l* and *m*: this Center may therefore be conceived to be at an Infinite Diſtance from *o* in the Line E F, and the ſtraight Line I y may be taken as a Portion of the Infinite Circle deſcribed from that Center, and *o* as one of the Points of Interſection of that Circle with E F. Hence A a, which here coincides with *o*y, is one of the Axes; and the other Interſection of this Infinite Circle with E F, which ſhould determine the Vaniſhing Point, to which a Line from y ſhould be drawn, and be parallel to the other Axe, being infinitely diſtant, that Axe muſt therefore be parallel to E F, wherefore B b drawn through *c* parallel to E F, is the Indefinite Axe Conjugate to the Axe A a.

Likewiſe that the Section is ſubcontrary, when I*o* and *o*y are equal, may be proved in this manner.

Fig. 69. N°. 1. It was ſhewn at Caſe 3. Prob. I. That when *k* Y is equal to the Height of the Eye, a ſubcontrary Section is produced: but when *k* Y is equal to the Height of the Eye, *o*y the Complement of the Image of *k*Y, will be equal to I*o* the Radial of *k* Y[f], which is here the ſame with the Diſtance of the Vaniſhing Line E F, conſequently when I*o* and *o*y are equal, the Section is ſubcontrary.

C O R. 4.

Fig. 70. N°. 4. If I*o* be greater than y*o*, B b will be the larger or Tranſverſe Axe; if I*o* be leſs than y*o*, B b will be the ſmaller Axe.

For if I*o*, or its equal *do*, be increaſed or diminiſhed, sβ and *c*w, and conſequently *c*b will be increaſed or diminiſhed in the ſame Proportion, whilſt *c*a remains the ſame ſo long as A a is ſuppoſed to be the given Image of the Diameter of the forming Circle.

P R O B. III.

The Image of any Diameter of an Original Circle which lies wholly on one ſide of the Directing Line of its Plane being given; from the Image of any Point in that Diameter produced without the Circle, to draw two Tangents to the *Ellipſis* formed by the Image of the Circle.

Fig. 71. N°. 1, 2. Let E F be the Vaniſhing Line of the Plane of the Circle, and I*o* its Diſtance, and let *ab* be the given Image of the Diameter, and *l* the Image of a Point in that Diameter produced, from whence the Tangents are to be drawn, and *x* the Vaniſhing Point of *ab*.

Find

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[a] Lem. 19.

[b] Cor. 1.

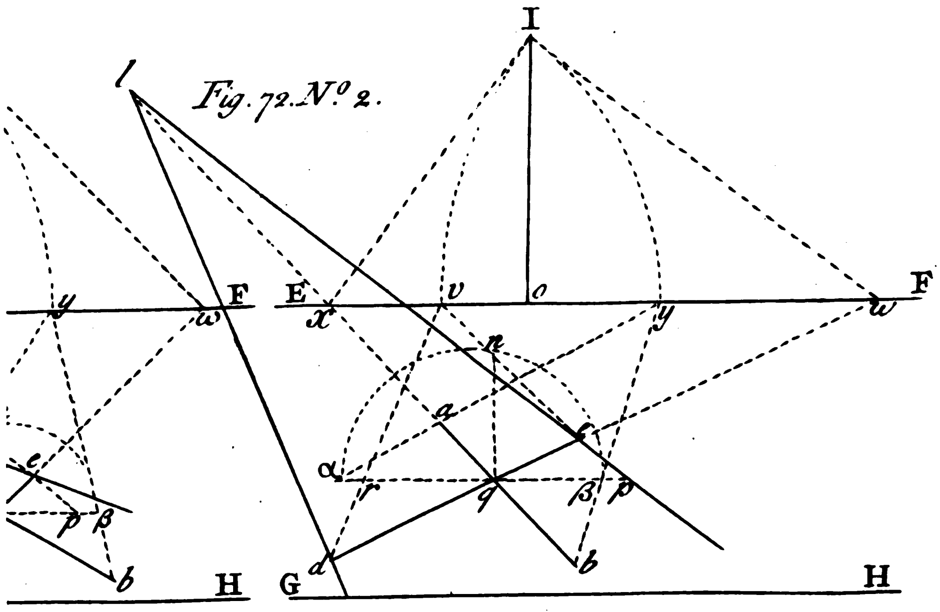
[c] Caſe 2. Prob. 1.

[d] Cor. 2.

[e] Ellip. Art. 7.

[f] Theor. 24. B I.

Fig. 72. No 2.



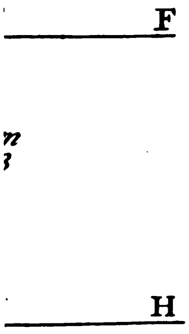


Fig. 73.

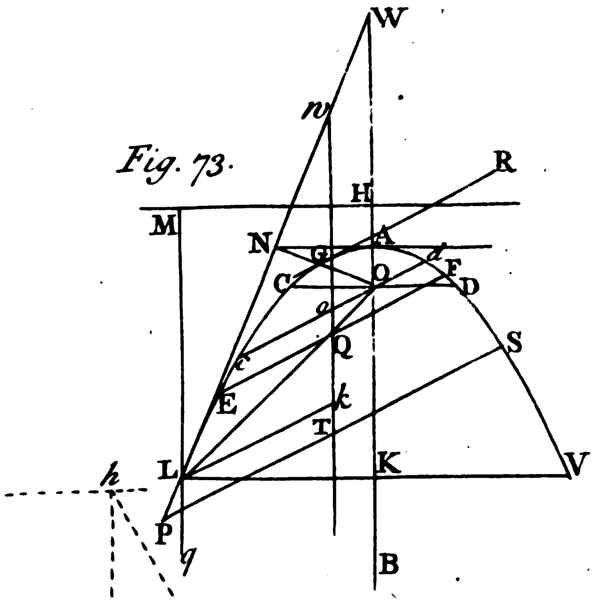
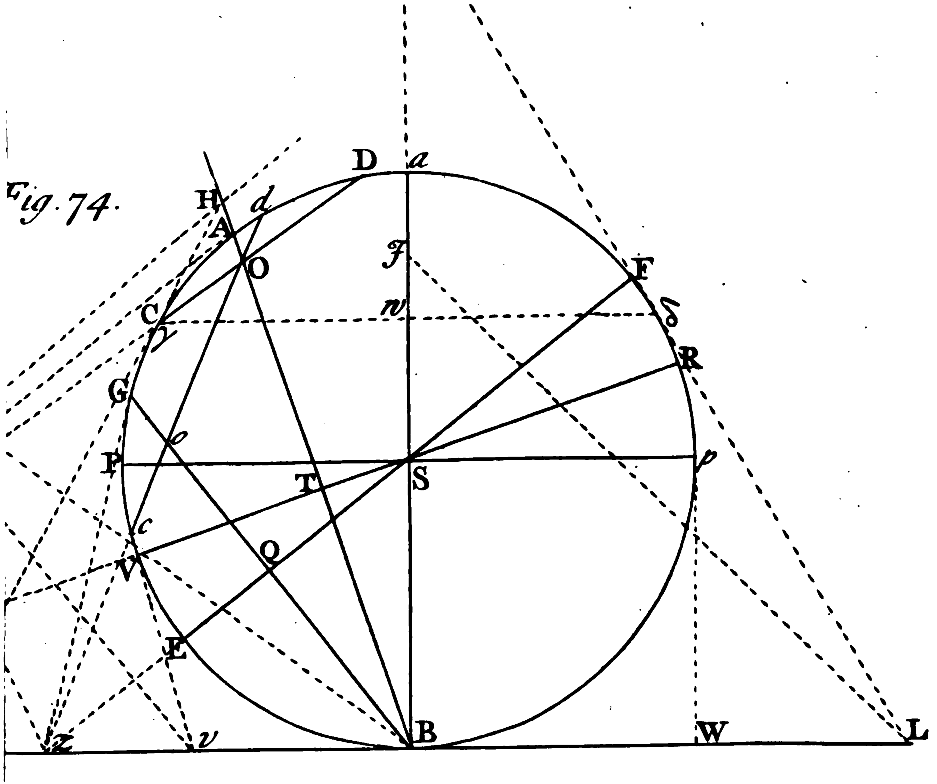


Fig. 74.



J. Mynde sc.

Find a Point q between a and b, ſo that the whole Line lb may be Harmonically divided in the Points l, a, q, and b [a], or (which is the ſame) conſider l as a Vaniſhing Point, and from thence make aq and qb repreſent equal Lines [b]; produce ab to its Vaniſhing Point x, and having drawn the Radial xI, and Iw perpendicular to it, cutting EF in w, draw wq; through q draw $\alpha\beta$ parallel to EF, and find the proportional Meaſures αq and $q\beta$ of the Parts aq and qb of the given Diameter [c]; and having found qn, a mean Proportional between αq and $q\beta$ [d], from q on the Line $\alpha\beta$ ſet off qr and qp, each equal to qn; then in the Line wq, make qe and qd repreſent Lines equal to the Originals of qp and qr [e], and draw le and ld, and theſe will be the Tangents required.

Dem. Becauſe the Chord of the Tangents to the Original Circle from the Original of the Point l cuts the Original of ab in a Point, ſo that the Original of the whole Line lb is Harmonically divided by that Point, and the Originals of l, a, and b [f]; the Image lb is therefore alſo Harmonically divided by the Images of thoſe Points [g]; conſequently the Point q taken between a and b, and which compleats the Harmonical Diviſion of lb, is the Image of that Point of the Original of the Diameter ab, through which the Chord of the Tangents from the Original of l paſſes [h], and the Original of that Chord being perpendicular to the Original of ab [i], de drawn through q to the Vaniſhing Point w, is therefore the Indefinite Image of that Chord; and the Semichord of the Tangents being a mean Proportional between the Segments of the Diameter of the Circle to which it is perpendicular [k], the Lines qe and qd, being by Conſtruction made to repreſent mean Proportionals between the Originals of aq and qb, are therefore the determinate Images of the Semichords deſired, and conſequently le and ld are the Images of Tangents to the Circle from the Original of the Point l, and are therefore Tangents to the *Ellipſis* formed by the Circle, in e and d from the Point l.

The ſame Method and Demonſtration holds good wherever the Point l be taken, if it be not either a Vaniſhing or Directing Point; in regard that the Original of the produced Diameter lb is always Harmonically divided by the Originals of a and q, and conſequently the Image lb is ſo too, whether the Original of the Point l be before or behind the Circle, or before or behind the Directing Plane [l]. *Q. E. I.*

C O R. 1.

If the Point l be a Vaniſhing Point, that is, if the Originals of the Tangents required be parallel to the given Diameter, it is evident they muſt touch the Circle in the Extremities of a Diameter perpendicular to the given Diameter, and therefore the Indefinite Image lb will be Harmonically divided in a and q [m]; but q then repreſents the Center of the Circle, and ed a Diameter of the Circle perpendicular to the Original of ab, the Extremities of which may be found by the Methods before propoſed [n].

C O R. 2.

If the Point l be a Directing Point, then ab muſt be biſected in q [o], which will give the Image of the Point in ab, through which the Chord of the Tangents from l paſſes; and the Extremities of this Chord being determined by the Method in this Problem, the Images of the Tangents themſelves muſt be drawn through thoſe Extremities parallel to the given Diameter ab [p].

C O R. 3.

If the entire Image of the Circle were given, the Work would be greatly ſhortened; for then the Points q and w being found as before, the Line wq will cut the given Image in e and d, the Extremities of the Chord of the Tangents from l, whence le and ld are determined without farther trouble.

This Problem is the ſame in effect with Problem XXVIII. *Book* II. *but although the Method here propoſed is more univerſal and convenient than what was there ſhewn, it could not be inſerted in that Place, it depending on Principles not then explained.*

P R O B. IV.

Any *Ellipſis* $a\alpha b\beta$ being given, thence to determine the Vaniſhing Line, Center, and Diſtance of a Plane, in which an Original Circle being placed, its Image ſhall be the given *Ellipſis*. Fig. 72.

In the firſt Place, this Curve may be produced by a Circle in any Plane, whoſe

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[a] Lem. 1.

[b] Cor. 6. Lem. 8.

[c] Cor. 1. Prob. 9. B. II.

[d] 13 El. 6.

[e] Cor. 1. Prob. 8. B. II.

[f] Lem. 11.

[g] Cor. 4. Lem. 8.

[h] Lem. 2.

[i] Cor. Lem. 4. B. II.

[k] 3 and 35 El. 3.

[l] Cor. 4. Lem. 8.

[m] Cor. 1. Lem. 8.

[n] Meth. 2. Prob. 24. B. II.

[o] Cor. 5. Lem. 8.

[p] Cor. 4. Theor. 12. B. I.

Vaniſhing Line neither cuts nor touches the given Figure; becauſe that Figure being then all on the ſame ſide of the Vaniſhing Line, the Original Circle which forms it, muſt lie all on the ſame ſide of the Directing Line of the Original Plane, the Image of which Circle is therefore an *Ellipſis*[a].

Having therefore taken any Line EF at pleaſure, not touching or cutting the given *Ellipſis*, for the Vaniſhing Line of the Plane of the forming Circle; the Center and Diſtance of that Vaniſhing Line may be found in this manner.

Draw in the *Ellipſis* any two Lines lm and nr parallel to EF, and terminated by the *Ellipſis* in the Points l, m, n, and r; biſect lm, and nr in q and p, and through q and p draw ab meeting EF in o; then ab will be a Diameter of the *Ellipſis*, to which lq, qm, np, and pr are Ordinates[b], and theſe being parallel to EF, ab is therefore a Diameter of the *Ellipſis* Conjugate to a Diameter of that Figure parallel to EF[c], and conſequently repreſents a Diameter of the forming Circle perpendicular to the Interſecting Line of its Plane[d], wherefore o, the Vaniſhing Point of ab, is the Center of the Vaniſhing Line EF.

Find s in the determinate Line ab, ſo that as and sb may repreſent equal Lines[e], and through s draw $\alpha\beta$ parallel to EF, and s will repreſent the Center, and $\alpha\beta$ a Diameter of the forming Circle, parallel to the Interſecting Line of its Plane; laſtly $a\alpha$ or $b\beta$ being drawn cutting EF in d, take oI perpendicular to EF and equal to od, and oI will be the Diſtance of the Vaniſhing Line EF: for the Originals of sb and $s\beta$ being equal, βb muſt terminate in d, the Point of Diſtance of the Vaniſhing Point o[f].

The Center and Diſtance of the Vaniſhing Line of the Plane of the forming Circle, and the Images ab and $\alpha\beta$ of two Diameters of that Circle, the one perpendicular, and the other parallel to the Interſecting Line of that Plane, being thus found, the forming Circle itſelf may lie in any Plane which hath EF for its Vaniſhing Line; but if any Line GH be drawn parallel to EF, and taken as the Interſecting Line of the Plane of the forming Circle, the particular Plane is thereby determined, in which the forming Circle lies, which therefore may be deſcribed by finding the Original of either of its Diameters repreſented by ab or $\alpha\beta$[g]. *Q. E. I.*

Of the Parabola.

Fig. 73. 1. Let LAV be a *Parabola*. Then if any two parallel Lines EF and PS be drawn in, and terminated by the Curve in E, F, P, and S, and EF and PS be biſected in Q and T, a Line TG drawn through Q and T, is called a *Diameter of the Parabola*; and the Point G where it meets the Curve, is called the *Vertex* or *Extremity of that Diameter*.

2. If any Line LV drawn perpendicular to GT, and bounded by the *Parabola* in L and V, be biſected in K, a Line AB drawn through K parallel to GT, and therefore perpendicular to LV, is the *Axe*, and its Extremity A is the *Vertex of the Parabola*; and all Lines drawn in the Curve parallel to LV, as CD, and terminated by the Curve, are biſected by AB, and each Moiety of ſuch Lines is an *Ordinate to the Axe* AB.

3. All Diameters of the *Parabola*, as AB, GT are parallel to each other, and none of them, though infinitely produced, can ever meet the *Parabola* in any other Point beſides its proper Vertex, as A or G.

4. If any Line EF drawn through any Diameter GT, and terminated by the Curve in E and F, be biſected by that Diameter in Q, then EQ and QF are *Ordinates to that Diameter*; and all Ordinates to the ſame Diameter are parallel to each other; but no Diameter hath its Ordinates at Right Angles to it, except only the Axe AB.

5. A Line drawn through the Extremity of any Diameter, parallel to the proper Ordinates of that Diameter, is a *Tangent to the Curve in that Point*; as AN drawn through A the Extremity of the Axe AB, parallel to KV, and GR drawn through G the Extremity of the Diameter GT, parallel to its proper Ordinate QF, are Tangents to the *Parabola* in A and G.

6. If from any Point L in the *Parabola* an Ordinate LK be drawn to the Axe AB, cutting it in K, take AW on the Axe produced beyond its Vertex A, equal to AK, and a Line LW will be a Tangent to the *Parabola* in L: likewiſe if from the ſame Point L an Ordinate Lk be drawn to any Diameter GT, cutting it in k, the Parts Gk, Gw of this Diameter intercepted between its Vertex G, the Ordinate Lk, and the Tangent LW, will be equal.

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[a] Con. Sec. Art. 9.

[b] Ellip. Art. 1.

[c] Ellip. Art. 4.

[d] Cor. 3. Part firſt of Prob. 1.

[e] Cor. 3. Prob. 8. B. II.

[f] Cor. 1. Prob. 8. B. II.

[g] Prob. 2. B. II.

7. If a Tangent at L cut the Axe in W, biſect LW in N, or, which is the ſame, produce the Tangent AN at the Vertex A, till it cut LW in N, and draw NO perpendicular to LW; the Point O where it cuts the Axe, is the ***Focus of the Parabola***, of which there is only one.

8. If through the *Focus* O, a Line CD be drawn perpendicular to the Axe AB; either Moiety OC or OD of that Line will be the double of AO, and the whole Line CD will be the Meaſure of the *Parameter* of the Axe: likewiſe a double Ordinate *cd* to any other Diameter GT drawn through the *Focus* O, is the Meaſure of the Parameter of that Diameter, and either Moiety *co* of that Ordinate is the double of *o*G: and therefore AO : CO :: CO : CD. and G*o* : *co* :: *co* : *cd*.

9 If the Axe AB be produced to H, until AH be equal to AO, or, which is the ſame, until OH be equal to CO; then a Line HM drawn through H parallel to AN or CD, is called the *Directrix* of the *Parabola*; and therefore a Tangent at C will meet the Axe in H.

10. If from the Point of Contact L of any Tangent LW, produced till it cut the Axe AB in W, a Line LO be drawn to the *Focus* O, the Lines LO and OW will be equal; and conſequently the Angle WLO, made by the Tangent LW with LO, will be equal to the Angle LWO, made by the Tangent with the Axe AB, or with any other Diameter GT; and the Line LO is equal to LM, drawn from L perpendicular to the Directrix MH; and either of them is equal to one fourth Part of the Parameter of the Diameter L*q*, which paſſes through L.

11. If an Ordinate EQ to any Diameter GT, cut it in Q, the Part GQ of that Diameter, intercepted between its Vertex and the Ordinate, is called the ***Abſciſſa***; and the Square of the Ordinate EQ is always equal to the Rectangle between the *Abſciſſa* GQ and the Parameter *cd* of that Diameter: that is, $EQ^2 = GQ \times cd$; or GQ : EQ :: EQ : *cd*.

PROB. V.

An Original Circle being given, touching the Directing Line of its Plane; therein to determine the Originals of the Axe and its Ordinates and Parameter, and of the Vertex, *Focus*, and Directrix of the *Parabola* formed by the Image of the Circle; and alſo the Originals of any other Diameters, and their proper Ordinates and Parameters, and the Angle made by any Diameter with its Ordinates.

1. To find the Originals of the Axe, and its Ordinates and Parameter, and of the Vertex, *Focus*, and Directrix of the *Parabola*:

Let BP*ap* be the Original Circle, touching the Directing Line KB in B; S the Fig. 74.
Center of the Circle, and IK the Eye's Director.

From the Point of Contact B to I draw BI, and perpendicular to it draw IN, cutting the Directing Line in N; from N as a Center with the *Radius* NB, draw an Arch cutting the Circle in A, and draw BA; biſect the Angle NIB by the Line IM, cutting KB in M, and from M as a Center with the *Radius* MB, draw an Arch cutting the Circle in C, and draw MC cutting AB produced in H, and from N through C and H draw ND and NH: then AB will be the Original of the Axe, and A that of its Vertex, O the Original of the *Focus*, and CD the Original of the Meaſure of the Parameter of the Axe; CO and OD will be the Originals of two Ordinates to the Axe, and N the Directing Point of thoſe Ordinates, and NH will be the Original of the Directrix of the *Parabola*.

Dem. From N through the Center of the Circle S, draw NR cutting the Circle in V and R; then becauſe NB is a Tangent to the Circle in B from the Point N, and NA is taken equal to it, NA is alſo a Tangent to the Circle in A from the ſame Point N; AB is therefore the Chord of the Tangents from N, conſequently the Images of CO and OD are equal, as are alſo the Images of VT and TR[a]; wherefore AB is the Original of a Diameter of the *Parabola*, to which the Images of CO, OD, VT, and TR are Ordinates[b]; but becauſe of the Directing Points N and B, the Angle NIB being by Conſtruction Right, the Images of CD and VR are perpendicular to the Image of AB; wherefore AB being the Original of a Diameter whoſe Ordinates are perpendicular to it, AB is the Original of the Axe[c], and A is therefore the Original of the Vertex of the *Parabola*[d].

Again, becauſe MB is a Tangent to the Circle in B from the Point M, and MC is

[a] Cor. 5. Lem. 8. and Lem. 11.

[b] Parab. Art. 1. and 4.

[c] Parab. Art. 4.

[d] Parab. Art. 2.

taken equal to it, MC is therefore alſo a Tangent to the Circle in C from the ſame Point M; but by Reaſon of the Directing Points N, B, and M, the Angle NIB being biſected by IM, the Image of the Triangle CHO is an Iſoſceles Triangle, having its Sides repreſented by CO and OH equal [a], conſequently the Images of the Angles HCO and CHO are equal; but the Angle HCO is that which the Tangent MC makes with the Original of the Ordinate CO, and the Angle CHO is that which the ſame Tangent makes with the Original of the Axe BA, and the Images of theſe two Angles being equal, the Ordinate repreſented by CO, therefore paſſes through the *Focus* [b]; and conſequently the Point O where CO cuts AB, is the Original of the *Focus*, and CD is therefore the Original of a double Ordinate to the Axe paſſing through the *Focus*, and conſequently the Original of the Meaſure of the Parameter of the Axe [c]: and laſtly, becauſe the Images of CO and OH are equal, NH, whoſe Image is parallel to that of CO (becauſe of their Directing Point N) is the Original of the Directrix of the *Parabola* [d]. *Q. E. I.*

2. To find the Original of any other Diameter of the *Parabola* with its Ordinates and Parameter.

From any Point *z* in the Directing Line NB as a Center, with the *Radius* *z*B, deſcribe an Arch cutting the Circle in G, and draw GB, and from *z* through S and O draw EF and *cd*; then GB will be the Original of a Diameter of the *Parabola*, to which the Images of *co*, *od*, EQ, and QF will be Ordinates, having *z* for their Directing Point; and a Line *cd* drawn through *z* and O the Original of the *Focus*, will be the Original of the Meaſure of the Parameter of the Diameter repreſented by GB.

Dem. For *z*B being a Tangent to the Circle from *z*, and *z*G being equal to it, *z*G is alſo a Tangent to the Circle from *z*, wherefore GB is the Chord of the Tangents from that Point, and conſequently the Images of *co* and *od* are equal, as are alſo the Images of EQ and QF [e]; and becauſe of the Directing Point *z*, the Images of *cd* and EF being parallel, GB is therefore the Image of a Diameter of the *Parabola*, to which the Images of *co*, *od*, EQ, and QF are Ordinates [f]; and the double Ordinate repreſented by *cd*, which paſſes through O the Original of the *Focus*, is the Meaſure of the Parameter of the Diameter repreſented by GB [g].

Or if GB the Original of a Diameter were given, the Directing Point *z* of its Ordinates is found by drawing through S a Diameter of the Circle EF perpendicular to GB, which will cut NB in the ſame Point *z*, as is ſufficiently evident.

And laſtly, if the Original of the *Focus* O were not known, the Line *cd* may be found in this manner.

Biſect the Angle *z*IB by the Line I*v*, and from *v* as a Center with the *Radius* *v*B, deſcribe an Arch which will cut the Circle in the ſame Point *c* as before, through which and *z* the Line *cd* is drawn.

The Demonſtration of this laſt Practice depends on the ſame Property of the *Parabola*, whereby the Original of the *Focus* was found [h]; the Angle *vcz*, made by the Tangent *vc* with the Line *cd*, repreſenting an Angle equal to *z*I*v*, which by Conſtruction is equal to the Angle *v*IB, which is the Angle made by the Images of the Tangent *vc* and the Axe AB, and conſequently *cd* paſſes through the Original of the *Focus* O. *Q. E. I.*

COR. 1.

All Lines as B*a*, BA, BG, drawn in the Circle from the Point of Contact B, are the Originals of Diameters of the *Parabola*, and each of them is perpendicular to, and biſected by that Diameter of the Circle which paſſes through the Directing Point of its proper Ordinates.

For by reaſon of the Directing Point B, the Images of B*a* and BG, and of all other Lines which can be drawn in the Circle from B, are parallel to the Image of AB, which is the Axe of the *Parabola*, and are therefore Diameters of that Figure [i]; and the Original of every Diameter of the *Parabola* being the Chord of the Tangents to the Circle from the Directing Point of its proper Ordinates, as already ſhewn, a Line drawn through that Directing Point and the Center of the Circle, is therefore perpendicular to, and biſects it [k].

COR. 2.

The Diameter B*a* of the Circle which is perpendicular to the Directing Line, is always the Original of a Diameter of the *Parabola*, and the Diameter P*p* of the Circle, which is parallel to the Directing Line, is the Original of a double Ordinate to that Diameter; but no other Diameter of the Circle beſides B*a*, can be the Original of a Diameter of the *Parabola*.

For

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[a] Cor. 1. Prob. 15. and Schol. Prob. 21. B. II.

[b] Parab. Art. 10.

[c] Parab. Art. 8.

[d] Parab. Art. 9.

[e] Lem. 11. and Cor. 5. Lem. 8.

[f] Parab. Art. 1. and 4.

[g] Parab. Art. 8.

[h] Parab. Art. 10.

[i] Parab. Art. 3.

[k] Cor. Lem. 4. B. II.

For the perpendicular Diameter of the Circle neceſſarily paſſes through B[a], and is therefore the Original of a Diameter of the *Parabola*[b]; and it is evident, that the Image of P*p*, and of all other Lines parallel to it and terminated by the Circle, are biſected by the Image of B*a*, and are therefore the Originals of Ordinates to that Diameter[c], the Directing Point of which is infinitely diſtant; and in regard no other Diameter of the Circle, beſides B*a*, can cut the Directing Line in B, no other Diameter of the Circle can be the Original of a Diameter of the *Parabola*, ſeeing its Image cannot be parallel to the Image of AB[d].

COR. 3.

The Image of that Moiety of the Original of any Diameter of the *Parabola*, which is fartheſt from its Directing Point B, is equal to its Complement, but the Image of the other Moiety is indefinite.

For AB being biſected by VR in T, the Image of A will biſect the Diſtance between the Image of T and the Vaniſhing Point of AB, or, which is the ſame, the Image of AT will be equal to its Complement[e]; and for the ſame reaſon, B*a* being biſected in S by the Diameter P*p*, and BG being biſected in Q by the Diameter EF, the Images of S*a* and QG will be equal to their reſpective Complements; but becauſe B is a Directing Point, the Image of which is infinitely diſtant, the Images of SB, TB, and QB will be indefinite: the ſame may be ſhewn of the Parts of the Original of any other Diameter of the *Parabola*, they being all biſected by that Diameter of the Circle which paſſes through the Directing Point of their reſpective Ordinates[f].

COR. 4.

The Original BA of the Axe of the *Parabola*, is not only perpendicular to the Diameter VR of the Circle which paſſes through N, but their Images are alſo perpendicular.

COR. 5.

The infinitely diſtant Extremities of the Indefinite Sides of the *Parabola* formed by the Image of the Circle, become ultimately parallel to its Diameters.

For the Originals of thoſe infinitely diſtant Extremities being at B, their Images become ultimately parallel to the Director IB[g], to which the Images of BG, BA, B*a*, which are the Originals of Diameters of the *Parabola*, are parallel.

CASE 2.

If the Center of the Circle be in the Line of Station, that is, if B be the Foot of the Eye's Director, and *J* the Place of the Eye; the Diameter *a*B of the Circle, which coincides with the Line of Station, will be the Original of the Axe of the *Parabola*.

Becauſe in this Situation of the Circle, the Image of *a*B will coincide with the Vertical Line, and the Images of all Lines drawn in the Circle parallel to the Directing Line, will be perpendicular to the Image of *a*B, and biſected by it[h].

In this Caſe, the Originals of the *Focus*, Parameter, and Directrix of the *Parabola*, may be found in this manner.

On either ſide of B ſet off BL on the Directing Line, equal to *J*B the Height of the Eye, and from L as a Center, with the *Radius* LB, draw an Arch cutting the Circle in δ, through which draw δγ parallel to LB cutting *a*B in *w*; and *w* will be the Original of the *Focus*, and γδ the Original of the Parameter of the Axe; and a Line drawn through *b*, the Interſection of Lδ with *a*B, parallel to γδ, will be the Original of the Directrix.

For it is evident the Image of the Triangle *bw*δ is Similar to the Triangle *J*BL, and therefore that the Image of Lδ makes equal Angles with the Images of γδ and *a*B, whence alſo the Images of δ*w* and *wb* are equal.

The Originals of the Ordinates to any other Diameter of the *Parabola* are found as in the firſt Caſe. *Q. E. I.*

COR.

If the Height of the Eye were equal to SB the *Radius* of the Original Circle, then S would be the Original of the *Focus*, and P*p* the Original of the Meaſure of the Parameter of the Axe, and the Directrix of the *Parabola* would coincide with the Vaniſhing Line.

For if BW be made equal to BS, and from W an Arch be deſcribed with the *Radius* WB, it will cut the Circle in *p*; whence P*p* will be the Original of the Parameter and S the Original of the *Focus*[i]; and in regard W*p* is parallel to *a*B, their In-

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[a] 19 El. 3.

[b] Cor. 1.

[c] Parab. Art. 4.

[d] Cor. 4. Theor. 12. B. I.

[e] Cor. 3. Lem. 8. or Theor. 26. B. I.

[f] Cor. 1.

[g] Con. Sec. Art. 11.

[h] Parab. Art. 4.

[i] Caſe 2.

terſection which ſhould mark the Point b, is at an infinite Diſtance, the Image of which Interſection is therefore the Vaniſhing Point of aB, and the Directrix being a Line drawn through that Point parallel to the Image of Pp[a], it muſt therefore coincide with the Vaniſhing Line.

PROB. VI.

The determinate Image of the perpendicular Semidiameter of a Circle which touches the Directing Line of its Plane, being given; thence to determine the Axe, and its Parameter, and the Vertex, *Focus*, and Directrix of the *Parabola* formed by the Image of the Circle; and alſo to find any other Diameter of the *Parabola* with its Vertex and Ordinates.

1. To determine the Axe, Parameter, Vertex, *Focus*, and Directrix.

Fig. 75. Let sa be the Image of the Semidiameter of the forming Circle, and s the Image of its Center, EF the Vaniſhing Line of its Plane, o its Center, and Io its Diſtance.

Through s draw vx perpendicular to sa cutting EF in x, and having drawn the Radial Ix, draw Iy perpendicular to it cutting EF in y, from y draw yt parallel to sa cutting vx in t, and biſect yt in A; then At will be the Axe of the *Parabola*, and A its Vertex.

Through a draw aw parallel to EF cutting At in w, biſect aw by the perpendicular bF cutting At in F, and F will be the *Focus*.

Through F draw cd perpendicular to At, and make Fc and Fd each equal to the double of FA, and cd will be the Meaſure of the Parameter of the Axe, and alſo a double Ordinate to it; and having taken Ab equal to AF, through b draw a Line parallel to cd, and that will be the Directrix.

Dem. Becauſe as is a Diameter of the *Parabola*[b], it is therefore parallel to the Axe, wherefore vx, which repreſents an Indefinite Diameter of the forming Circle, being drawn perpendicular to as, is alſo perpendicular to the Axe, and conſequently repreſents the Diameter VR in the Original Plane; Fig. 74. and becauſe of the Vaniſhing Points x and y, the Lines xt and yt, which by Conſtruction are perpendicular, have alſo perpendicular Originals, yt is therefore the Indefinite Axe of the *Parabola*[c], and t repreſents the Point T in the Original Plane; conſequently ty being biſected in A, the Point A is the Vertex of the *Parabola*[d].

Again, becauſe aw is the Image of a Tangent to the Original Circle in the Point repreſented by a, it is therefore a Tangent to the *Parabola* in that Point; and that Tangent cutting the Axe At in w, and being biſected in b, the Perpendicular bF cuts the Axe in F the *Focus* of the *Parabola*[e], whence the Parameter and Directrix are found as above directed[f]. *Q. E. I.*

COR. 1.

If the Extremities v and r of the Indefinite Diameter vx of the Circle be found, the Point t will biſect it, and tv and tr will be Ordinates to the Axe At.

Fig. 74. For in the Original Plane, the Line NR being Harmonically divided in the Points N, V, T, and R[g], and N being a Directing Point, the Image of VR will be biſected by the Image of T[h]; wherefore in the Picture, Fig. 75. tv and tr are equal, and v and r being Points in the Image of the Circle, they are therefore Points of the *Parabola*, and conſequently tv and tr which are perpendicular to the Axe At, are Ordinates to it.

COR. 2.

If through y and r a Line yr be drawn, it will be a Tangent to the *Parabola* in r, and if yr be biſected in l, a Perpendicular to it drawn through l will cut the Axe in the ſame Point F as before, which is the *Focus* of the *Parabola*.

For by reaſon of the Vaniſhing Points y and x, the Originals of yr and xv are perpendicular, and r being the Image of the Extremity of a Diameter of the Circle, the Original of yr is a Tangent to the Circle in the Original of the Point r, wherefore yr is a Tangent to the *Parabola* in that Point; and this Tangent cutting the Axe in y, and being biſected in l, the Perpendicular lF meets the Axe in the *Focus* F[i].

Here the Tangent yr cutting the Axe At in y, and rt being an Ordinate to the Axe from the Point of Contact r, yA and At are equal[k].

COR.

[a] Parab. Art. 9.

[b] Cor. 2. Prob. 5.

[c] Cor. 4. Prob. 5.

[d] Cor. 3. Prob. 5.

[e] Parab. Art. 7.

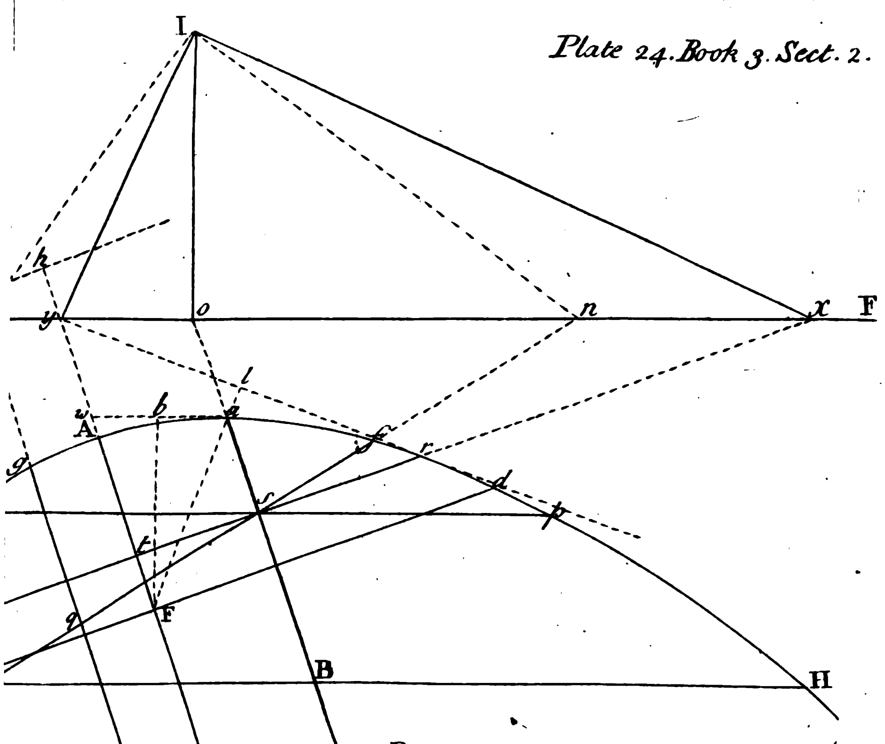
[f] Parab. Art. 8. and 9.

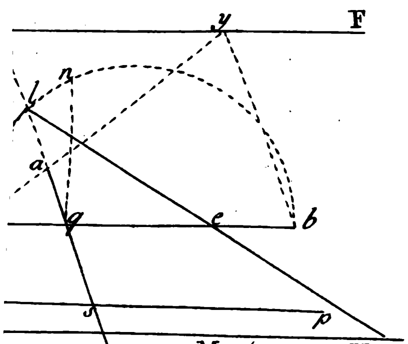
[g] Lem. 11.

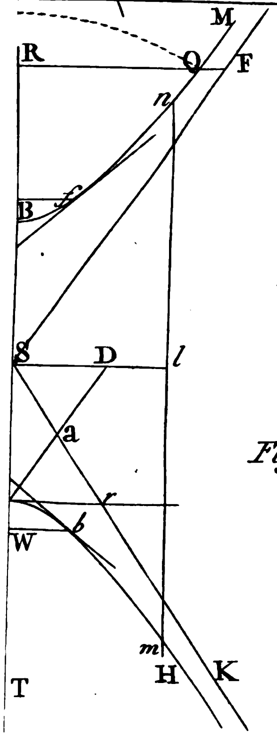
[h] Cor. 5. Lem. 8.

[i] Parab. Art. 7.

[k] Parab. Art. 6.







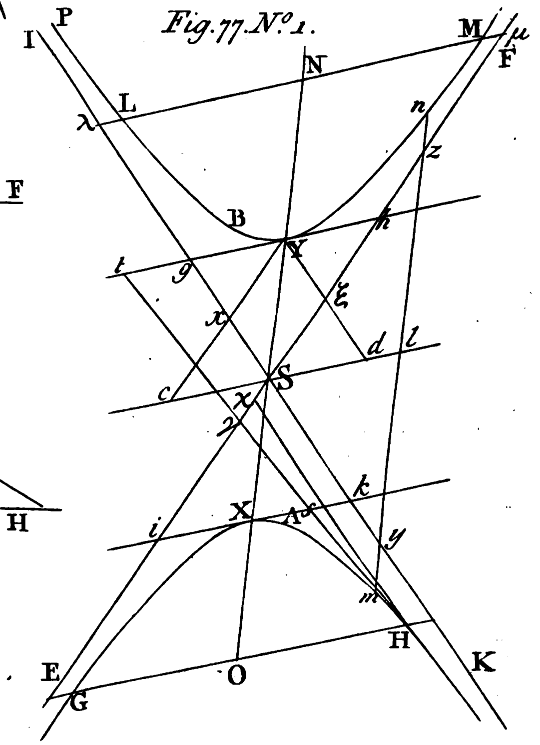


Fig. 77. No. 1.

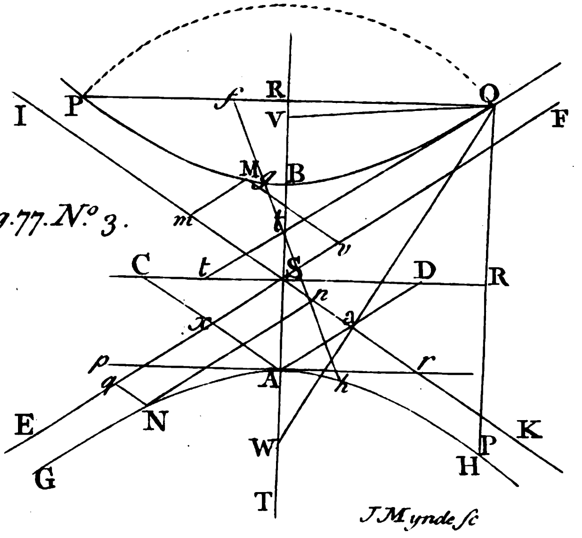


Fig. 77. No. 3.

J. Mynde sc.

COR. 3.

If through s a Line Pp be drawn parallel to EF, make sP and sp each equal to Io the Diſtance of the Vaniſhing Line, and Pp will be the Image of the Diameter of the forming Circle, which is parallel to the Directing Line, and conſequently a double Ordinate to the Diameter as of the *Parabola*[a].

Fig. 74. For Sp in the Original Plane being equal to S B the Complement of aS from S, the Image of Sp is equal to the Diſtance of the Vaniſhing Point of aB[b], which is the Center of the Vaniſhing Line; wherefore in the Picture, sp and Io are equal.

2. To find any other Diameter of the *Parabola* with the Poſition of its Ordinates.

Take in the Vaniſhing Line EF any two Points m and n, whoſe Radials make together a Right Angle, and through either of thoſe Points, as m, draw mq parallel to sa, and from n through s draw ns repreſenting an Indefinite Diameter of the forming Circle cutting mq in q, and find its Extremities e and f[c]; then biſect mq in g, and gq will be a Diameter of the *Parabola*, g the Vertex, and eq and qf Ordinates to that Diameter.

Dem. For mq drawn parallel to sa, is an Indefinite Diameter of the *Parabola*[d], the Original of which is perpendicular to that Diameter of the forming Circle which paſſes through the Directing Point of its Ordinates[e]; ef is therefore the Image of that Diameter, ſeeing the Originals of mq and ef are perpendicular, becauſe of the Vaniſhing Points m and n; wherefore eq and qf are Ordinates to the Diameter mq of the *Parabola*, and mq being biſected in g, g is the Vertex of that Diameter[f]. Q. E. I.

CASE 2.

If the Original of the given Semidiameter sa were in the Line of Station, its Image would be perpendicular to EF, vr would coincide with Pp, and the Points s and t would be the ſame, the Point y would coincide with o, and a would be the Vertex, and as the Axe of the *Parabola*, whence the *Focus*, Directrix, and Parameter might be found as before.

PROB. VII.

The Image of any Diameter of an Original Circle which touches the Directing Line of its Plane, being given; from the Image of any Point in that Diameter produced without the Circle, to draw two Tangents to the *Parabola* formed by the Image of the Circle.

CASE 1.

In an Original Circle which touches the Directing Line, and whoſe Image is therefore a *Parabola*[g], all the Diameters, except that which paſſes through the Point of Contact, muſt cut the Circle in two Points, each of which may be repreſented; ſo that the Images of all thoſe Diameters are determinate; and the Images of Tangents from any Point in any of thoſe Diameters produced without the Circle, may be found by the Method in Prob. III. Q. E. I.

CASE 2.

If the Point from whence the Tangents are to be drawn, be in the perpendicular Diameter of the Circle, the Images of the Tangents from that Point are found in this manner.

Fig. 76. Let EF be the Vaniſhing Line of the Plane of the Circle, Io its Diſtance, and as the Indefinite Image of the perpendicular Diameter of the Circle, and s the Image of its Center; and let l be a Point in that Diameter produced beyond its Extremity a, from whence the Tangents are required to be drawn.

Take aq equal to al, and through q draw de parallel to EF, and on that Line find $q\alpha$ and qb, the proportional Meaſures of the Original of aq, and of its Complement from the Original of q[h]: then having found qn the mean Proportional between αq and qb, and taken qd and qe, each equal to qn, from l draw ld and le, and theſe will be the Tangents required.

Dem. Let b (Fig. 74.) repreſent the Original of l, it is evident, that the perpendicular Diameter aB of the Circle, is Harmonically divided in b, a, B, and w, where $y\delta$ the Chord of the Tangents from b cuts aB[i]; and B being a Directing Point, the Images of wa and ab are equal[k]: therefore (in Fig. 76.) aq being made equal to al, q repreſents the Point in the Semidiameter as, through which the Chord of the Tan-

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[a] Cor. 2. Prob. 5.

[b] Prob. 10. B. II.

[c] Cor. 2. Meth. 2. Prob. 24. B. II.

[d] Parab. Art. 3.

[e] Cor. 1. Prob. 5.

[f] Cor. 3. Prob. 5.

[g] Con. Sec. Art. 12.

[h] Cor. 1. Prob. 9 and 10. B. II.

[i] Lem. 11.

[k] Cor. 5. Lem. 8.

gents from l paſſes, and therefore qa repreſents the Segment wa (Fig. 74.) and the Indefinite Line qs repreſents the other Segment wB of the perpendicular Diameter aB; conſequently the proportional Meaſures of the Originals of aq, and of the Indefinite Line qs, being found, the reſt of the Conſtruction is the ſame as before[a]. *Q. E. I.*

C O R.

If the Point l were beyond the Vaniſhing Line, its Original being then behind the Directing Line, the Conſtruction would be the ſame; for ſtill the Directing Point of the perpendicular Diameter of the Circle, which is one of its Extremities, being one of the Points of Harmonical Diviſion, the Images of l and q will be equally diſtant from a[b].

S C H O L.

In this Caſe, the Original of the Point l cannot be a Directing Point, in regard that if it be in the perpendicular Diameter of the Circle, it muſt be the ſame with the Extremity B of that Diameter, to which the Directing Line is a Tangent, which therefore can have no Repreſentation: but if the Original of l be in any other produced Diameter of the Circle, it may be a Directing Point; but then only one Tangent can be drawn from thence to the *Parabola*, becauſe the Original of one of the Tangents to the Circle from any Point of the Directing Line muſt coincide with that Line, and can have no Image; but the other Tangent from that Point may be repreſented, and its Image will therefore be a Tangent to the *Parabola*: and from hence it appears, that there cannot be two Tangents drawn to a *Parabola* parallel to each other.

P R O B. VIII.

Fig. 75. **Any *Parabola* GAH being given; thence to determine the Vaniſhing Line, Center, and Diſtance of a Plane, in which an Original Circle being placed, its Image ſhall be the given *Parabola*.**

The Original of the given *Parabola* may be a Circle in any Original Plane touching the Directing Line of that Plane[c], the Vaniſhing Line of which muſt neither touch nor cut the given Figure.

Having therefore taken any Line EF, not touching or cutting the given *Parabola*, for the Vaniſhing Line of the Plane of the forming Circle, draw any two Lines Pp and GH parallel to EF, and terminated by the *Parabola* in P, p, G, and H; biſect Pp and GH in s and B, and draw Bs cutting the *Parabola* in a, and the Line EF in o, take as in the Line Bo equal to ao, and through s draw Pp parallel to EF (if that be not one of thoſe already drawn) terminated by the Curve in P and p, and from o erect oI perpendicular to EF, and equal to sP or sp: then s will repreſent the Center, and as the *Radius* of a forming Circle in a Plane whoſe Vaniſhing Line is EF, o its Center, and Io its Diſtance; and Pp will repreſent the Diameter of that Circle which is parallel to the Directing Line. If then any Line GH parallel to EF be taken as the Interſecting Line of the Original Plane, the Original of Pp being found in that Plane, and on it as a Diameter a Circle being deſcribed, the Image of that Circle will produce the given *Parabola*.

Dem. For by the Conſtruction as is a Diameter of the *Parabola*, to which GB, BH, Ps, and sp are Ordinates, and theſe being parallel to the Vaniſhing Line EF, sa is therefore the Indefinite Image of that Diameter of the forming Circle which is perpendicular to the Directing Line[d], and its Vaniſhing Point o is therefore the Center of the Vaniſhing Line; and as being made equal to ao, s is the Image of the Center of the forming Circle[e], and conſequently Pp is the Image of that Diameter of the Circle which is parallel to the Directing Line; wherefore Io being made equal to sp, Io is the Diſtance of the Vaniſhing Line[f]. *Q. E. I.*

Of the Hyperbolas or oppoſite Sections.

Fig. 77. N°. 1. Let GAH and PBM be two oppoſite *Hyperbolas*.

1. Draw any two parallel Lines LM, GH, either in the ſame or in the oppoſite Sections, terminated by them in L, M, G, and H; biſect LM and GH in N and O, and draw NO cutting the oppoſite Sections in Y and X; through Y and X draw gh and ik parallel to LM, and biſect XY in S.

Then S is the Center, and XY a firſt Diameter of the *Hyperbolas*; and LM, GH, and all other Lines parallel to them drawn within either of the Sections, and terminated

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[a] Prob. 3.

[b] Cor. 5. Lem. 8.

[c] Con. Sec. Art. 12.

[d] Cor. 2. Prob. 5.

[e] Cor. 3. Prob. 5.

[f] Cor. 3. Prob. 6.

nated by it, will be biſected by the Diameter XY, and each Moiety LN, NM, GO, OH, &c. of ſuch Lines is an Ordinate to that Diameter, and gh and ik drawn through the Extremities Y and X of any firſt Diameter XY parallel to its Ordinates, are Tangents to the oppoſite Sections in X and Y.

2. If through the Center S a Line cd be drawn parallel to the Ordinates of any firſt Diameter XY, and through any Point l in the Line cd, a Line mn be drawn parallel to XY, cutting the oppoſite Sections in m and n; then cd is an indeterminate ſecond Diameter of the Sections, and XY and cd, with reſpect to each other, are called *Conjugate Diameters*, and the Line mn drawn through cd parallel to its Conjugate Diameter XY, and all other Lines parallel to mn, drawn on either Side of S, will each cut the oppoſite Sections in two Points m and n, &c. and be biſected by cd; and each Moiety ml, ln, &c. of ſuch Lines is an Ordinate to the ſecond Diameter cd.

Fig. 77. N°. 2.

3. From S as a Center, with any *Radius* large enough to cut either of the Sections in two Points, draw an Arch cutting the *Hyperbola* PBM in P and Q, and having drawn PQ, biſect it in R; through R and S draw RT, which will be perpendicular to PQ, cutting the Sections in A and B, and through S and A draw CD and pr perpendicular to AB, and conſequently parallel to PQ:

Then AB will be a firſt Diameter, to which PR and RQ are Ordinates, A and B are the Vertices of that Diameter, and pr, drawn through A parallel to PQ, is a Tangent to the *Hyperbola* GAH in A; CD is a ſecond Diameter Conjugate to the Diameter AB, and any Line mn drawn through CD parallel to AB, will cut the oppoſite Sections in two Points m and n, and be biſected by CD in l, and each Moiety of that Line is an Ordinate to the Second Diameter CD: theſe two Conjugate Diameters are called the *Axes*, of which AB is the firſt or Tranſverſe Axe, and CD the indeterminate ſecond Axe, and are the only two Conjugate Diameters of the *Hyperbolas* which are perpendicular to each other and to their reſpective Ordinates; and the Extremities A and B of the Tranſverſe Axe AB, are alſo called the *Vertices of the oppoſite Sections.*

4. Biſect the Angle pAT, made by the Tangent pA with the Axe AB, by the Line AG, cutting the *Hyperbola* GAH in G, and from G to B, the other Extremity of the Axe AB, draw GB, cutting Ap in p; then Ap is the Length or Meaſure of the Parameter of the Axe AB.

5. Biſect Ap in v, and take Aq and Ar on the Tangent Ap on both Sides of A, each equal to a mean Proportional between vA the Semiparameter and SA the Semiaxe, and through S and the Points q and r draw EF and IK, and theſe are the *Aſymptotes*; which Lines, though indefinitely produced both ways, can never touch or cut either of the Sections, though they will approach nearer and nearer to them; ſo that the Aſymptotes EF and IK may be conſidered as Tangents to the oppoſite *Hyperbolas* at an infinite Diſtance; and any Line AD, drawn parallel to either of the Aſymptotes EF, can cut but one of the *Hyperbolas* GAH, and that only in one Point A.

6. From either of the Extremities A, of the firſt Axe AB, draw AD parallel to the Aſymptote EF, and AC parallel to the other Aſymptote IK, cutting the ſecond Axe CD in D and C; then DC will be the determinate ſecond Axe, which Axe will be biſected in S, and will be equal to qr, the Tangent at the Vertex A, terminated by the Aſymptotes.

7. The Lines AD and AC drawn from A parallel to the Aſymptotes, which determine the Extremities D and C of the ſecond Axe CD, are equal, and each of them is biſected by one of the Aſymptotes; thus AD drawn parallel to the Aſymptote EF, is biſected in a, by the other Aſymptote IK, and either Moiety Aa of that Line, is equal to Sa, the Segment of the Aſymptote IK, intercepted between the Center S and the Line AD; and the Square of Sa is called the *Power of the Hyperbolas.*

8. The Angles ESK and ISF made by the Aſymptotes, and within which both the Sections intirely lie, are called the *Inward Angles of the Aſymptotes*; and the contrary Angles ESI and KSF are called the *Outward Angles*; the Inward Angles are biſected by the Tranſverſe Axe AB, and the Outward by the ſecond Axe CD.

Fig. 77. N°. 1.

9. All Lines as XY drawn through S within the Angles ESK and ISF, are firſt Diameters; and every ſuch Diameter produced both ways, will cut each of the oppoſite Sections in one Point only as X and Y, which Points are the Extremities of that Diameter, and every firſt Diameter is biſected by the Center S: and all Lines as cd drawn through S without the Angle ESK, that is, within the Angle ESI, or its oppoſite

KSF,

KSF, are indeterminate ſecond Diameters, and none of theſe can ever meet or cut either of the Sections, though indefinitely produced.

10. Every firſt Diameter hath a ſecond Diameter Conjugate to it parallel to its Ordinates, and the Extremities of this ſecond Diameter are determined by Lines drawn from either Extremity of its Conjugate firſt Diameter parallel to the Aſymptotes; and every ſecond Diameter thus terminated, is biſected by the Center S. Thus Yc and Yd drawn from the Extremity Y of the firſt Diameter XY, parallel to the Aſymptotes EF and IK, cut cd the indeterminate ſecond Diameter Conjugate to XY in c and d its Extremities, and cd is biſected in S.

11. The Lines Yc and Yd are biſected in x and ξ by the reſpective Aſymptotes IK and EF, and the Rectangle betwen the Segment Sx of the Aſymptote IK, and either Moiety xc of the Line Yc, is equal to the Rectangle between the Segment $S\xi$ of the Aſymptote EF, and either Moiety ξd of the Line Yd; and either of theſe Rectangles is equal to the Square of Sa (Fig. N°. 2.) the Power of the *Hyperbolas.*

12 The two Aſymptotes and any two Conjugate Diameters are always Harmonical Lines; thus (Fig. N°. 2.) IK, EF, AB, and CD are Harmonical Lines, the Line AD parallel to EF being biſected in a by the other three; likewiſe (Fig. N°. 1.) IK, EF, XY, and cd are Harmonical Lines, the Line Yc parallel to EF being biſected in x by the other three[a].

13. If any two Conjugate Diameters of the *Hyperbola* be equal, every Diameter of thoſe *Hyperbolas* will be equal to its Conjugate, and the Aſymptotes will be perpendicular, and the Sections are then ſaid to be *Equilateral.*

14. If any firſt Diameter be longer than its Conjugate, every firſt Diameter will be longer than its Conjugate, and the inward Angle of the Aſymptotes will be Acute; if any firſt Diameter be ſhorter than its Conjugate, every firſt Diameter will alſo be ſhorter than its Conjugate, and the inward Angle of the Aſymptotes will be Obtuſe.

Fig. 77. N°. 2.

15. If from S a Diſtance be ſet off both ways on the Tranſverſe Axe AB at V and W, equal to AC the Diſtance between the Extremities of the firſt and ſecond Axes; each of the Points V and W is called the *Focus* of that *Hyperbola* within which it falls. And if through V or W a double Ordinate ef or ab to the Axe AB be drawn, either of them is the Meaſure of the Parameter of that Axe, and is therefore equal to Ap[b].

16. If from the Center S, there be taken on the ſecond Axe CD, a Diſtance SZ equal to the Semitranſverſe Axe SA, two Tangents Zb and Zf drawn from Z to the oppoſite *Hyperbolas,* will touch them in b and f, one of the Extremities of the double Ordinates ab and ef drawn through the *Foci* W and V; and if the like Diſtance were ſet off on the other Side of S, Tangents from thence would touch the Sections in a and e, the other Extremities of thoſe Ordinates.

Fig. 77. N°. 1.

17. If from any Point Y in an *Hyperbola* PBM, a Line Yx be drawn parallel to either of the Aſymptotes EF, and cutting the other Aſymptote IK in x, take xg in that Aſymptote equal to xS, and a Line drawn through g and Y will be a Tangent to the *Hyperbola* in Y; and every Tangent gh, terminated by the Aſymptotes in g and h, is biſected in Y its Point of Contact with the Section, and is parallel and equal to cd the ſecond Diameter Conjugate to the firſt Diameter XY, drawn through the Point of Contact Y. Thus alſo χH being drawn parallel to IK, and $\gamma\chi$ being taken equal to χS, tH drawn through γ and H, is a Tangent to the *Hyperbola* GAH in H.

18. If two Tangents gh and ik, at the Extremities Y and X of any firſt Diameter XY, be cut in t and s by any other Tangent Ht to either of the *Hyperbolas*; the Semidiameter Sc Conjugate to the Diameter XY will be a mean Proportional between Xs and Yt, the Segments of the Tangents ik and gh, intercepted between the Points of Contact X and Y, and the other Tangent Ht.

19. If any Line LM or mn cut both the Aſymptotes in λ and μ, or y and z, and alſo cut either the ſame or the oppoſite Sections in two Points L and M, or m and n, the Parts $L\lambda$ and $M\mu$, or my and zn of that Line, intercepted between the Section or Sections and the Aſymptotes, will be equal, and conſequently λM will be equal to $L\mu$, and mz to yn.

20. If from any Point M or m in either *Hyperbola,* a Line LM or mn be drawn parallel to any Diameter cd or XY, cutting the ſame or the oppoſite Sections in L and M, or m and n, and the Aſymptotes in λ and μ; or y and z; the Rectangle between λL and $L\mu$, or its equal λM, will be equal to the Square of the Semidiameter Sd parallel to it; or the Rectangle between my and yn, or its equal mz, will be equal

to

[a] Lem. 7.

[b] Art. 4.

to the Square of the Semidiameter S Y which is parallel to it; that is, the Semidiameter S *d* will be a mean Proportional between λ L and λ M, and the Semidiameter S Y will be a mean Proportional between *m y* and *m z*.

21. The Parameter of any Diameter is a third Proportional to that Diameter and its Conjugate, putting the Diameter, whoſe Parameter it is, as the firſt Term. Thus if A*p* be the Parameter of the Diameter A B, to which C D is the Conjugate, then $AB : CD :: CD : Ap$; or $AB^2 : CD^2 :: AB : Ap$. Fig. 77. N°. 2.

22. If an Ordinate G O be drawn to any firſt Diameter X Y produced within the Curve, cutting it in any Point O; then a mean Proportional between Y O and X O will be to the Ordinate G O, as the Diameter X Y is to its Conjugate *c d*; that is, $\sqrt[2]{YO \times XO} : GO :: XY : cd :: SY : Sd$. and putting *p* for the Parameter of the Diameter X Y, $YO \times XO : GO^2 :: XY^2 : cd^2 :: XY : p$. Fig. 77. N°. 1.

23. If an Ordinate *n l* be drawn to any ſecond Diameter *c d*, cutting it in *l*; then, as the Hypotenuſe of a Right angled Triangle, whoſe Sides are S *d* and S *l*, is to the Ordinate *l n*, ſo is the Diameter *c d* to the firſt Diameter X Y Conjugate to it: that is, $\sqrt[2]{Sd^2 + Sl^2} : ln :: cd : XY :: Sd : SY$; and *p* being put for the Parameter of the Diameter X Y as before, $Sd^2 + Sl^2 : ln^2 :: cd^2 : XY^2 :: p : XY$.

24. If from any Point Q of an *Hyperbola*, two Lines Q V, Q W be drawn to the *Foci* V and W, a Line Q *t* drawn from Q biſecting the Angle V Q W, will be a Tangent to the *Hyperbola* in the Point Q. Fig. 77. N°. 3.

25 If from any Point Q of an *Hyperbola*, an Ordinate Q R be drawn to any Diameter A B or C D, and from the ſame Point Q, a Tangent Q *t* be drawn cutting that Diameter in *t*; then the Semidiameter S B or S D, to which Q R is an Ordinate, will be a mean Proportional between S R and S *t*, the Segments of that Semidiameter by the Ordinate and Tangent; obſerving that the Points R and *t* fall both on the ſame Side of the Center S, when Q R is an Ordinate to a firſt Diameter A B; but that they fall one on each Side of the Center, when Q R is an Ordinate to a ſecond Diameter CD. And if Q R be produced till it cut the ſame or the oppoſite *Hyperbola* in P; *t* P drawn from the correſponding Point *t* will alſo be a Tangent to the Section in P.

26. If QP be a double Ordinate to any firſt Diameter A B, and *t* be the Point where the Tangents in Q and P meet that Diameter, then the Diameter A B will be Harmonically divided in A, *t*, B, and R its Interſection with QP; and if through *t* any Line *fb* be drawn cutting the oppoſite Sections in *g* and *b*, and the Ordinate Q P in *f*, the Line *f b* will alſo be Harmonically divided in *b*, *t*, *g*, and *f*.

27 If from any two Points M and N, either in the ſame or in the oppoſite *Hyperbolas*, two Lines M *m*, N *n* be drawn parallel to either of the Aſymptotes E F, and terminated by the other Aſymptote I K in *m* and *n*, and from the ſame Points M and N, two other Lines M *v*, N *q* be drawn parallel to the Aſymptote I K, and terminated by the Aſymptote E F in *v* and *q*; and if from A the Vertex of the *Hyperbola* GAH, two Lines A *x*, A a be in like manner drawn parallel to the Aſymptotes, and terminated by them in *x* and a; the Parallelograms S *q* N *n*, S *v* M *m*, and S *x* A a will be equal, that is, $Sq \times qN = Sv \times vM = Sa \times aA = Sa^2$; which laſt is therefore called the *Power of the Hyperbolas* [a].

28. The oppoſite *Hyperbolas* are every way Similar and equal, and all Diameters of the one are alſo Diameters of the other, and the Ordinates and Tangents at the Extremities of the ſame Diameter in both Sections are parallel to each other.

P R O B. IX.

An Original Circle being given cutting the Directing Line in two Points; therein to determine the Originals of the Axes, Center, and Aſymptotes of the oppoſite *Hyperbolas* formed by the Image of the Circle, and the other Lines and Points relating to theſe Sections, before deſcribed.

1. To determine the Originals of the Aſymptotes, the Center, the Axes, and their Ordinates, and the Vertices of the Oppoſite Sections.

Let A F B G be the Original Circle, cutting the Directing Line LT in F and G, and let IK be the Eye's Director. Fig. 78. N°. 1.

Draw FS and GS Tangents to the Circle in F and G, meeting in S, and from I

I i draw

[a] Art. 7.

draw IF and IG; biſect the Angle FIG by the Line IM, cutting the Directing Line in M, and draw SM cutting the Circle in A and B; from I draw IL perpendicular to IM, cutting the Directing Line in L, and draw LS, and from A draw AF and AG, cutting LS in C and D:

Then SF and SG will be the Originals of the Aſymptotes, and S the Original of the Center of the Oppoſite *Hyperbolas*; SA will be the Complement of the Original of one Moiety of the Tranſverſe Axe[a], and SB the Original of the other Moiety, and A and B the Originals of its Extremities, or the Vertices of the oppoſite Sections; and L will be the Directing Point of the Ordinates to that Axe; laſtly CD will be the Original of the ſecond or Conjugate Axe, and M will be the Directing Point of its Ordinates; and FIG will be equal to the Inward Angle of the Aſymptotes.

Dem. Becauſe F and G are Directing Points, whoſe Images are infinitely diſtant[b], the Images of SF and SG which touch the Circle in thoſe Points, are therefore Tangents to the Image of the Circle at an infinite Diſtance; SF and SG are therefore the Originals of the Aſymptotes, and the Point S where they croſs, is the Original of the Center of the oppoſite *Hyperbolas*[c]; and becauſe FG is the Chord of the Tangents to the Circle from S, any Line, as SA, drawn from S within the Angle FSG, will cut the Circle in two Points A and B, one before and the other behind the Directing Line, and will be Harmonically divided by thoſe Interſections and the Line FG[d]; and the Interſection M of SA with FG being conſtantly a Directing Point, the Images of A and B will therefore be at an equal Diſtance from the Image of S[e]; but the Images of A and B are Points in the oppoſite Sections, and S being the Image of their Center, SA is therefore the Complement of the Original of one Moiety of a firſt Diameter, and SB is the Original of the other Moiety, and A and B the Originals of the Extremities of that Diameter[f]. And becauſe the Angle FIG is biſected by IM, to which IL is perpendicular, the Line LG is Harmonically divided in L, F, M, and G[g], therefore M is a Point in the Chord of the Tangents to the Circle from L[h]; but FG being the Chord of the Tangents from S, and L being a Point in that Chord, the Chord of the Tangents from L muſt alſo paſs through S[i]; wherefore AB which paſſes through M and S, is that Chord; conſequently the Images of all Lines drawn from L, and terminated both ways by the Circle, are biſected by the Image of AB[k], wherefore all ſuch Lines are double Ordinates to the Diameter repreſented by SBA[l]: but becauſe the Angle LIM is Right, the Images of all Lines proceeding from L are perpendicular to the Image of SBA, therefore this Line repreſenting a Diameter whoſe Ordinates are perpendicular to it, it is the Original of the Tranſverſe Axe, and A and B the Originals of the Vertices of the oppoſite Sections[m]; and L being the Directing Point of the Ordinates to that Axe, it is alſo the Directing Point of the ſecond or Conjugate Axe, which is parallel to them[n], wherefore LS is the Indefinite Original of the ſecond Axe; and becauſe of the Directing Points F and G, the Images of AF and AG being parallel to the Images of SF and SG, which are the Aſymptotes, and paſſing through the Image of A, the Vertex of one of the Sections, they therefore cut the ſecond Axe in its Extremities[o]; wherefore C and D, where AF and AG cut LS, are the Originals of the Extremities of the ſecond Axe. And laſtly, becauſe the Images of all Lines drawn through M are parallel to the Image of AB, therefore M is the Directing Point of the Ordinates to the ſecond Axe, theſe being parallel to the firſt Axe[p]; and the Angle FIG being that which the Images of SF and SG make together, incloſing the Image of the Circle, it is therefore the Inward Angle of the Aſymptotes. *Q. E. I.*

COR. 1.

If inſtead of the Vertex A made uſe of to determine the Points C and D by the Lines AF and AG, the Vertex B had been uſed, and the Lines GB and FB had been drawn; theſe two Lines would have cut LS in the ſame Points C and D, and thereby have determined the Original of the ſecond Axe, as before.

For LG and SA being both Harmonically divided, as before ſhewn, and Interſecting in M a Point of Diviſion common to both, and L and S the ſecond Points of Diviſion in each from the common Point M, being joined by the Line LS, the Lines AF and GB, which join the other Points, muſt meet the Line LS in the ſame Point C; and for the ſame reaſon, the Lines AG and FB muſt meet LS in the ſame Point D (although that Point be here out of reach) ſince thoſe Lines do not happen to be parallel[q].

COR. 2.

It is evident, that all Lines drawn through S within the Angle FSG, and conſequently

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[a] Schol. Theor. 4. and Def. 24. B. I.

[b] Cor. 1. Theor. 4. B. I.

[c] Hyperb. Art. 5.

[d] Lem. 11.

[e] Cor. 5. Lem. 8.

[f] Hyperb. Art. 9.

[g] Lem. 5 and 8.

[h] Cor. Lem. 11.

[i] Cor. 2. Lem. 14.

[k] Lem. 11. and Cor. 5. Lem. 8.

[l] Hyperb. Art. 1.

[m] Hyperb. Art. 3.

[n] Hyperb. Art. 3.

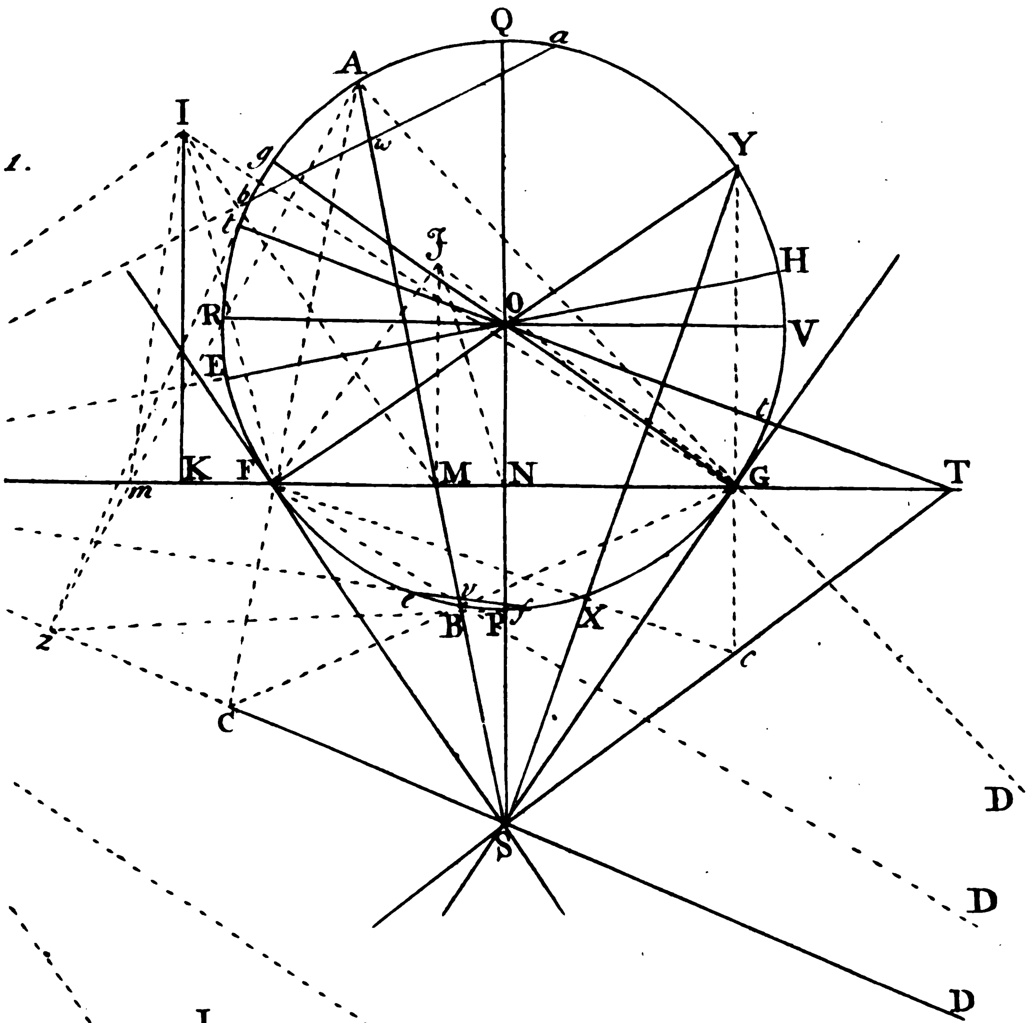
[o] Hyperb. Art. 6.

[p] Hyperb. Art. 2.

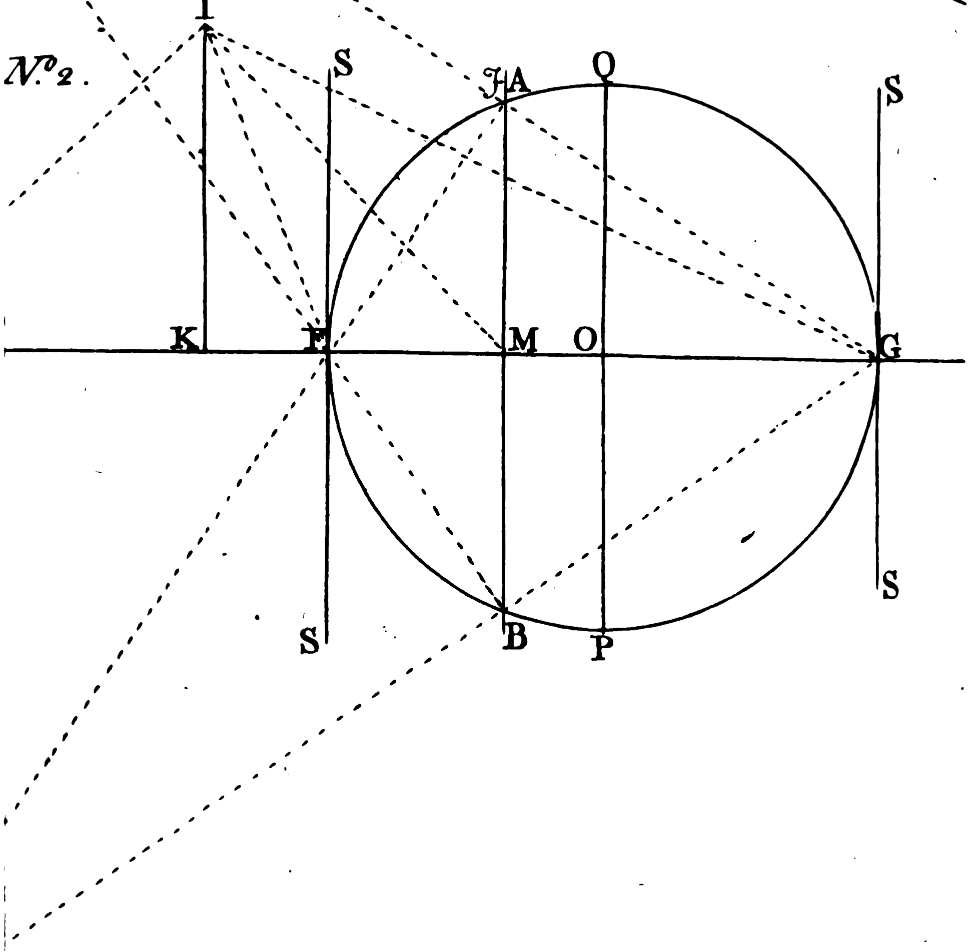
[q] Lem. 9.

Plate 25. Book 3 Sect. 2.

1.



N°2.



J. Mynde sc.

quently QP the perpendicular Diameter of the Circle, which muſt paſs through S, are the Indefinite Originals of firſt Diameters; and all Lines drawn through S without that Angle, are the Originals of Indefinite ſecond Diameters. Thus SXY is the Complement of the Original of a firſt Diameter, and TS of a ſecond Diameter; but no Diameter of the Circle can be the Original of a firſt Diameter of the *Hyperbolas* beſides PQ, ſeeing no other Diameter of the Circle can paſs through S.

2. To find the Directing Point of the Ordinates, and alſo the Original of the Diameter Conjugate to any firſt Diameter whoſe Original is given.

Let SXY be the given Original of a firſt Diameter.

Through O the Center of the Circle, draw OT perpendicular to XY cutting the Directing Line in T, and draw TS; then T will be the Directing Point of the Ordinates, and TS will be the Original of the Indefinite ſecond Diameter Conjugate to the Diameter repreſented by SXY.

Dem. For OT being perpendicular to XY, Tangents to the Circle in X and Y muſt meet in ſome Point of the Line OT[a], if XY do not paſs through the Center of the Circle, in which laſt Caſe the Tangents in X and Y would be parallel to OT; and becauſe FG is the Chord of the Tangents from S, the Line XY which paſſes through S, muſt be a Chord to Tangents from ſome Point in the Line FG[b]; the Interſection therefore of the Tangents to the Circle at X and Y being ſomewhere in the Line OT, and alſo in the Line FG, it muſt be in the Point T where theſe Lines Interſect; wherefore XY is the Chord of the Tangents from T, and conſequently the Images of all Lines drawn from T, and terminated both ways by the Circle, are biſected by the Image of XY, and are therefore double Ordinates to the Diameter repreſented by SXY; and TS which hath the ſame Directing Point T, is therefore the Original of the Indefinite ſecond Diameter Conjugate to it[c].

It is evident alſo, that Lines drawn through F and G and the Points X or Y, the Originals of either of the Extremities of the Diameter SXY, will determine the Originals of the Extremities of the ſecond Diameter, by their Interſections with TS[d]. Thus either FX or YG will give the Point *c*, which determines S*c* the Original of one Moiety of the Diameter Conjugate to SXY; and FY or GX would determine the other Extremity of S*c* were it within reach. *Q. E. I.*

COR. 1.

The Original of every firſt Diameter of the *Hyperbolas*, is always the Chord of the Tangents to the forming Circle from the Directing Point of its proper Ordinates; and the Diameter of the Circle which paſſes through that Directing Point, is always perpendicular to that Chord; and the Image of that Diameter is parallel to the Ordinates, Tangents, and Conjugate Diameter, and is therefore a double Ordinate to the firſt Diameter propoſed.

Thus SXY is the Chord of the Tangents from T, to which the Diameter of the Circle *tt* which paſſes through T is perpendicular; and the Image of *tt* is parallel to the Ordinates, Tangents, and ſecond Diameter Conjugate to the firſt Diameter repreſented by SXY, becauſe of their common Directing Point T, and is therefore a double Ordinate to that Diameter[e]. The ſame is true of the Diameter EH, which paſſes through L the Directing Point of the Ordinates to the Axe repreſented by SBA.

COR. 2

If the perpendicular Diameter QP of the forming Circle, be propoſed as the Original of a firſt Diameter of the *Hyperbolas*; the Tangents in Q and P being parallel to the Directing Line, their Point of Concourſe T with that Line becomes infinitely diſtant, ſo that the Diameter RV of the forming Circle which is parallel to the Directing Line, becomes the Original of a double Ordinate to the firſt Diameter repreſented by SPQ; and the Original of the Diameter Conjugate to it, paſſes through S parallel to RV, ſeeing it ought to meet RV, and likewiſe the Tangents, in a Point T in the Line LT, which Point is here at an infinite Diſtance.

COR. 3.

If from any Point without the forming Circle, and within the Angle FSG, two Tangents be drawn to the Circle, their Images will both be Tangents to the ſame *Hyperbola*; but if the propoſed Point be without the Angle FSG, the Images of the Tangents drawn from thence, will be Tangents, one of them to the one *Hyperbola*, and the other to its oppoſite; and if the propoſed Point be any where in SF or SG, there can be but one Tangent drawn from thence to the Circle, the Image of which

can

[a] Lem. 4. B. II.

[b] Cor. 2. Lem. 14.

[c] Hyperb. Art. 2.

[d] Hyperb. Art. 10.

[e] Hyperb. Art. 2.

can touch either of the *Hyperbolas*; and laſtly, no Tangent to one *Hyperbola* can be a Tangent to its oppoſite.

For if the Point from whence the Tangents are to be drawn, be within the Angle F S G, it is evident the Points of Contact muſt both fall either in the Arch F Q G, or in the Arch F P G of the forming Circle; and F Q G being the Original of one of the *Hyperbolas*, and F P G the Original of the other[a], the Images of thoſe Tangents muſt therefore both touch the ſame *Hyperbola*: and if the propoſed Point be without the Angle F S G, it is likewiſe evident that one of the Tangents muſt touch the Arch F Q G, and the other the Arch F P G, and therefore that their Images muſt be Tangents to the oppoſite *Hyperbolas*. But if the propoſed Point be any where in S F or S G, that Line is itſelf one of the Tangents, and its Image being one of the Aſymptotes, whoſe Point of Contact with the *Hyperbola* is at an infinite Diſtance[b], the other Tangent from the propoſed Point, is the only one which can produce a Tangent to the *Hyperbola*; and laſtly, as no Tangent to the Circle can touch it in more than one Point, the Image of ſuch a Tangent can only touch one or other of the *Hyperbolas*, but cannot touch them both.

C O R. 4.

No Line parallel to any firſt Diameter, or to either of the Aſymptotes of the *Hyperbolas*, can be a Tangent to either of them, but if produced, muſt neceſſarily cut one or both of them; all Lines parallel to any firſt Diameter cutting both the Sections, and thoſe parallel to either of the Aſymptotes cutting one of the Sections in one Point.

For the Directing Points of all firſt Diameters being ſomewhere between F and G[c], all Lines parallel to thoſe Diameters, muſt alſo have their Directing Points between F and G[d], and muſt therefore if produced cut both the Sections, each in one Point; and all Lines parallel to either of the Aſymptotes, having either F or G for their Directing Point, thoſe Lines muſt therefore cut one or other of the Sections in one Point, but cannot cut them both, ſeeing the Point F or G one of the Interſections of the Originals of thoſe Lines with the forming Circle hath no Image.

3. Laſtly, to determine the Originals of the *Foci*.

Biſect the Angle L I M by the Line I *m*, cutting the Directing Line in *m*, and through *m* and either of the Extremities A, of the Original of the Tranſverſe Axe, as happens to be moſt convenient, draw A *m* till it cut C D the Original of the Conjugate Axe in *z*; from *z* draw two Tangents to the Circle *z b* and *z f*, touching it in *b* and *f*, and from L through *b* and *f*, draw *b a* and *e f*, cutting A B in *w* and *v*; then the Points *w* and *v* will be the Originals of the *Foci*, and the Lines *a b* and *e f* will be each the Original of the Meaſure of the Parameter of the Tranſverſe Axe.

Dem. Becauſe of the Directing Points L, *m*, and M, the Angle L I M being biſected by I *m*, the Complement of the Image of the Triangle S *z* A will be a Triangle in the Picture, lying part on the one ſide, and part on the other of the Vaniſhing Line, which Triangle will be Iſoſceles[e]; ſo that the Complement of the Image of S A, which makes one Side of that Triangle, will be equal to the Image of S *z*, which is another Side of that Triangle; but the Complement of the Image of S A is one Moiety of the Tranſverſe Axe as already ſhewn, therefore the Image of S *z* will be equal to one Moiety of the Tranſverſe Axe; and the Lines *z b* and *z f* being Tangents to the Circle in *b* and *f* from the Point *z*, their Images are therefore Tangents to the oppoſite Sections in the Images of thoſe Points from the Image of *z*; and L being the Directing Point of the Ordinates to the Tranſverſe Axe, the Lines *a b* and *e f* drawn through L and the Points of Contact *b* and *f*, are the Originals of the Ordinates which paſs through the *Foci*[f], wherefore the Points *v* and *w*, where they cut the Original of the Tranſverſe Axe, are the Originals of the *Foci*, and *a b* and *e f* are each the Original of the Meaſure of the Parameter of the Tranſverſe Axe[g]. *Q. E. I.*

C A S E. 2.

Fig. 78. N°. 2.

If O the Center of the Circle, be in the Directing Line L G, the Points F and G being then the Extremities of the Diameter F O G, the Tangents F S and G S will be parallel to each other and perpendicular to the Directing Line; ſo that the Point S will be infinitely diſtant, which Point is therefore repreſented by the Vaniſhing Point of F S and G S, which is the Center of the Vaniſhing Line, and alſo the Center of the Sections. And all Lines which ought to paſs through S, muſt be drawn parallel to F S or G S, ſeeing their Images muſt all have the ſame Vaniſhing Point; wherefore the An-

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[a] Con. Sec. Art. 13.

[b] Hyperb. Art. 5.

[c] Cor. 2. Part firſt of this Prob.

[d] Cor. 5. Theor. 12. B. I.

[e] Schol. Prob. 4. B. II. and Gen. Cor.

[f] Hyperb. Art. 16.

[g] Hyperb. Art. 15.

gle FIG being biſected by IM, and IL being drawn perpendicular to it, AB and CD drawn through M and L perpendicular to the Directing Line, will be the Complements of the Originals of the Tranſverſe and Conjugate Axes, and L will be the Directing Point of the Ordinates to the Tranſverſe Axe, and M the Directing Point of the Ordinates to its Conjugate; and the Points C and D, which are the Originals of the Extremities of the ſecond Axe, will be determined by Lines drawn from A and B through F or G as before. *Q. E. I.*

COR. 1.

All Lines drawn through any Point between F and G parallel to AB, are the Complements of the Originals of firſt Diameters; and the Tangents to the Circle from the Extremities of any of the Lines thus drawn, other than the Diameter PQ, muſt meet in ſome Point in the Directing Line, through which Point the Original of the ſecond Diameter Conjugate to that firſt Diameter muſt paſs, parallel to AB; and the Originals of the Extremities of ſuch ſecond Diameter are found by Lines drawn through the Extremities of its correſponding firſt Diameter and the Point F or G.

COR. 2.

Hence the Diameter PQ of the Original Circle which is perpendicular to the Directing Line, is ſtill the Complement of the Original of a firſt Diameter; but in regard the Tangents to the Circle in Q and P are parallel to LG, the Directing Point of the Original of the Diameter Conjugate to it, is infinitely diſtant, ſo that this Diameter hath no real Original. However as a Line drawn from I to this infinitely diſtant Directing Point, which ſhould determine the Angle which the Image of that ſecond Diameter makes with the Vaniſhing Line[a], may be conceived to be parallel to LG, it follows, that the Diameter Conjugate to the Diameter repreſented by QP, muſt coincide with the Vaniſhing Line, the Center of the *Hyperbolas* through which it muſt paſs, being in that Line[b], and it being parallel to its Director, which is parallel to LG. And as Lines drawn from Q and P through F and G, can cut the Directing Line only in F and G; the Extremities of this ſecond Diameter are found by Lines drawn in the Picture, from the Extremities of the firſt Diameter whoſe Originals are Q and P, parallel to the correſponding Aſymptotes, which laſt are parallel to the Directors IF and IG of their Originals FS and GS. But in this Poſition of the Circle, the Diameter FG coinciding with the Directing Line, it can have no Image.

[a] Theor. 12. B. I.

[b] Caſe 2.

CASE 3.

Fig. 78. N°. 1.

If the Center of the Circle be in the Line of Station, but not in the Foot of the Eye's Director, that is, if N be the Point of Station; then QP the Diameter of the Circle perpendicular to the Directing Line, coinciding with the Line of Station SPQ, will be the Complement of the Original of the firſt or Tranſverſe Axe; in regard that the Images of all Lines drawn in the Circle parallel to the Directing Line, will be perpendicular to the Image of QP, and biſected by it; and the Diameter RV which is parallel to the Directing Line, will therefore be the Original of a double Ordinate to the Axe: the Original of the Conjugate Axe muſt therefore paſs through S the Original of the Center of the *Hyperbolas*, parallel to the Directing Line; and its Extremities are determined by Lines drawn from F and G through Q or P, as before.

Fig. 78. N°. 2.

But if the Center of the Circle O, be the Foot of the Eye's Director, the Tangents FS and GS being then parallel, the Point S is infinitely diſtant, and therefore the Conjugate Axe hath no real Original; becauſe that Original ought to be a Line paſſing through the infinitely diſtant Point S, parallel to the Directing Line, no ſuch Line as which can be drawn: however as the Image of this infinitely diſtant Point, is the Vaniſhing Point of QP, which is the Center of the Vaniſhing Line, the Vaniſhing Line itſelf muſt be the Image of the imaginary Original of the ſecond Axe; and the Extremities of this ſecond Axe are found by Lines drawn in the Picture through the Images of Q and P, which are the Vertices of the oppoſite *Hyperbolas*, parallel to the Aſymptotes, as before. *Q. E. I.*

But here as in the preceeding Caſe, the Diameter FG of the Circle, coinciding with the Directing Line, it can have no Image.

COR. 1.

If the Foot of the Eye's Director, or Point of Station, fall within the Circle, and the Height of the Eye be taken a mean Proportional between the Segments of the Directing Line, made by the Circle and the Point of Station; then the oppoſite *Hy-*

perbolas formed by the Image of the Circle, will be Equilateral, that is, the Aſymptotes will be perpendicular to each other, and the Tranſverſe and Conjugate Axes will be equal[a].

[a] Hyperb. Art. 13.

Fig. 78. N°. 1, 2.

For if F G be the Directing Line, and M be the Point of Station, and the Height of the Eye M *J* be a mean Proportional between F M and M G, the Segments of the Directing Line; it is evident, a Semicircle deſcribed on F G as a Diameter, will paſs through *J*, and therefore, that the Angle F *J* G, which is the Angle made by the Aſymptotes, will be a Right Angle.

C O R. 2.

If the Height of the Eye be taken greater than a mean Proportional between the Segments of the Directing Line terminated as before, the inward Angle of the Aſymptotes will be Acute, and the Tranſverſe Axe will be longer than the Conjugate. If the Height of the Eye be leſs than the mean Proportional between the Segments of the Directing Line, the inward Angle of the Aſymptotes will be Obtuſe, and the ſecond Axe will be longer than the Tranſverſe[b].

[b] Hyperb. Art. 14.

C O R. 3.

If the Point of Station be out of the Circle, no Equilateral *Hyperbolas* can be produced, whatever Height of the Eye be taken.

For if K be the Point of Station, it is evident the Angle F I G muſt be Acute, whatever Point I in the Perpendicular K I be taken for the Place of the Eye; ſeeing that if K do not fall ſomewhere between F and G, no Point in K I can cut a Semicircle drawn on F G as a Diameter[c].

[c] Cor. 1.

P R O B. X.

The Images of the Extremities of the perpendicular Diameter of a Circle which cuts the Directing Line, being given; thence to determine the Center, Aſymptotes, and Axes, or any other Conjugate Diameters of the *Hyperbolas* formed by the Image of the Circle.

Fig. 79. N°. 1.

Let E F be the Vaniſhing Line, o its Center, and oI its Diſtance, and let a and b be the Images of the Extremities of the perpendicular Diameter of the forming Circle; then if ab be drawn, it will paſs through o, and be a firſt Diameter of the *Hyperbolas*[d].

[d] Cor. 2. Part firſt of Prob. 9.

1. To find the Center and Aſymptotes of the *Hyperbolas*, and the Diameter Conjugate to the Diameter ab.

Through a and b draw lb and gq parallel to EF, and on either of theſe Lines as lb, find the proportional Meaſures ab, al, of the Complements of the Originals of oa and ob[e]; biſect lb in r, and take at equal to either Moiety rb of that Line, and draw to cutting gq in g; biſect ab in S, and through S draw cd parallel to E F, and having taken Sd a mean Proportional between at and gb[f], and made Sc equal to Sd, draw bd and bc, and through S draw ef parallel to bc, and ik parallel to bd; then S will be the Center, and ef and ik the Aſymptotes of the *Hyperbolas*, and cd will be the ſecond Diameter Conjugate to the Diameter ab, and c and d its Extremities.

[e] Cor. 1. Prob. 11. B. II.

[f] 13 El. 6.

Dem. Becauſe ab is a firſt Diameter of the *Hyperbolas*, the Point S which biſects it, is their Center[g]; and becauſe the Originals of gq and lb, which paſs through the Originals of b and a, the Extremities of a Diameter of the forming Circle, are perpendicular to that Diameter, they are therefore Tangents to the Circle in thoſe Points[h]; wherefore gq and lb are Tangents to the *Hyperbolas* in b and a, and conſequently cd drawn through S parallel to thoſe Tangents, is the Indefinite ſecond Diameter Conjugate to the Diameter ab[i]. And becauſe ab and al are the Proportional Meaſures of the Complements of the Originals of oa and ob, lb is the proportional Meaſure of a Diameter of the forming Circle; wherefore at, which is equal to the half of lb, is the proportional Meaſure of a *Radius* of that Circle; but the Original of at being a Tangent to the Circle in the Extremity a of its Diameter repreſented by ab, and being the proportional Meaſure of a *Radius* of that Circle, the Original of ot, which is parallel to the Original of ab, and paſſes through the Original of t, muſt touch the forming Circle in the Extremity of a *Radius* parallel to the Original of at; conſequently ot is a Tangent to one of the *Hyperbolas*; and this Tangent cutting the Tangent bg at the other Extremity b of the Diameter ab in g, the Semidiameter Conjugate

[g] Hyperb. Art. 1.

[h] 18 El. 3.

[i] Hyperb. Art. 1, 2.

to

Plate 26. Book 3. Sect. 2.

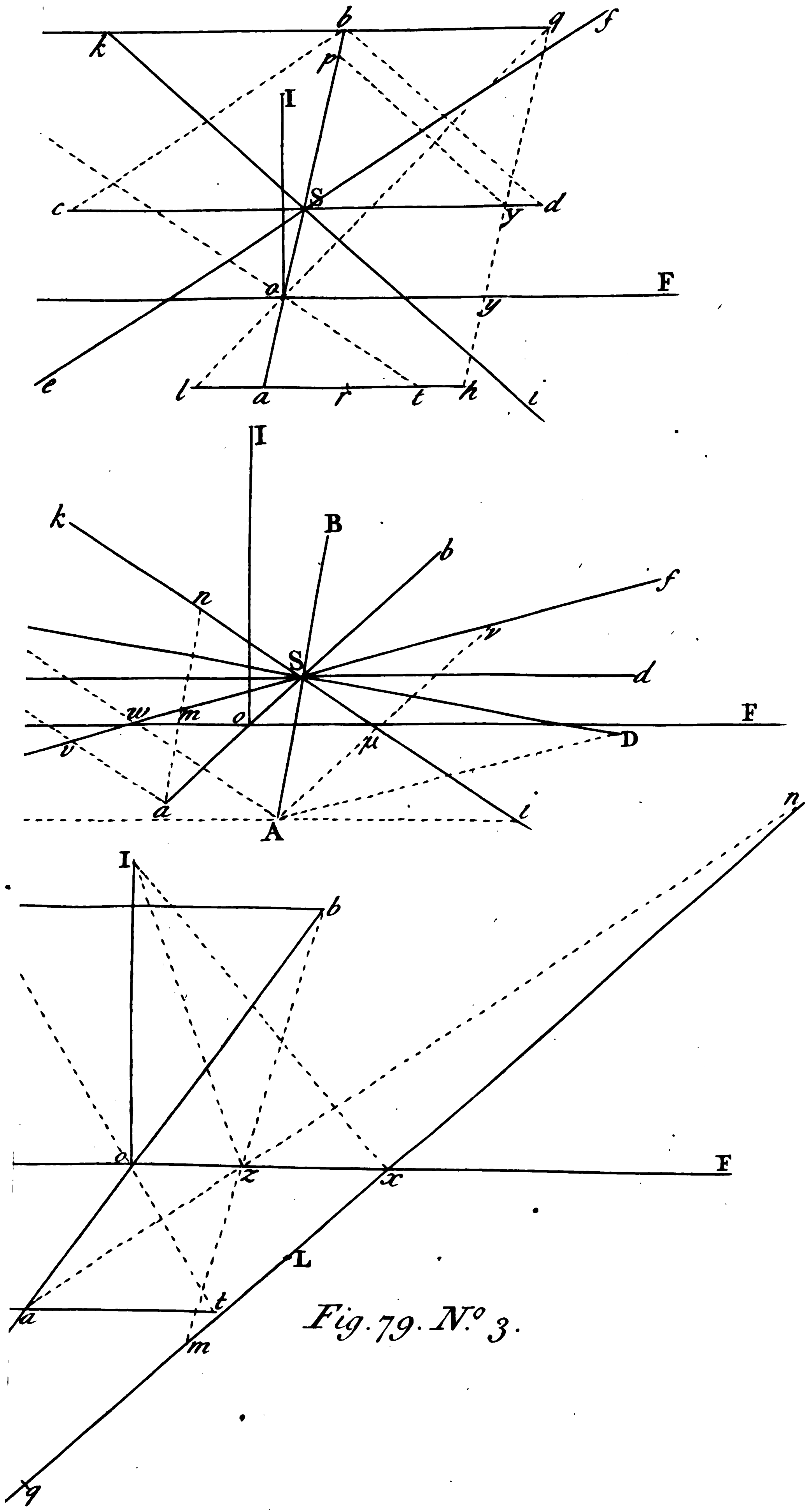

Fig. 79. No 3.

J Mynde sc

to the Diameter ab, is a mean Proportional between bg and at[a]. Wherefore Sd and Sc being each made equal to this mean Proportional, the Conjugate Diameter cd is rightly determined; and ef and ik drawn through S parallel to bc and bd, are therefore the Aſymptotes[b]. *Q. E. I.*

COR. 1.

The Semidiameter Sd is to Io the Diſtance of the Eye, as the Semidiameter Sb is to a mean Proportional between ob and oa, the Segments of the Diameter ab by the Vaniſhing Line.

Take Sp a mean Proportional between ob and oa, it muſt be proved that $\Big\}$ $Sd : Io :: Sb : Sp.$

In the Similar Triangles qoy, qlb, $oy = Io^{c} : lb :: qo : gl :: bo : ba$

Therefore $Io : \frac{1}{2}lb = at :: bo : \frac{1}{2}ba = Sb$

And becauſe al is the proportional Meaſure of the Complement of the Original of bo, and ab is the like Meaſure of the Complement of the Original of oa, therefore $al : ab :: ao : ob^{d}$

And by Compoſition $al : al + ab = lb :: ao : ao + ob = ab$

And $al : \frac{1}{2}lb = at :: ao : \frac{1}{2}ab = Sb$

But in the Similar Triangles bqo, ola, $bq = Io^{e} : al :: bo : oa$

And in the Similar Triangles gbo, oat, $gb : at :: bo : oa$

Therefore $Io : gb :: al : at$

And by Parity of Reaſon $Io : gb :: ao : Sb$

But it was before ſhewn that $Io : at :: bo : Sb$

Therefore multiplying theſe Proportionals by each other $\Big\}$ $Io^{2} : gb \times at = Sd^{2\,f} :: ao \times bo = Sp^{2} : Sb^{2}$

And extracting the Roots $Io : Sd :: Sp : Sb$

And by Converſion $Sd : Io :: Sb : Sp.$

COR. 2.

Hence if from S, a Diſtance Sp be ſet off on Sb, equal to a mean Proportional between ob and oa, and from S a Diſtance Sy be ſet off on Sd equal to Io, draw py, and parallel to it draw bd, and thereby the Semidiameter Sd will be determined.

For in the Similar Triangles bSd, pSy, $Sd : Sy = Io :: Sb : Sp.$

COR. 3.

If Io be a mean Proportional between oa and ob, then the Semidiameters Sb and Sd will be equal, and the *Hyperbolas* will be Equilateral[g].

For by the laſt Corollary $Sd : Io :: Sb : Sp$

If therefore $Io = Sp$

Then $Sd = Sb.$

COR. 4.

As the Director $\mathcal{J}N$ of the perpendicular Diameter PQ of the forming Circle, is to NG the mean Proportional between the Segments PN, NQ, of that Diameter by the Directing Line, ſo is the Semidiameter of the *Hyperbolas* formed by the Complement of the Image of PQ to the Semidiameter Conjugate to it. Fig. 78. N°. 1.

Let ab be the Complement of the Image of PQ, and conſequently a firſt Diameter of the *Hyperbolas* formed by the Image of the Circle. Fig. 79. N°. 1.

Becauſe ob is the Complement of the Image of PN, $ob : Io :: \mathcal{J}N : PN^{h}$

And becauſe oa is the Complement of the Image of NQ, $oa : Io :: \mathcal{J}N : NQ$

Therefore multiplying theſe Proportionals in order $oa \times ob : Io^{2} :: \mathcal{J}N^{2} : PN \times NQ$

And extracting their Roots $\sqrt[2]{oa \times ob} = Sp^{i} : Io :: \mathcal{J}N : NG$

But $Sp : Io :: Sb : Sd^{k}$

Therefore $\mathcal{J}N : NG :: Sb : Sd.$

COR. 5.

If the Point of Station M fall within the Circle, and the Eye's Director $\mathcal{J}M$ be a mean Proportional between FM and MG, the Segments of the Directing Line by the Circle and the Point of Station; the ſemiconjugate Diameters Sb and Sd of the *Hyperbolas* formed by the Circle, will be equal, and the *Hyperbolas* will be Equilateral. Fig. 78. N°. 1. Fig. 79. N°. 1.

For if $\mathcal{J}M$ be a mean Proportional between MF and MG, a Semicircle deſcribed on FG as a Diameter will paſs through $\mathcal{J}$, and N being the Center of that Semicircle, the Director $\mathcal{J}N$ will be equal to NG.

But

[a] Hyperb. Art. 18.

[b] Hyperb. Art. 10.

[c] Prob. 11. B. II.

[d] Cor. 2. Prob. 11. B. II.

[e] Prob. 10. B. II.

[f] Hyperb. Art. 18.

[g] Hyperb. Art. 13.

[h] Theor. 24. B. I.

[i] Cor. 1.

[k] Cor. 1.

But by the laſt Corollary $JN : NG :: Sb : Sd.$
If therefore $JN = NG$
Then $Sb = Sd.$

2. To determine the Axes.

METHOD 1.

Fig. 79. N°. 2. Biſect the inward Angle of the Aſymptotes *eSi* by the Line AB, to which through S draw CD perpendicular, and from *a* draw *an* parallel to AB, cutting the Aſymptotes in *m* and *n*; make SA and SB each equal to a mean Proportional between *am* and *an* [a], and draw AD, AC, parallel to the Aſymptotes *ef* and *ik*, cutting CD in D and C; then AB and CD will be the Axes.

For AB biſecting the inward Angle of the Aſymptotes, it is the Indefinite Tranſverſe Axe [b], and *a* being a Point in the *Hyperbola*, from whence a Parallel *an* to the Diameter AB is drawn, cutting the Aſymptotes in *m* and *n*, the Semidiameter SA is a mean Proportional between *am* and *an* [c]. Laſtly becauſe CD is perpendicular to AB, and terminated in D and C by AD and AC drawn parallel to the Aſymptotes, CD is the determinate ſecond Axe [d]. *Q.E.I.*

METHOD 2.

The Line *ac* parallel to the Aſymptote *ik* being given, cutting the other Aſymptote *ef* in *v*, from S ſet off S*w*, a mean Proportional between S*v* and *va*, and through *w* draw AC parallel to *ac*, and make *w*A, *w*C, each equal to *w*S, and thereby A and C, one Extremity of each Axe will be found [e], whereby the Length and Poſition of the Axes are determined. *Q.E.I.*

3. To determine any two Conjugate Diameters.

METHOD 1.

Fig. 79. N°. 2. Having the Aſymptotes *ef* and *ik*, and any Point A in either of the *Hyperbolas* given; through their Center S, draw any Line *ab* within the Angle of the Aſymptotes for an Indefinite firſt Diameter, and through A draw Aμ parallel to *ab*, cutting the Aſymptotes in μ and ν, and make S*a* and S*b*, each a mean Proportional between Aμ and Aν, which will give the Extremities *a* and *b* of the firſt Diameter *ab* [f]; then through either Extremity *a* of the Diameter *ab*, draw *ac* parallel to either Aſymptote *ik*, cutting the other Aſymptote *ef* in *v*, and make *vc* equal to *va*, and through S draw *cd* making S*d* equal to S*c*, and thereby *cd* the determinate ſecond Diameter Conjugate to the Diameter *ab* will be found [g].

Or if the indeterminate ſecond Diameter *cd* be firſt drawn; through A draw *ei* parallel to *cd*, cutting the Aſymptotes in *e* and *i*, and make S*c* and S*d*, each a mean Proportional between A*e* and A*i*, and *c* and *d* will be the Extremities of the Diameter *cd*; and the Diameter *ab* Conjugate to it, is found by drawing *ca* parallel to *ik* cutting *ef* in *v*, and making *va* equal to *cv*, and through S drawing *ab*. *Q.E.I.*

METHOD 2.

If any Indefinite firſt Diameter of the *Hyperbolas* be given, and a Vaniſhing Point be found, whoſe Radial may be perpendicular to the Radial of the given Diameter, the Indefinite Image of a Diameter of the forming Circle, paſſing through that Vaniſhing Point, will be parallel to the Diameter of the *Hyperbolas* Conjugate to the Diameter given [h]. If then the Image of ſuch a Diameter of the forming Circle be found, through S the Center of the *Hyperbolas* draw a Parallel to it, and that will be an Indefinite Diameter of the *Hyperbolas* Conjugate to that which is given.

SCHOL.

This laſt Method ſuppoſes, that the Image of a Diameter of the forming Circle may be drawn through any given Point; but as in the Poſition of the forming Circle which produces the *Hyperbolas*, the Image of its Center is frequently out of reach, it being at an infinite Diſtance, when the Center of the Circle is in the Directing Line; it may not be improper, in the following Corollaries, to ſhew how the Indefinite Image of any Diameter of the forming Circle, paſſing through any given Point, and alſo the Extremities of that Diameter may be found, the Images of the Extremities of the perpendicular Diameter of the forming Circle being given.

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[a] Lem. 23.

[b] Hyperb. Art. 8.

[c] Hyperb. Art. 20.

[d] Hyperb. Art. 6.

[e] Hyperb. Art. 7 and 11.

[f] Hyperb. Art. 20.

[g] Hyperb. Art. 11.

[h] Cor. 1. Part ſecond of Prob. 9.

COR. 1.

The Images *a* and *b* of the Extremities of the perpendicular Diameter of the forming Circle being given; thence to find the Indefinite Image of another Diameter of that Circle, which ſhall paſs through a given Point L. Fig. 79. N°. 3.

Through *a* and *b* draw any two parallel Lines *rt*, *gb*, and from any Point *g* in *gb*, through *o* the Vaniſhing Point of *ab*, draw *go* cutting *rt* in *t*, take *ar* equal to *at*, and draw *gr* till it cut *ab* in a Point C (if that Point be within reach) and C will be the Image of the Center of the forming Circle.

For the Complement of the Original of *ab*, which is the perpendicular Diameter of the forming Circle, being biſected by its Center, and paſſing through the Directing Line, its Indefinite Image *ab* (when that Center is not in the Directing Line) will be Harmonically divided by C, *a*, and *b*, the Images of the Center and of the Extremities of that Diameter, and its Vaniſhing Point *o*[a]; but by this Conſtruction, if the Line *ab* be cut by *gr* in any Point C, it will be Harmonically divided in *b*, *o*, *a*, and C[b]; which laſt Point is therefore the Image of the Center of the forming Circle, through which and the Point L the propoſed Diameter muſt be drawn; but if the Line *gr* do not meet *ab* in C within a convenient Diſtance, the required Diameter may be ſtill found, by drawing it through the given Point L, ſo as to tend to the ſame Point with *ab* and *gr*[c].

COR. 2.

The Indefinite Image C*n* of the propoſed Diameter of the forming Circle being found; thence to determine its Extremities.

Biſect the Angle *o*I*x*, made by the Radials of the Diameters repreſented by *ab* and C*n*, by the Line I*z*, and from *a* and *b* the Extremities of the Diameter *ab*, through the Point *z*, draw *az* and *bz* cutting C*n* in *n* and *m*; and theſe will be the Images of the Extremities of the Diameter repreſented by C*n*[d].

SCHOL.

If the Center of the forming Circle be in the Directing Line, the ſecond Method before propoſed[e] cannot be uſed: for in this Situation of the forming Circle, the Originals of all firſt Diameters of the *Hyperbolas*, being perpendicular to the Directing Line[f], the Diameter of the Circle which is perpendicular to them, coincides with that Line, and hath no Image. In this Caſe alſo, the Images of all other Diameters of the forming Circle will be parallel, the Center of the Circle being their common Directing Point.

CASE 2.

If S the Center of the *Hyperbolas*, coincide with *o* the Center of the Vaniſhing Line, which it will do, when the Center of the forming Circle is in the Directing Line[g]; the Diameter *cd*, Conjugate to the given Diameter *ab*, will coincide with the Vaniſhing Line EF[h]; and S*d* the Moiety of that Diameter, will be equal to I*o*; in regard that S*b* and S*a* being equal, the mean Proportional between them is alſo equal to them; wherefore the Points *p* and *b* coinciding, the Points y and *d* alſo coincide[i]. Fig. 79. N°. 4.

And as *ai* is here the proportional Meaſure of the Complement of *ao*, which is a *Radius* of the forming Circle, and *ae* the proportional Meaſure of *ob*, being equal to *ai*; the Aſymptotes *ik* and *ef* repreſent Tangents to the forming Circle, at the Extremities of its Diameter which coincides with the Directing Line; that is, they are Tangents to the *Hyperbolas* at an infinite Diſtance.

And here the Axes are found in the ſame manner as before. Likewiſe if I*o* be equal to *a*S, the *Hyperbolas* will be Equilateral[k]. *Q. E. I.*

CASE 3.

If the Center of the forming Circle be in the Line of Station, the Indefinite Image AB of the perpendicular Diameter of the Circle, will coincide with the Vertical Line; and the Diameter Conjugate to it, being parallel to the Vaniſhing Line, or coinciding with it, AB and CD become the Axes; and when S coincides with *o*, CD is part of the Vaniſhing Line. *Q. E. I.* Fig. 79. N°. 5.

PROB. XI.

The Images of the Extremities of any Diameter of an Original Circle which produces two oppoſite *Hyperbolas*, being given; from the Image of any Point in that Diameter, produced without the

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[a] Cor. 1. Lem. 8.

[b] Lem. 1.

[c] Prob. 18. B. II.

[d] Cor. 2. Meth. 2. Prob. 24. B. II.

[e] Part third of this Prob.

[f] Cor. 1. Caſe 2. Prob. 9.

[g] Caſe 2. Prob. 9.

[h] Cor. 2. Caſe 2. Prob. 9.

[i] Cor. 2. Part firſt of this Prob.

[k] Cor. 3. Part firſt of this Prob.

Circle, to draw two Tangents to the *Hyperbolas* formed by the Image of the Circle.

CASE 1.

If the propoſed Diameter of the Circle have a determinate Image, that is, if the Images of both its Extremities be on the ſame ſide of the Vaniſhing Line; the Image of the Point in that Diameter, through which the Chord of the Tangents paſſes, and thence the Images of the Tangents themſelves, are found as at Prob. III. *Q. E. I.*

CASE 2.

If the propoſed Diameter of the Circle be either of thoſe which have one of their Extremities in the Directing Point of either of the Aſymptotes; as FY or Gg, Fig. 78. N°. 1. the Image of the Point in that Diameter, through which the Chord of the Tangents paſſes, is found as at Caſe 2. Prob. VII.

This Caſe cannot happen but when the Center of the forming Circle is out of the Directing Line; for if it be in the Directing Line, as in Fig. 78. N°. 2. the Diameter FG of the Circle, which has its Extremities in the Directing Points of the Aſymptotes, coincides with that Line, and hath no Image.

But when this Caſe doth happen, one Moiety of the propoſed Diameter muſt have a determinate Image, which will be equal to its Complement, and the Image of the other Moiety will be Indefinite, as was ſhewn of the perpendicular Diameter of a Circle which forms a *Parabola*[a]; and in this Caſe likewiſe, the Image of the propoſed Diameter will be parallel to that Aſymptote, through the Directing Point of which the Original of that Diameter paſſes; and the Radials of that Aſymptote and Diameter will be perpendicular, in regard that SF and SG are perpendicular to FY and Gg. *Q. E. I.*

Fig. 78. N°. 1.

CASE 3.

Fig. 79. N°. 3.

If the propoſed Diameter of the Circle cut the Directing Line, the Images m and n of its Extremities, will fall one on each Side of the Vaniſhing Line, and L the Image of the Point from whence the Tangents are to be drawn, muſt neceſſarily fall between m and n; the Original of every Point in the Indefinite Image Cn of the propoſed Diameter, which falls without the Part mn either way, being a Point within the forming Circle; from whence therefore no Tangents can be drawn: and for the ſame reaſon, the Image of that Point of the propoſed Diameter, through which the Chord of the Tangents paſſes, muſt fall out of mn: and as the Original Diameter of the Circle, produced to the Point from whence the Tangents are to be drawn, is Harmonically divided by that Point, by its own Extremities, and by its Interſection with the Chord of the Tangents[b]; ſo the Indefinite Image Cn of that Diameter, will likewiſe be Harmonically divided in n, L, m, and the Point ſought[c]; which Point falling on the outſide of m or n, it muſt be an extreme Point in the Harmonical Diviſion of Cn, and will therefore fall on the outſide of m, that Extremity of mn which is neareſt to L; in regard that, of a Line Harmonically divided, the middle Part muſt be leſs than either of the Extremes[d].

If then a Point q be found in Cn, ſo that qn may be Harmonically divided in q, m, L, and n[e]; q will be the Image of that Point in the Diameter of the forming Circle, through which the Chord of the Tangents from the Original of L paſſes; and the Indefinite Image of that Chord being drawn, its Extremities are determined in the ſame manner as at Prob. III. whether thoſe Extremities fall both on the ſame Side, or one on each Side of the Vaniſhing Line: obſerving that when thoſe Extremities fall both on the ſame Side of the Vaniſhing Line, the Tangents found, are both Tangents to the ſame *Hyperbola*; but if otherwiſe, they are Tangents to the oppoſite *Hyperbolas*. *Q. E. I.*

SCHOL.

In this Problem, the Indefinite Image Cn of the Diameter of the forming Circle, which paſſes through L the Image of the Point from whence the Tangents are to be drawn, is ſuppoſed to be given; ſeeing if L were given, with the Image of the perpendicular Diameter of the forming Circle, or indeed of any other, the Image Cn of the Diameter which paſſes through the Original of L, may be found by the Corollaries of Method 2. Part 3. of the laſt Problem.

PROB. XII.

Fig. 80.

Two oppoſite *Hyperbolas* Qbm, nar, with their Center S, and Aſymptotes

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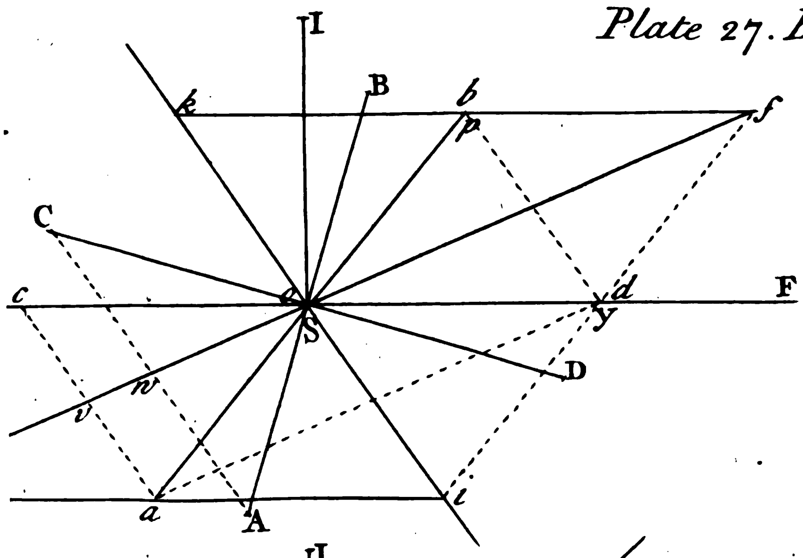
[a] Prob. 7.

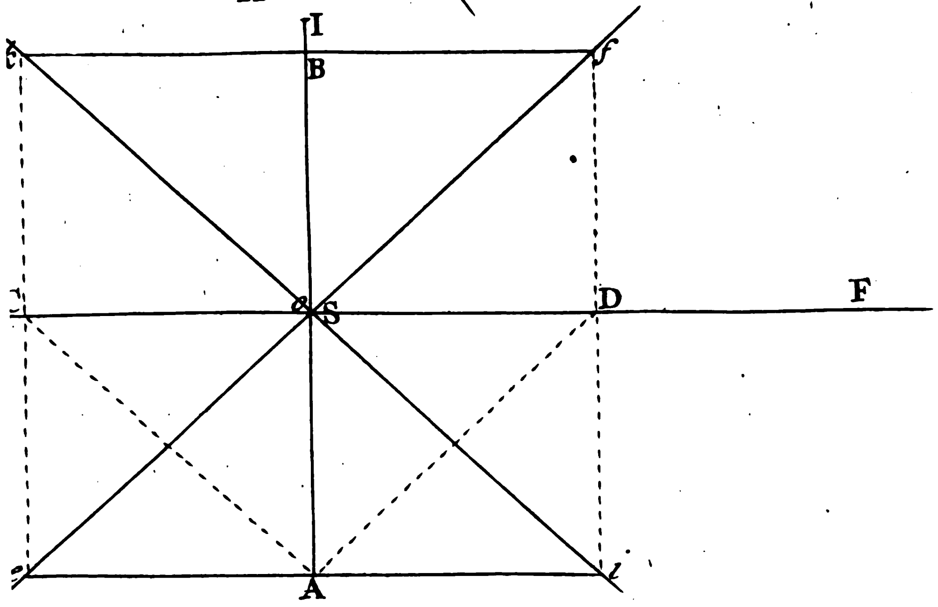
[b] Lem. 11.

[c] Cor. 4. Lem. 8.

[d] Cor. 1. Lem. 1.

[e] Lem. 1.





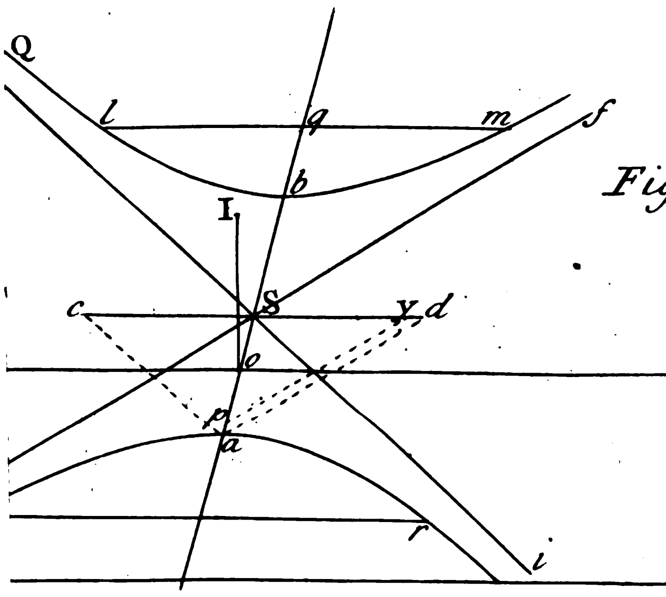


Fig. 80.

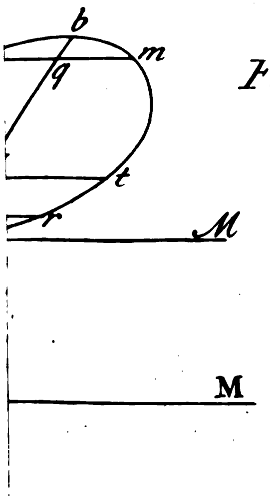
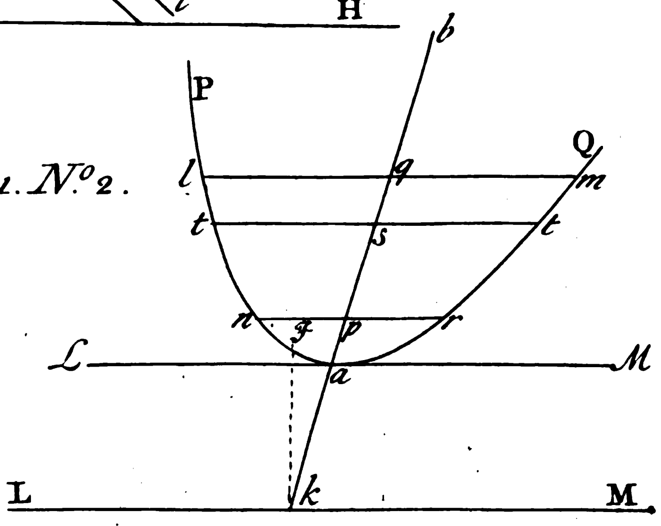


Fig. 81. No. 2.



J Mynde sc.

ſymptotes ef, ik, being given; thence to find the Vaniſhing Line, Center, and Diſtance of a Plane, in which an Original Circle being placed, its Image ſhall be the given *Hyperbolas.*

The Original of the given *Hyperbolas,* may be a Circle in any Plane, whoſe Directing Line is cut by that Circle [a], the Vaniſhing Line of which Plane muſt neither cut nor touch either of the Sections, and muſt therefore paſs between them.

Having therefore taken any Line EF, paſſing between the given *Hyperbolas,* for the Vaniſhing Line of the Plane of the forming Circle, draw any Line lm in either of the Sections parallel to EF, and terminated by the Section in l and m; biſect lm in q, from whence through S draw qS, cutting the Sections in a and b, and the Line EF in o.

Then becauſe ab is a firſt Diameter of the *Hyperbolas,* to which lm parallel to EF is a double Ordinate, ab is therefore the Complement of the Image of the perpendicular Diameter of the forming Circle [b], and conſequently o is the Center of the Vaniſhing Line.

Then find cd, the Diameter of the *Hyperbolas* Conjugate to the Diameter ab [c], and having drawn ad, take on Sa, a Diſtance Sp, equal to a mean Proportional between ob and oa, and draw py parallel to ad, cutting Sd in y; then make oI equal to Sy, and oI will be the Diſtance of the Vaniſhing Line [d]. If then any Line GH parallel to EF, be taken as the Interſecting Line, and an Original Plane be conſtructed accordingly, the Originals of a and b being found in this Plane, and joined by a ſtraight Line, that Line will be the Diameter of a Circle in that Plane, the Image of which Circle will produce the given *Hyperbolas.* *Q.E.I.*

GENERAL COROLLARY.

In the firſt, fifth, and ninth Propoſitions of this Section, which relate to the Original Plane, the Diſtance of the Picture not being concerned in the Demonſtrations, but only the Height of the Eye, and the Situation of the Original Circle with reſpect to the Directing Line; the Originals of the Axes, Diameters, Ordinates, *&c.* of any of the Sections produced by the Image of the Circle in any of the Situations before deſcribed, will be the ſame, at whatever Diſtance the Picture be placed from the Directing Plane, while it remains parallel to that Plane, and the Place of the Eye is not changed; and therefore the Interſecting Line of the Original Plane is not marked in thoſe Figures, as being unneceſſary.

In the ſecond, ſixth, and tenth Propoſitions, which relate to the Picture, the Height of the Eye not being concerned in the Demonſtrations, but only its Diſtance from the Picture, and the Situation of the Vaniſhing Line with reſpect to the given Image of the perpendicular Diameter of the forming Circle; the Axes, Diameters, Ordinates, *&c.* of any of the Sections found in the Picture from the given Image, will be the ſame, at whatever Diſtance from the Vaniſhing Plane, the Plane of the forming Circle lies; ſo as thoſe Planes be parallel, and the Diſtance of the Vaniſhing Line be not changed, and that the given Image, from whence the other Lines are found, be the Image of the perpendicular Diameter of the forming Circle in the Original Plane; and therefore the Interſecting Line in the Picture is not marked.

But in the fourth, eighth, and twelfth Propoſitions, where the Original Circle which produces a given Conick Section, is required; the Interſecting Line becomes neceſſary, in order to aſcertain the particular Original Plane, and the Original Circle in that Plane, of which the given Section in the Picture is the Image; in regard that the given Section may be the Image of a Circle in any Original Plane, parallel to the Vaniſhing Plane firſt ſuppoſed, while the Center and Diſtance of the Vaniſhing Line are not varied. But when the Interſecting Line is once taken, then the Height of the Eye as well as its Diſtance from the Picture become fixed, and the Situation of the Original Plane, and of the Original Circle in that Plane with regard to the Picture, are thereby ſettled; ſo that the given Curve in the Picture, can be the Image of no other Circle, nor of any other Figure in that Original Plane.

Nevertheleſs in all theſe Propoſitions, the Angle of Inclination of the Picture to the Original Plane is left undetermined; the ſame Image being produced by the ſame Original, and *vice verſa,* whatever Inclination be given to the Picture with regard to the Original Plane, while the ſame Diſtance and Height of the Eye are retained [e].

[a] Con. Sec. Art. 15.

[b] Cor. 2. Part firſt of Prob. 9.

[c] Method 1. Part third of Prob. 10.

[d] Cor. 2. Prob. 10.

[e] Gen. Cor. after Prob. 30. B. II.

SECTION III.

Of the Tranſmutation of the Conick Sections into each other, by the Rules of Stereography.

IT has been ſhewn in the preceeding Section, that an Original Circle will, by its Image, produce either a Circle or *Ellipſis*, a *Parabola*, or two oppoſite *Hyperbolas*, according as that Circle lies, either wholly on one Side of the Directing Line, or touches, or cuts that Line; and that any Conick Section in the Picture, may be the Image of a Circle in an Original Plane, in a certain Poſition with regard to the Directing Line; which Poſition is determined by the Species of that Section. It ſhall be ſhewn in the following Theorems, that any Conick Section in an Original Plane, may by its Image produce any Conick Section whatever, according to the Situation of the Original Section propoſed, with reſpect to the Directing Line of its Plane; and that any Conick Section in the Picture, according to its Situation with reſpect to the Vaniſhing Line of its Plane, may be the Image of any Conick Section whatever in an Original Plane.

PROP. XIII. Theor.

If any Conick Section in an Original Plane, neither touch nor cut the Directing Line of that Plane, the Image of that Section will be either a Circle or an *Ellipſis*.

1. When the Original Section given, is a Circle or *Ellipſis*.

This has already been ſhewn with reſpect to a Circle, and the ſame is likewiſe evident of an *Ellipſis*; in regard that if an *Ellipſis* lie wholly on one ſide of the Directing Plane, every Point in that *Ellipſis* having a real Image, the intire Image muſt be a Figure returning into itſelf; and being the Section of the Elliptical Cone by the Picture, it muſt therefore be a Conick Section, and conſequently either an *Ellipſis* or a Circle[a].

But as it hath been ſhewn, that an Original Circle which doth not touch or cut the Directing Line, cannot have a Circle for its Image, unleſs the Center of the Original Circle lie in the Line of Station, and even then, only at a certain determinate Height of the Eye[b]; ſo an Original *Ellipſis* cannot have a Circle for its Image, unleſs the Directing Point of that Diameter of the *Ellipſis*, whoſe Ordinates are parallel to the Directing Line, be in the Point of Station, and that only, at one certain Height of the Eye: in all other Poſitions, an *Ellipſis*, as well as a Circle, muſt have an *Ellipſis* for its Image, while the Original Figure lies wholly on the ſame Side of the Directing Line.

Fig. 81. N°. 1. *Dem.* Let $atbt$ be an *Ellipſis* in the Original Plane, and LM the Directing Line; draw any two Lines lm, nr, in the *Ellipſis*, parallel to LM, and having biſected them in q and p, through q and p draw ab, which will be a Diameter of the *Ellipſis*, to which lm and nr will be double Ordinates[c]; and becauſe the Image of ab biſects the Images of lm and nr, the Image of ab will alſo be a Diameter of the produced Curve, to which the Images of lm and nr will be double Ordinates: produce ab to its Directing Point k, and find a Point s between a and b, ſo that the Line kb may be Harmonically divided in k, a, s, and b[d], and through s draw tt parallel to LM; then the Images of as and sb being equal[e], the Image of tt, which is parallel to thoſe of lm and nr, will be a Diameter of the produced Curve, Conjugate to the Diameter repreſented by ab[f]: now in regard that no two parallel Lines in a Circle, can be biſected by any Line beſides a Diameter of that Circle perpendicular to them[g], the Image thus produced, cannot be a Circle, unleſs the Image of ab be perpendicular to the Images of lm, nr, and tt, which it cannot be, unleſs k the Directing Point of ab, be alſo the Point of Station; and therefore if k be not the Point of Station, the Image produced muſt be an *Ellipſis*.

But if the Directing Point k be the Point of Station, and the Height of the Eye Jk, be taken in the ſame Proportion to st, as ka to sa, or as kb to bs[h]; the Images of ab and tt will not only be perpendicular, but equal to each other, and the Figure produced, will be a Circle.

For when k is the Point of Station, the Image of ab being perpendicular to the Va-

[a] Con. Sec. Art. 18.

[b] Caſe 3. Prob. 1.

[c] Ellip. Art. 1.

[d] Lem. 1.

[e] Cor. 5. Lem. 8.

[f] Ellip. Art. 4.

[g] 3 El. 3.

[h] 12 El. 6.

niſhing Line[a], it is alſo perpendicular to the Images of *lm*, *nr*, and *tt* which it biſects, and therefore *ab* and *tt* are the Originals of the Axes of the Figure produced[b]; and becauſe by the Suppoſition, *Jk* is to *st*, as *ka* to *sa*, the Images of *as* and *st*, and conſequently of *ab* and *tt* are equal[c], and therefore the Figure produced is a Circle[d]. But at any other Height of the Eye in the Line *Jk*, the Figure produced will be an *Ellipſis*, of which the Images of *ab* and *tt* will be the Axes; ſeeing they cannot then be equal. *Q.E.D.*

2. When the given Original Section is a *Parabola.*

Let P*a*Q be a *Parabola* in the Original Plane, and LM the Directing Line. Fig. 81. N°. 2.

Draw any two Lines *lm*, *nr* in the *Parabola*, parallel to LM, and having biſected them in *q* and *p*, through *q* and *p* draw *ab*, which will be a Diameter of the *Parabola*, to which *lm* and *nr* will be double Ordinates[e]; and becauſe the Image of *ab* biſects the Images of *lm* and *nr*, the Image of *ab* will alſo be a Diameter of the produced Curve, to which the Images of *lm* and *nr* will be double Ordinates; and in regard the Image of the infinitely diſtant Extremity of *ab*, is at the Vaniſhing Point of that Line, the Image of the Indefinite Diameter *ab* will be a determinate Line in the Picture, terminated by the Image of *a* and that Vaniſhing Point; which Vaniſhing Point is alſo the Vaniſhing Point of all other Diameters of the Original *Parabola*, they being all parallel to each other[f]; and the infinitely diſtant Extremities of the Indefinite Sides *a*P and *a*Q of the *Parabola*, becoming ultimately parallel to its Diameters[g], the Images of thoſe infinitely diſtant Extremities muſt therefore meet and unite at the ſame Vaniſhing Point; and all other Points in the *Parabola*, except thoſe infinitely diſtant Extremities of *a*P and *a*Q, having real Images, the Figure produced muſt therefore be a Figure returning into itſelf, and conſequently either an *Ellipſis* or a Circle, which will touch the Vaniſhing Line in one Point, *viz.* the Vaniſhing Point of *ab*; and the Image of *ab* will therefore be a determinate Diameter of that Figure.

Produce *ab* to its Directing Point *k*, and take *as* equal to *ak*, and through *s* draw *tt* parallel to LM; then becauſe *sa* is equal to its Complement *ak*, the Image of *sa* will be equal to its Complement[h], wherefore the Image of *s* will biſect the Diameter of the produced Curve, whoſe Original is *ab*, and conſequently *tt* drawn through *s* parallel to the double Ordinates *lm*, *nr*, will be the Original of a Diameter of the produced Curve, Conjugate to the Diameter repreſented by *ab*. But for the reaſon given in the Demonſtration of the firſt Part of this Propoſition, the Image thus produced cannot be a Circle, unleſs *k* the Directing Point of *ab* be the Point of Station; therefore if *k* be not the Point of Station, the Figure produced muſt be an *Ellipſis*.

But if *k* be the Point of Station, and the Height of the Eye *Jk* be taken equal to *st*, then the Image produced will be a Circle: for the Images of *ab* and *tt* will then be perpendicular to each other, and the Images of *as* and *st*, and conſequently of *ab* and *tt*, which are the Axes, will be equal[i]. But at any other Height of the Eye in the Line *Jk*, the Figure produced will be an *Ellipſis*, of which the Images of *ab* and *tt* will be the Axes, which cannot then be equal. *Q.E.D.*

3. When the given Sections are two oppoſite *Hyperbolas.*

Let Pa*m*, *n*bQ be two oppoſite *Hyperbolas* in an Original Plane, neither touching nor cutting the Directing Line LM, and let *ef* and *ik* be their Aſymptotes, and S their Center. Fig. 81. N°. 3.

Draw any two Lines *lm*, *nr*, either in the ſame or in the oppoſite *Hyperbolas*, parallel to LM, and having biſected them in *q* and *p*, through *q* and *p* draw ab, and through S draw cd parallel to *lm*, and terminated in c and d by bc and bd drawn parallel to the Aſymptotes; then ab will be a firſt Diameter of the *Hyperbolas*, to which *lm* and *nr* will be double Ordinates[k], and cd will be the ſecond Diameter Conjugate to the Diameter ab[l]: and becauſe the Image of ab biſects the Images of *lm* and *nr*, the Complement of the Image of ab (which is a determinate Line in the Picture, paſſing through the Vaniſhing Line) will alſo be a Diameter of the produced Curve, to which the Images of *lm* and *nr* will be double Ordinates: and becauſe the Indefinite Side a*m* of the *Hyperbola* Pa*m*, ultimately coincides with the Aſymptote *ef*[m], the Image of the infinitely diſtant Extremity of a*m*, is at the Vaniſhing Point of *ef*; and for the ſame reaſon, the Image of the infinitely diſtant Extremity of the Indefinite Side aP of the ſame *Hyperbola*, is at the Vaniſhing Point of the Aſymptote *ik*; wherefore the Image of the *Hyperbola* Pa*m*, will be a Curve paſſing through the Image of a, and terminated at the two Vaniſhing Points of the Aſymptotes. After the ſame manner, the Image of the oppoſite *Hyperbola* *n*bQ, will be a Curve on the con-

[a] Cor. 3.

[b] Theor. 12. B. I.

[c] Ellip. Art. 3.

[d] Theor. 32. B I.

[d] Ellip. Art. 7.

[e] Parab. Art. 1. and 4.

[f] Parab. Art. 3.

[g] Cor. 5. Caſe 1. Prob. 5.

[h] Theor. 26. B. I.

[i] Cor. 1. Theor. 32. B. I.

[k] Hyperb. Art. 1 and 2.

[l] Hyperb. Art. 10.

[m] Hyperb. Art. 5.

trary Side of the Vaniſhing Line, paſſing through the Image of b, and terminated at the ſame two Vaniſhing Points; wherefore theſe two Images will together compoſe a Curve Line returning into itſelf, and conſequently be either an *Ellipſis* or a Circle, cutting the Vaniſhing Line in two Points: but in regard the Complement of the Image of a b is a Diameter of this Curve, to which the Images of *lm* and *nr* are double Ordinates, the Image produced cannot be a Circle, unleſs this Diameter and its Ordinates be perpendicular, which they cannot be, unleſs k the Directing Point of a b, be the Point of Station; therefore if k be not the Point of Station, the Figure produced muſt be an *Ellipſis*.

But if k be the Point of Station, and the Height of the Eye $\mathcal{J}k$ be ſuch, that a mean Proportional between b k and k a, may be to $\mathcal{J}k$, as the Semidiameter S a, is to its Semiconjugate S d, then the Curve produced will be a Circle.

Fig. 81. N°. 4. Let EF be the Vaniſhing Line, and I*o* its Diſtance, and let *a b* be the Complement of the Image of a b, which in this Caſe muſt be perpendicular to the Vaniſhing Line, and paſs through its Center *o*.

Fig. 81. N°. 3, 4. Then becauſe *a o* is the Complement of the Image of k a, $ka : \mathcal{J}k :: Io : ao$[a]

And becauſe *o b* is the Complement of the Image of k b, $kb : \mathcal{J}k :: Io : ob$

Therefore multiplying theſe Proportionals $ka \times kb : \mathcal{J}k^2 :: Io^2 : ao \times ob$

And extracting their Roots $\sqrt[2]{ka \times kb} : \mathcal{J}k :: Io : \sqrt[2]{ao \times ob}$

But by the Suppoſition $\sqrt[2]{ka \times kb} : \mathcal{J}k :: Sa : Sd$

Therefore by parity of Reaſon $Sa : Sd :: Io : \sqrt[2]{ao \times ob}$

Fig. 81. N°. 3. Through a draw the Tangent a*t*, cutting the Aſymptote *ef* in *t*, then a*t* will be parallel and equal to S d[b].

Now becauſe S a and S*t* meet in S, and cut a*t*, a Line parallel to the Directing Line, in a and *t*; S a is to a*t*, as the Radial of S a is to the Diſtance between the Vaniſhing Points of S a and S*t*[c], which let be *z*:

But a*t* is equal to S d, and the Radial of S a is I*o*, therefore $Sa : Sd :: Io : z$

And it having been already ſhewn that $Sa : Sd :: Io : \sqrt[2]{ao \times ob}$

Therefore $z = \sqrt[2]{ao \times ob}.$

Fig. 81. N°. 4. On *a b* as a Diameter, with the Center C, deſcribe a Circle *a c b d*, cutting EF in *x* and *y*, then *oy* will be a mean Proportional between *a o* and *o b*, that is, $oy = \sqrt[2]{ao \times ob}$; and therefore $oy = z$, and conſequently *y* is the Vaniſhing Point of the Aſymptote *ef*: and after the ſame manner it may be ſhewn, that *x* is the Vaniſhing Point of the other Aſymptote *i k*; wherefore *x* and *y* are Points in the Images of the oppoſite *Hyperbolas*, and *x y* being biſected in *o*, is therefore a double Ordinate to the Diameter *a b* of the produced Curve, to which it is alſo perpendicular; the Diameter *a b* is therefore one of the Axes, and the Ordinate *oy*, being a mean Proportional between *oa* and *o b* the Segments of the Axe *a b*, that Axe is equal to its Conjugate *c d*, and the Curve produced is therefore a Circle[d]; and conſequently the Circle *a c b d* is the Image of the oppoſite *Hyperbolas* propoſed. But at any other Height of the Eye in the Line $\mathcal{J}k$, the Length of the Axe *a b* of the produced Curve will be changed, while *ox* and *oy* will continue the ſame; and therefore *o y* cannot then be a mean Proportional between *o a* and *o b*, which laſt at all Heights of the Eye are ſtill in Proportion to each other as k b to k a[e]; and therefore at any other Height of the Eye, the Curve produced muſt be an *Ellipſis*, of which *a b* will be one of the Axes, and *x y* a double Ordinate to it; whence the other Axe may be eaſily determined[f]. *Q. E. D.*

GENERAL COROLLARY.

From this Propoſition it follows;

1. That the Original of an *Ellipſis* or Circle in the Picture, which doth not touch or cut the Vaniſhing Line, muſt be either an *Ellipſis*, or a Circle in the Original Plane, which doth neither touch nor cut the Directing Line.

2. That the Original of an *Ellipſis* or Circle in the Picture, which touches the Vaniſhing Line, muſt be a *Parabola* in the Original Plane, which neither touches nor cuts the Directing Line.

3. And that the Original of an *Ellipſis* or Circle in the Picture, which cuts the Vaniſhing

[a] Theor. 24. B. I.

[b] Hyperb. Art. 17.

[c] Cor. 2. Theor. 32. B. I.

[d] Ellip. Art. 13 and 16.

[e] Cor. 2. Prob. 11. B. II.

[f] Ellip. Art. 13.

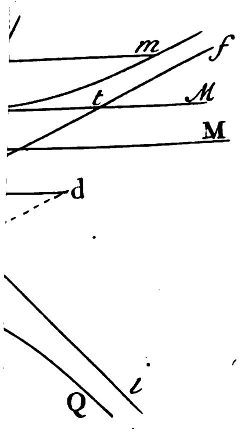


Fig. 81. No. 4.

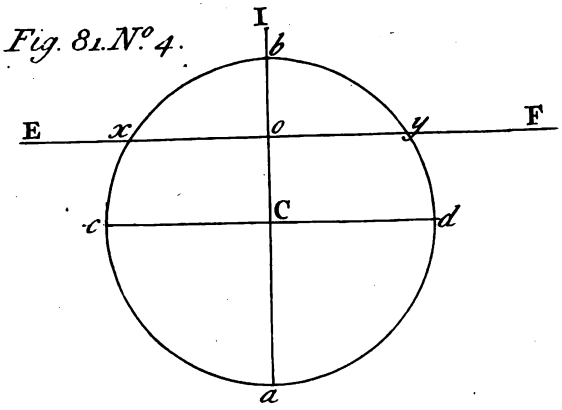


Fig. 83. No. 1.

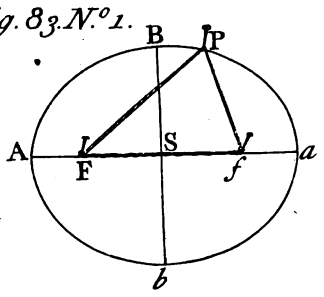


Fig. 82. No. 2.

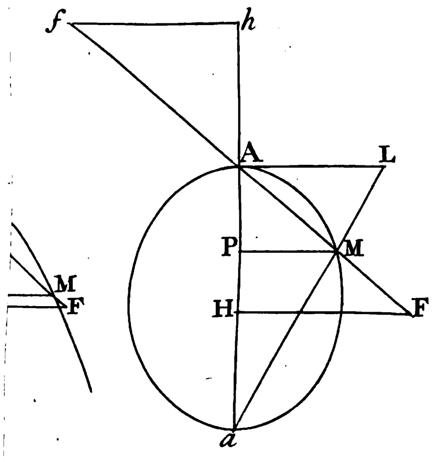
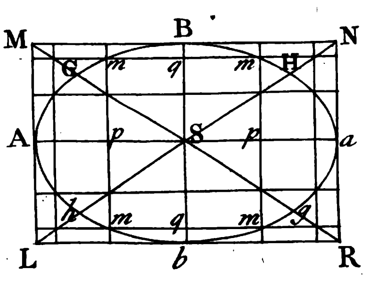


Fig. 83. No. 2.



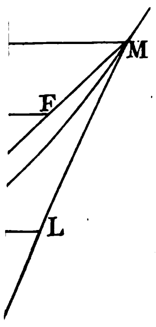


Fig. 83. No. 4.

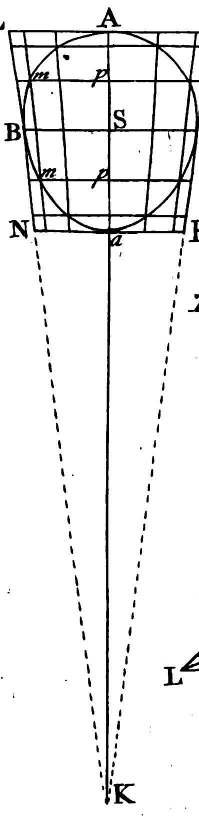
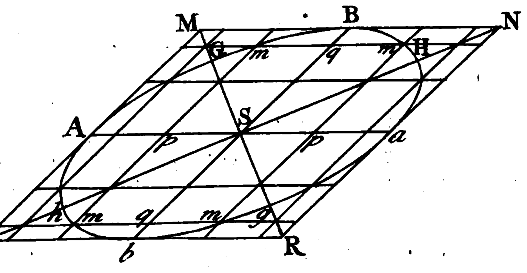
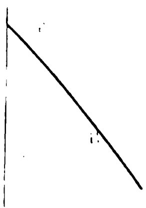


Fig. 83. No. 3.





J. Mynde sc.

niſhing Line, muſt be two oppoſite *Hyperbolas* in the Original Plane, the one lying wholly on one Side, and the other on the other Side of the Directing Line.

PROP. XIV. Theor.

If any Conick Section in an Original Plane touch the Directing Line, the Image of that Section will be a *Parabola*.

1. When the Original Section is a Circle or *Ellipſis*.

Fig. 81. N°. 1. This has already been ſhewn with reſpect to an Original Circle, and the ſame is equally evident of an *Ellipſis*; ſeeing every Point of an *Ellipſis atbt*, which touches the Directing Line *LM* in a Point *a*, hath a real Image, ſave only the Point *a*; and the Images of thoſe Parts of the Original *Ellipſis*, which lie infinitely near that Point, muſt become ultimately parallel to the Director of that Point, which agrees with the Property of the *Parabola* already deſcribed[a]. *Q. E. D.*

2. When the Original Section is a *Parabola*.

Fig. 81. N°. 2. Let P*a*Q be the Original *Parabola*, touching the Directing Line *LM* in *a*; the Images of the Indefinite Sides *a*P, *a*Q, meeting at the Vaniſhing Point of the Diameter *ab*[b], thereby form one continued Figure, which would be compleat and return into itſelf, could the Point of contact *a* be repreſented; but the Image of this Point being at an infinite Diſtance, and the Images of the Parts of the *Parabola*, which are infinitely near to *a*, becoming ultimately parallel to the Director of the Point *a*, the Image produced is therefore a *Parabola*, touching the Vaniſhing Line in one Point, *viz.* the Vaniſhing Point of *ab*. *Q. E. D.*

3. When the given Sections are two oppoſite *Hyperbolas*.

Fig. 81. N°. 3. Let Pa*m*, *n*bQ be two oppoſite *Hyperbolas*, and *LM* the Directing Line, touching the *Hyperbola* Pa*m* in a.

The *Hyperbola n*bQ which doth not touch or cut the Directing Line, forms a Curve in the Picture, terminated at two Points in the Vaniſhing Line, *viz.* the Vaniſhing Points of the Aſymptotes, at which two Points the Indefinite Sides a*m* and aP of the *Hyperbola* Pa*m* alſo terminate[c]; and therefore the Images of both *Hyperbolas* would together form a Figure returning into itſelf, could the Point a be repreſented; but as the Image of this Point is infinitely diſtant, and the Images of the Parts of the *Hyperbola* Pa*m*, which lie infinitely near that Point, becoming ultimately parallel to its Director, the entire Image of both *Hyperbolas* taken together, doth therefore form a *Parabola*, cutting the Vaniſhing Line in two Points, *viz.* the Vaniſhing Points of the Aſymptotes *ef* and *ik*.

Or if the Directing Line be one of the Aſymptotes, as *ef* (which may be taken to be a Tangent to the oppoſite *Hyperbolas* at an infinite Diſtance) the Section produced will be a *Parabola*, whoſe Diameters will be parallel to the Vaniſhing Line; and conſequently the Vaniſhing Line will be one of thoſe Diameters.

For the Images of the Indefinite Sides aP and bQ of the oppoſite *Hyperbolas*, uniting at the Vaniſhing Point of *ik*, they muſt together compoſe one continued Figure; and the Indefinite Sides a*m* and b*n* becoming ultimately parallel to the Directing Line *ef*, their Images muſt become ultimately parallel to the Vaniſhing Line, which will therefore be a Diameter of the Figure produced, and which conſequently will be a *Parabola*, according to the Property of that Section before mentioned. *Q. E. D.*

GENERAL COROLLARY.

Hence it follows;

1. That the Original of a *Parabola* in the Picture, which neither touches nor cuts the Vaniſhing Line, is either a Circle or an *Ellipſis* in the Original Plane, touching the Directing Line.
2. That the Original of a *Parabola* in the Picture, which touches the Vaniſhing Line, is a *Parabola* in the Original Plane, touching the Directing Line.
3. And that a *Parabola* in the Picture, cutting the Vaniſhing Line in two Points, is produced by two oppoſite *Hyperbolas* in the Original Plane, one of which only touches the Directing Line.

Or if a *Parabola* in the Picture cut the Vaniſhing Line only in one Point, it muſt be produced by two oppoſite *Hyperbolas* in the Original Plane, having the Directing Line for one of their Aſymptotes.

PROP.

[a] Con. Sec. Art 11.

[b] Prop. 13. Part ſecond.

[c] Prop. 13. Part third.

PROP. XV. Theor.

If any Conick Section in the Original Plane cut the Directing Line, the Image of that Section will be two oppoſite *Hyperbolas*.

1. When the Original Section is a Circle or an *Ellipſis*.

Fig. 81. N°. 1. This has already been ſhewn with reſpect to an Original Circle, and the ſame is alſo evident of an *Ellipſis*; for if *nr* be taken as the Directing Line, cutting the Original *Ellipſis atbt* in two Points *n* and *r*, the Image of the Part *nbr* muſt form a Figure whoſe Sides will be Indefinite, the Extremities of which Sides muſt ultimately become parallel to the Directors of the Points *n* and *r*, and will therefore ultimately coincide with the Images of Tangents to the *Ellipſis* in *n* and *r*, and conſequently that Figure will be an *Hyperbola*, of which thoſe Tangents will be the Originals of the Aſymptotes: after the ſame manner, the Part *nar* of the *Ellipſis* will form another *Hyperbola* having the ſame Aſymptotes with the other, and conſequently thoſe *Hyperbolas* will be oppoſite, and one of them will lie wholly on the one Side, and the other on the other Side of the Vaniſhing Line. *Q. E. D.*

2. When the given Section is a *Parabola.*

The Directing Line may cut the given *Parabola* either in two Points, or only in one.

Fig. 81. N°. 2. If *nr* be taken as the Directing Line, cutting the *Parabola* P*a*Q in two Points *n* and *r*; then the Images of the Indefinite Sides *n*P, *r*Q, uniting at the Vaniſhing Point of the Diameter *ab*, that Image will be one continued Figure, whoſe Sides will be Indefinite, the Extremities of which Sides will ultimately become parallel to the Directors of the Points *n* and *r*, and will therefore ultimately coincide with the Images of Tangents to the *Parabola* in the Points *n* and *r*, which Figure will therefore be an *Hyperbola*, to which thoſe Tangents are the Originals of the Aſymptotes; and the continued Part *nar* of the *Parabola* will form another *Hyperbola*, having the ſame Aſymptotes, and conſequently oppoſite to the other; and of theſe *Hyperbolas*, that which is formed by the Indefinite Sides *n*P, *r*Q of the *Parabola*, will touch the Vaniſhing Line in the Vaniſhing Point of *ab*, the other *Hyperbola* will fall altogether on the contrary Side of that Line.

If the Directing Line cut the *Parabola* only in one Point, it muſt then be a Diameter of that Section; if then any Diameter *ab* of the *Parabola* P*a*Q, be taken as the Directing Line, cutting the *Parabola* in *a*; the Image of the Indefinite Side *a*P of the *Parabola*, will be Indefinite at both ends, and the infinitely diſtant Extremity of that Image, whoſe Original is at *a*, will ultimately coincide with the Image of *LM* the Tangent to the *Parabola* at the Point *a*, and the other infinitely diſtant Extremity of that Image muſt ultimately coincide with the Vaniſhing Line; in regard that the Indefinite Side *a* P of the *Parabola*, the farther it is produced beyond P, becomes farther diſtant from the Directing Line *ab*, to which it becomes ultimately parallel, ſo that its Image can never paſs the Vaniſhing Line, though it ſtill approaches nearer to it; the Figure therefore thus produced, having two Indefinite Sides, one ultimately coinciding with the Vaniſhing Line, and the other ultimately coinciding with the Image of the Tangent *LM*, it is an *Hyperbola*, of which the Vaniſhing Line and the the Image of *LM* are the Aſymptotes: and after the ſame manner, the other Indefinite Side *a*Q of the *Parabola* forms the oppoſite *Hyperbola*, having the ſame Lines for its Aſymptotes as the other. *Q. E. D.*

3. When the given Sections are two oppoſite *Hyperbolas*.

The Directing Line may either cut one of the *Hyperbolas* in two Points, or each *Hyperbola* in one Point, or only one of them in one Point.

Fig. 81. N°. 3. Firſt then, let *nr* be the Directing Line, cutting the *Hyperbola n*b*Q* in two Points *n* and *r*, through which two Points, draw two Tangents to that *Hyperbola*.

Then the Images of thoſe Tangents will be Tangents to the Image of *n*b*r*, and alſo to the Images of the Parts Q*r* and *bn*, at the infinitely diſtant Extremities of thoſe Images, the Originals of which are *r* and *n*; conſequently the Image of the Part *n*b*r* of the *Hyperbola n*b*Q* will be a compleat *Hyperbola*, of which the Images of the Tangents at *r* and *n* are the Aſymptotes; and for the ſame reaſon, the Images of the Remainder Q*r* and *bn* of that *Hyperbola*, will be two Indefinite Sides of an oppoſite *Hyperbola* having the ſame Aſymptotes as the other, but terminated each at a Point in the Vaniſhing Line; the Image of the infinitely diſtant Extremity of *r*Q being at the Vaniſhing Point of the Original Aſymptote *ik*, and the Image of the infinitely diſtant Extremity of *nb* being at the Vaniſhing Point of the other Aſymptote *ef*, at which two Vaniſhing Points the Images of the infinitely diſtant Extremities of the oppoſite *Hyperbola* Pa*m* alſo terminate; and conſequently the Image of the *Hyperbola* Pa*m*, together with the Images of *r*Q and *nb*, form an *Hyperbola*, cutting the Vaniſhing Line in two Points, and oppoſite to that formed by the Part *n*b*r* of the Original *Hyperbola* *n*bQ, they having both the ſame Aſymptotes.

In the next place, take any Line ab for the Directing Line, cutting the oppoſite *Hyperbolas* Pa*m*, *n*bQ, each in one Point a and b, and through a and b draw two Tangents to the Sections.

Then the Images of theſe Tangents, will be Tangents to the Images of the Indefinite Sides Pa, *m*a, and *n*b, Qb of the oppoſite Sections, at the infinitely diſtant Extremities of thoſe Images, the Originals of which are a and b; and conſequently the Images of every one of theſe Indefinite Sides will be Portions of H*yperbolas*, to which the Images of the Tangents at a and b are the Aſymptotes; but the Images of the infinitely diſtant Extremities of bQ and aP unite at the Vaniſhing Point of *ik*, therefore the Images of theſe two together form one compleat *Hyperbola*, cut by the Vaniſhing Line in one Point only; and for the ſame reaſon, the Images of the Indefinite Sides a*m* and b*n*, which unite at the Vaniſhing Point of *ef*, together form the oppoſite *Hyperbola*, which is alſo cut by the Vaniſhing Line in one Point.

Laſtly, let the Directing Line cut only one of the *Hyperbolas* in one Point; which it can only do, when it is parallel to one of the Aſymptotes[a]; and let bd parallel to the Aſymptote *ef*, be the Directing Line propoſed, cutting the *Hyperbola* *n*bQ in b, and through b draw a Tangent to the *Hyperbola* *n*bQ.

Then becauſe the Aſymptote *ef* is parallel to the Directing Line bd, its Image will be parallel to the Vaniſhing Line; and in regard the Indefinite Sides a*m* and b*n* of the oppoſite *Hyperbolas* Pa*m*, *n*bQ, ultimately coincide with the Aſymptote *ef*, the Images of thoſe Sides muſt ultimately coincide with the Image of *ef*; wherefore the Image of *ef* will be one of the Aſymptotes of the *Hyperbolas* to be produced, and the Image of the Tangent at b will be the other Aſymptote; conſequently the Image of the Indefinite Side b*n* of the *Hyperbola* *n*bQ, whoſe infinitely diſtant Extremity beyond *b* ultimately coincides with the Image of *ef*, and whoſe other Extremity, the Original of which is b, ultimately coincides with the Image of the Tangent at b, forms one compleat *Hyperbola*: and the Image of the Indefinite Side bQ of the ſame *Hyperbola* *n*bQ, forms part of the oppoſite *Hyperbola*, terminated at the Vaniſhing Point of *ik*, where it is met by the Image of the infinitely diſtant Extremity of the Side aP of the *Hyperbola* Pa*m*; and the Image of the infinitely diſtant Extremity of the other Side a*m* of this *Hyperbola*, ultimately coinciding with the Image of *ef*, the intire Image of Pa*m*, together with the Image of the Indefinite Side bQ, jointly form an *Hyperbola*, oppoſite to that which is formed by the Indefinite Side b*b* of the Original *Hyperbola* *n*bQ; and which laſt formed *Hyperbola* will be cut by the Vaniſhing Line in one Point only, that Line being parallel to the Aſymptote whoſe Original is *ef*, as already ſhewn. *Q. E. D.*

GENERAL COROLLARY.

Hence it follows;

1. That the Original of two oppoſite *Hyperbolas* in the Picture, neither of which touches or cuts the Vaniſhing Line, is either a Circle or an *Ellipſis* in the Original Plane, cutting the Directing Line.

2. That the Original of two oppoſite *Hyperbolas* in the Picture, one of which only touches the Vaniſhing Line, is a *Parabola* in the Original Plane, cutting the Directing Line in two Points.

Or if two oppoſite *Hyperbolas* in the Picture, have the Vaniſhing Line for one of their Aſymptotes, their Original is a *Parabola* in the Original Plane, of which the Directing Line is one of the Diameters.

3. That the Originals of two oppoſite *Hyperbolas* in the Picture, one of which cuts the Vaniſhing Line in two Points, are two oppoſite *Hyperbolas* in the Original Plane, one of which cuts the Directing Line in two Points.

Or if two oppoſite *Hyperbolas* in the Picture, be each of them cut by the Vaniſhing Line in one Point, their Originals are two oppoſite *Hyperbolas* in the Original Plane, each of them cutting the Directing Line in one Point.

Or laſtly, if of two oppoſite *Hyperbolas* in the Picture, only one of them cut the

[a] Hyperb. Art. 5.

Vaniſhing Line in one Point, their Originals are two oppoſite *Hyperbolas* in the Original Plane, one of which only cuts the Directing Line in one Point.

SCHOL.

It would not be difficult, from what has been ſhewn in the foregoing Propoſitions of this Book, to demonſtrate many of the Properties of the Conick Sections on the ſame Principles; and to deduce Methods whereby to determine any Number of Conick Sections in different Planes, which may produce the ſame Image, or the Reverſe. Of what uſe this might be, for the more exact Deſcription of the ſeveral Projections of the Sphere, for advancing the Science of Conick Sections, or, in Aſtronomy, for determining the true Figures of the Orbits of Planets or Comets, from their obſerved Appearances, is left to the Learned in thoſe Sciences, who may improve the Hints here given for their Purpoſes, if they ſhall judge the Subject worthy of their farther Conſideration; but this Enquiry being wide of the Deſign of theſe Papers, and it having been already purſued farther than it may be thought was neceſſary, we ſhall here drop it, and conclude this Book with ſome few Methods of deſcribing the Conick Sections, from ſuch Lines in them firſt given, as are ſufficient to determine their Figure.

Of the Methods of deſcribing the Conick Sections.

LEM. 24.

A Diameter of any Conick Section being given, together with one of its Ordinates; thence to determine the Parameter of that Diameter.

1. For the *Parabola.*

Fig. 82. N°. 1. Let AP be the given Diameter, A its Vertex, and PM the Ordinate.

Take from A on the Diameter AP, a Part AH equal to PM, and draw HF parallel to PM, and terminated in F by a Line AM, drawn through A and M; and HF will be the Parameter deſired.

For in the Similar Triangles APM, AHF, $AP : PM :: AH = PM : HF$, that is $PM^2 = AP \times HF$, and therefore HF is the Parameter[a]. *Q. E. I.*

[a] Parab. Art. 11.

2. For the *Ellipſis* and *Hyperbola.*

Fig. 82. N°. 2, 3. Let the given Diameter be A*a*, and PM its Ordinate.

Through either Extremity A of the Diameter A*a*, draw the Tangent AL parallel to PM, and from the other Extremity *a* of the given Diameter, through M, draw *a*M cutting AL in L; take from A on the Diameter A*a*, a Part AH equal to AL, and draw HF parallel to PM, and terminated in F by a Line AM drawn through A and M, and HF will be the Parameter deſired.

For in the Similar Triangles *a*PM, *a*AL, $aP : PM :: aA : AL = AH$
And in the Similar Triangles APM, AHF, $AP : PM :: AH : HF$
And multiplying theſe Proportionals in order $\big\}\; aP \times AP : PM^2 :: aA \times AH : HF \times AH :: aA : HF.$
And conſequently HF is the Parameter of the Diameter A*a*[b]. *Q. E. I.*
Note, *the Diameter* A*a of the Hyperbola is ſuppoſed to be a firſt Diameter.*

[b] Ellip, Art. 13. and Hyperb. Art. 22.

COR.

The Point H may be taken on either Side of A; for if A*b* be made equal to AH, it is evident, that *hf*, drawn parallel to HF, and terminated in *f* by the Line AM, will be equal to HF.

PROP. XVI. PROB. XIII.

To deſcribe an *Ellipſis.*

METHOD 1.

The Conjugate Axes being given.

Fig. 83. N°. 1. Let A*a* and B*b* be the given Axes; find the *Foci* F and *f*, as already directed[c]; then having fixed two Pins in F and *f*, take a Thread and double it, tying the Ends together, and let it be of ſuch Length, as that being put over the two Pins F and *f*, a Pencil P put within the double of the Thread may reach to A or *a*; then this Pencil being moved along within the double of the Thread, always keeping the Thread extended, and bearing on the two Pins F and *f*, will by its Motion about thoſe Pins deſcribe the *Ellipſis* deſired.

Dem. For by the Conſtruction, the Thread F P *f* is equal to $2Fa = 2Ff + 2fa = Ff + Aa$;

[c] Ellip. Art. 9.

= F*f* + A*a*; and F*f* being always the ſame, wherever the Point P is taken, it follows, that FP + P*f* is every where equal to A*a*, and conſequently that the Point P is always a Point of the *Ellipſis*[a]. *Q. E. I.*

METHOD 2.

Any two Conjugate Diameters A*a*, B*b* of an *Ellipſis*, being given; thence to deſcribe the *Ellipſis*. Fig. 83. N°. 2, 3.

Through the Extremities of the given Diameters draw a Parallelogram MNLR, having its Sides reſpectively parallel to thoſe Diameters, and divide the Sides MN, ML of this Parallelogram, each in the ſame Proportion, as formerly directed for dividing the Sides of a Square circumſcribing a Circle[b]; and having by theſe Diviſions ſubdivided the Parallelogram MNLR as in the Figure, a Curve drawn through the Interſections of theſe Subdiviſions, correſponding to thoſe in the Square, through which the Circle paſſes, will be the *Ellipſis* deſired,

Dem. For it is evident, that every Ordinate *pm* to the Diameter A*a*, is to $\sqrt[2]{Ap \times pa}$, as SB to SA; and every Ordinate *qm* to the Diameter B*b*, is to $\sqrt[2]{Bq \times qb}$, as SA, to SB[c]; in regard that every Diviſion of A*a*, is to the correſponding Diviſion of B*b*, as SA to SB. *Q. E. I.*

COR. 1.

If the given Diameters A*a*, B*b*, be the Axes, the Parallelogram MNLR will be Rectangular; and the Diagonals MR, LN will coincide with G*g*, H*h*, the two Conjugate Diameters of the *Ellipſis* which are equal: and if the given Diameters A*a*, B*b*, be equal, the Parallelogram MNLR will be equilateral, and G*g* and H*h*, which coincide with the Diagonals MR, NL, will be the Axes; in all other Caſes the Parallelogram will neither have its Sides nor Angles equal. Fig. 83. N°. 2. Fig. 83. N°. 3.

COR. 2.

In Fig. N°. 3. every Ordinate *pm* to the Diameter A*a*, is equal to $\sqrt[2]{Ap \times pa}$, and every Ordinate *qm* to the Diameter B*b*, is equal to $\sqrt[2]{Bq \times qb}$, the Conjugate Diameters A*a* and B*b* being equal[d]; if therefore theſe Ordinates were ſet perpendicular to the Diameters on which they inſiſt, the Curve paſſing through their Extremities would be a Circle, of which A*a* and B*b* would be two Diameters.

COR. 3.

If either of the Sides MN of the Parallelogram be divided in the Proportion above directed, the Diviſions of the other Side ML may be found by either of the Diagonals MR; ſeeing the Diagonals are cut by the Lines drawn through the Diviſions of MN parallel to ML, in the ſame Points, through which the Parallels from the correſponding Diviſions of ML paſs.

SCHOL.

If any two Lines A*a*, B*b* be given, biſecting each other in S, and from any Point K in A*a* produced, there be drawn KB, K*b*, meeting LM and NR, drawn parallel to B*b*, in L, N, M, and R; then if LM and LN be divided in the Proportion already mentioned, and through the Diviſions of LN Parallels be drawn to B*b*, and theſe be cut by Lines drawn through the Diviſions of LM to the Point K; a Curve drawn through the Interſections of theſe Subdiviſions, correſponding to thoſe, through which the *Ellipſis* paſſes in the other Figures, will not be an *Ellipſis*, but an Oval, more properly ſo called, from its Reſemblance to the Shape of an Egg: and every Ordinate *mp* to the Diameter A*a* of this Curve, will be to the Ordinate in an *Ellipſis* to the ſame Diameter and in the ſame Point, as K*p* to KS; the *Ellipſis* being ſuppoſed to have the ſame Conjugate Diameters A*a* and B*b* with the Oval. So that the Ordinates beyond S are larger than the correſponding Ordinates of an *Ellipſis*, and decreaſe in their Proportions, as they come nearer to *a*; the Proportion of which Decreaſe depends on the Diſtance of the Point K, the Decreaſe being quicker as that Point is taken nearer, and ſlower as it is taken farther diſtant; the Curve becoming a perfect *Ellipſis*, when K is at an infinite Diſtance, that is, when LN and MR are parallel. Fig. 83. N°. 4.

After this manner, a regular Curve may be formed, which may more nearly agree with the Shape of a human Face, than an *Ellipſis*; and this perhaps may be beſt effected by taking KA to A*a* in the ſame Proportion as the uſual Height of a Man is to the Length of his Head, or as 7 to 1; and making B*b* perpendicular to A*a*, and equal

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[a] Ellip. Art. 9.

[b] Prob. 24. B. II.

[c] Ellip. Art. 13.

[d] Ellip. Art. 13.

equal to $\frac{5}{8}$ or $\frac{3}{4}$ of A*a*. But theſe Proportions varying in different Subjects, no conſtant certain Rule can be given.

It is left to Architects alſo to conſider, how far this Method may be ſerviceable for the Deſcription of *Ovolos* in the Capitals and Entablatures of Columns, a due Proportion between K A, A *a*, and B *b* being choſen.

Thus if K *a* be taken to KA as 3 to 5, and B*b* to A *a* as 2 to 3, an Oval will be formed, but little different in general from the *Ovolo* deſcribed in the uſual manner, by Portions of Circles from different Centers; than which, the Curve thus formed, muſt needs be more regular.

If inſtead of dividing L N Geometrically, as here directed, it were to be divided Stereographically, that is, ſo as to repreſent a Line divided in that Proportion from K taken as a Vaniſhing Point; the Curve then produced would be a true *Ellipſis*[a].

COR. 4.

Fig. 83. N°. 5.

If an Original Circle A B*ab*, having Q T for the Directing Line of its Plane, be circumſcribed by a *Trapezium* LMNR, formed by Tangents drawn from the Directing Points Q and T, of the Originals A *a* and B *b* of any two Conjugate Diameters of its Image[b], and that *Trapezium* be ſubdivided by Lines from Q and T in ſuch manner, that its Image may be a Parallelogram, whoſe Sides are divided in the ſame Proportion as before mentioned[c]; the Original Circle will paſs through the Interſections of the Subdiviſions of the *Trapezium*, correſponding to thoſe of the Parallelogram, through which the *Ellipſis* formed by the Image of the Circle doth paſs.

For the Image of this *Trapezium*, being a Parallelogram ſubdivided in the manner propoſed, the Curve which paſſes through the proper Subdiviſions of that Parallelogram, will be an *Ellipſis*[d], touching its Sides in the Images of A, B, *a*, and *b*; and as no two different Conick Sections can touch any four ſtraight Lines in the ſame four Points[e], the *Ellipſis* thus formed, muſt be the Image of the Original Circle; and therefore the Images of the proper Interſections of the Subdiviſions of the *Trapezium*, being Points of the *Ellipſis*, the Originals of thoſe Points are Points of the forming Circle.

METHOD 3.

Fig. 83. N°. 6.

Any Diameter A *a* of an *Ellipſis*, together with a double Ordinate *m* P *n* to that Diameter, and the Tangents *m o*, *n o*, at its Extremities being given, meeting the Diameter A *a* in *o*[f]; thence to deſcribe the *Ellipſis*.

Through A and *a* draw L M, N R parallel to *m* P *n*, cutting *o m* and *o n* in L, N, M, and R; and having through *o* drawn *d d* parallel to L M, conſider LMNR as the Image of a Square in a Plane whoſe Vaniſhing Line is *d d*, and ſubdivide that Image in ſuch manner, that it may repreſent a Square ſubdivided in the Proportion before mentioned; and a Curve drawn through the proper Interſections of theſe Subdiviſions, will be the *Ellipſis* required.

Dem. For the Curve thus determined, will be an *Ellipſis* touching the Sides of the *Trapezium* LMNR in A, *a*, *m*, and *n*[g]; which *Trapezium* by Conſtruction touching the required *Ellipſis* in the ſame four Points, the *Ellipſis* thus formed muſt be the *Ellipſis* ſought[h]. *Q. E. I.*

COR. 1.

The double Ordinate *m n* repreſents that Diameter of the forming Circle which is parallel to the Directing Line of its Plane.

For the Diameter A *a* of the *Ellipſis* being Harmonically divided in A, P, *a*, and *o*[i], if *o* be taken as the Vaniſhing Point of A *a*, the Parts A P and P *a* will repreſent equal Lines[k].

COR. 2.

The Point *o*, and thence the Tangents *o m*, *o n* may be found, if not given, by finding in A *a* produced, a Point *o*, ſo that the whole Line A *o* may be Harmonically divided in A, P, *a*, and *o*[l].

METHOD 4.

Fig. 83. N°. 7.

Any Diameter A *a* of an *Ellipſis*, together with one of its Ordinates P M, being given; thence to deſcribe the *Ellipſis*.

Through either Extremity A of the given Diameter, draw the Tangent G L, and having found the Parameter of the given Diameter[m], make A G equal to it; and having from G drawn G F parallel to A *a*, take in it any Diſtances G F, F F, &c. and from A on the Tangent G L, on the oppoſite Side to G, ſet off the like Diſtances A L, L L,

[a] Meth. 1. Prob. 24. B. II.

[b] Part ſecond of Prob. 1.

[c] Prob. 21. B. II.

[d] Meth. 2.

[e] Con. Sec. Art. 19.

[f] Ellip. Art. 31.

[g] Meth. 1. Prob. 24. B. II.

[h] Con. Sec. Art. 19.

[i] Ellip. Art. 11.

[k] Cor. 6. Lem. 8.

[l] Lem. 1.

[m] Lem. 24.

Plate 29. Book 3 Sect 3.

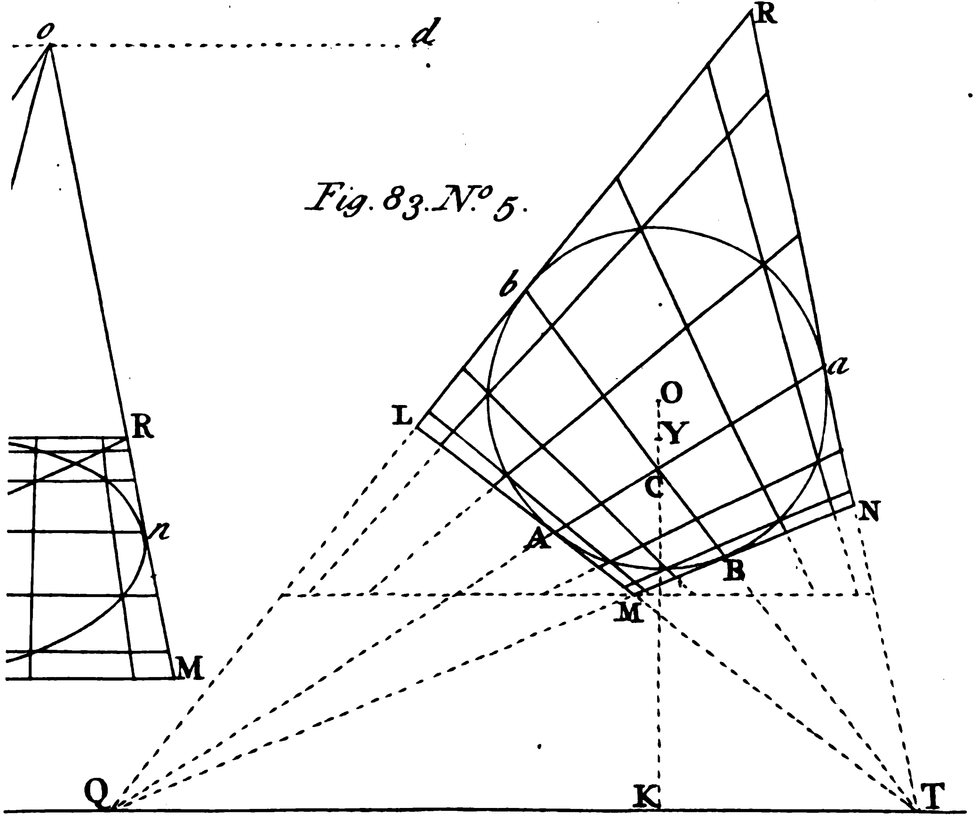
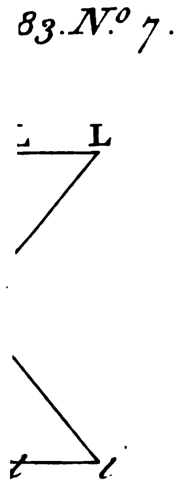


Fig. 83. No. 5.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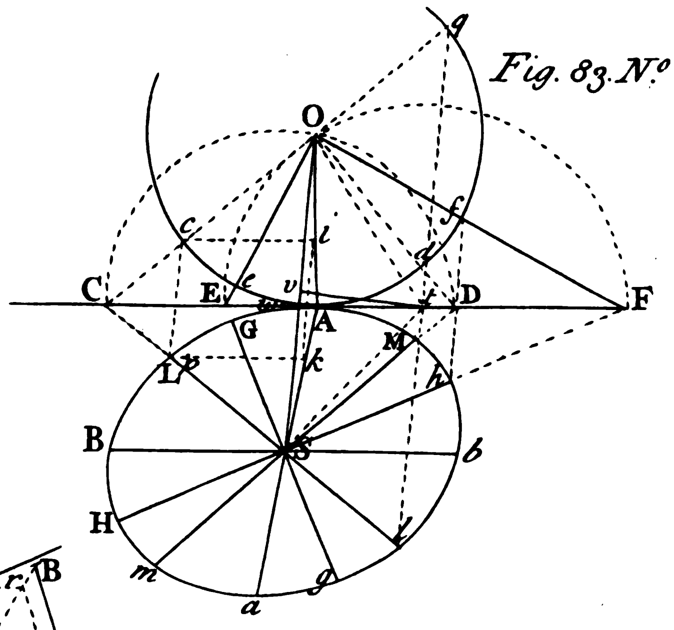


Fig. 83. No. 8.

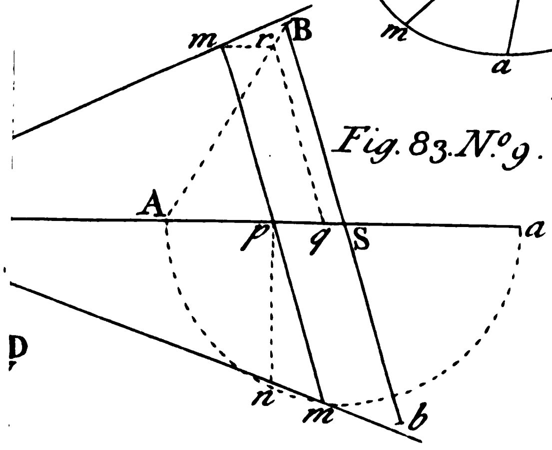


Fig. 83. No. 9.

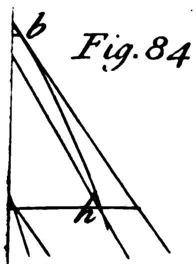


Fig. 84. No. 1.

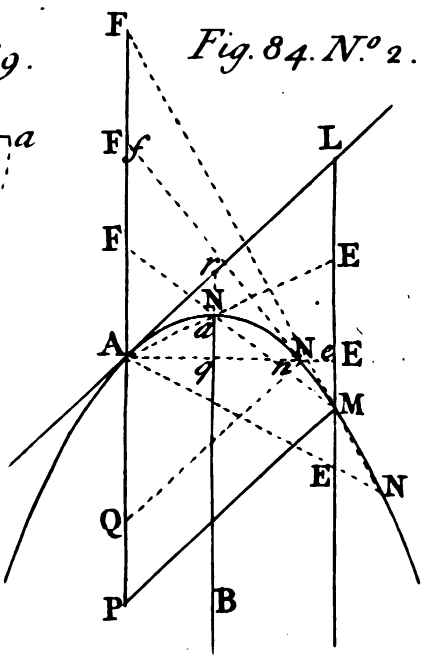


Fig. 84. No. 2.

J. Mynde sc.

LL, &c. all equal to each other, or ſo as that every GF may be equal to its correſponding AL; then from each of the Points F, through A, draw FA, FA, &c. and from the Points L, to the other Extremity *a* of the Diameter A*a*, draw L*a*, L*a*, &c. and the Interſections M, M, of each FA with its correſponding L*a*, will be Points of the *Ellipſis*; and if from every Point M thus found, a Line M*m* be drawn through the Diameter A*a*, parallel to the Tangent GL, cutting that Diameter in P, take P*m*, P*m*, each equal to its correſponding PM, and the Points *m* will alſo be Points of the *Ellipſis*, and a Curve Line drawn through the Points thus found, will be the *Ellipſis* deſired.

Dem. From any of the Points F, draw FH parallel to GL, cutting the Diameter A*a* produced in H, and from H to the correſponding Point L draw HL, and from the Point M, which correſponds to the others, draw the Ordinate PM.

Then in the Similar Triangles *a*PM, *a*AL, $aP : PM :: aA : AL = GF = AH$

And in the Similar Triangles APM, AHF, $AP : PM :: AH : HF = AG$

And multiplying theſe Proportionals $\Big\}$ $AP \times aP : PM^2 :: aA \times AH : AG \times AH :: aA : AG.$

And AG being by Conſtruction equal to the Parameter of the Diameter A*a*; PM is therefore an Ordinate to that Diameter[a], and conſequently M is a Point of the *Ellipſis*: the ſame may be ſhewn of every other Point M or *m*. *Q. E. I.*

COR.

In drawing the *Ellipſis* by this Method, as the Point M becomes farther diſtant from A, the Lines GF, AL become longer, ſo as at laſt to be inconvenient; but to remedy this, when the Points M are found as far as the Extremity of the Diameter SM Conjugate to A*a*, the Points M on the other Side of that Diameter may be found, by drawing through *a*, the Tangent *ag*, and proceeding in the ſame manner with it, as was before done with reſpect to the Tangent AG, as in the Figure.

METHOD 5.

Any two Conjugate Diameters A*a* and B*b* of an *Ellipſis*, being given; thence to deſcribe the *Ellipſis*. Fig. 83. No. 8.

Through the Extremity A, of either of the given Diameters A*a*, draw CF parallel to the other Diameter B*b*, and from A erect AO perpendicular to CF, and equal to the Semidiameter SB; from O as a Center with the *Radius* OA, deſcribe a Circle, and from O to S the Center of the *Ellipſis*, draw OS: This being done, through S draw any Line SC, cutting CF in C, and draw CO cutting the Circle in *c*, from whence draw *c*L parallel to OS, cutting SC in L, and take S*l* equal to SL; then L*l* will be a Diameter of the *Ellipſis*, and L and *l* its Extremities: and after the ſame manner, as many more Diameters may be found as are requiſite, through the Extremities of which a Curve being drawn, it will be the *Ellipſis* deſired.

Dem. Draw *ci*, L*k*, parallel to CF, cutting AO and AS in *i* and *k*, and draw *ik*; and on L*k* from *k*, ſet off *kp*, equal to a mean Proportional between *k*A and *ka*.

Then in the Similar Triangles OCS, *c*CL, $OC : cC :: SC : LC$

And in the Similar Triangles OCA, O*ci*, $OC : cC :: OA : iA$

And in the Similar Triangles SAC, S*k*L, $SC : LC :: SA : kA$

Conſequently $OA : iA :: SA : kA$

Wherefore the Triangles SAO, *k*A*i* are Similar, and *ik* is parallel to SO, and conſequently to *c*L, and therefore in the Parallelogram *ci*L*k*, the Sides *ci* and L*k* are equal.

Now becauſe $OA : iA :: SA : kA$, therefore $2OA - iA : iA :: 2SA - kA : kA$

But in the Circle *c*A*fq* $2OA - iA : ci :: ci : iA$

And by the Conſtruction $2SA - kA : kp :: kp : kA$

Therefore $ci : kp :: iA : kA$

But as already ſhewn $iA : kA :: OA : SA$

Therefore $ci : kp :: OA : SA.$

But $ci = Lk$, $kp = \sqrt[2]{Ak \times ka}$, and $OA = SB$

Therefore $Lk : \sqrt[2]{Ak \times ka} :: SB : SA$

Conſequently L*k* is an Ordinate to the Diameter A*a* of the *Ellipſis*[b], and L is therefore a Point of the *Ellipſis*, and conſequently one Extremity of the Diameter L*l*; and S*l* and SL being made equal, *l* is the other Extremity of that Diameter. *Q. E. I.*

[a] Ellip. Art. 12 and 13.

[b] Ellip. Art. 13.

COR. 1.

It is evident, that if CO be produced beyond O, till it cut the Circle in q, a Line ql drawn parallel to OS, will cut Ll in its Extremity l; for CL, OS, and ql being parallel, and cO and Oq being equal, SL muſt be equal to Sl.

COR. 2

If from any Point w in the Line CF as a Center, with the *Radius* wO, a Semicircle COD be deſcribed, cutting CF in C and D; two Lines Cl, Dm drawn from C and D through S, will be two Indefinite Conjugate Diameters of the *Ellipſis*; the Extremities of which may be determined by the above Method.

For in the Semicircle COD, $\mathrm{CA} \times \mathrm{AD} = \mathrm{AO}^2 = \mathrm{S}b^2$, and CF being a Tangent to the *Ellipſis* in A, the Diameters Ll and Mm which paſs through C and D are therefore Conjugate[a].

[a] Ellip. Art. 14.

COR. 3.

Any Indefinite Diameter Dm being given; thence to find its Conjugate, and the Poſition of their reſpective Ordinates.

Draw DO, and from O draw OC perpendicular to it, cutting CF in C; and Cl will be the Diameter Conjugate to Dm, the Ordinates to either of which Diameters are parallel to the other[b].

[b] Ellip. Art. 4.

For it is apparent the Point O is in a Semicircle, whoſe Diameter is CD. And thus if any Line in the *Ellipſis*, not a Diameter, be given, a Diameter may be found, to which the given Line is a double Ordinate; ſeeing that Diameter muſt be the Conjugate to the Diameter which is parallel to the given Line.

COR. 4.

The ſame things being ſuppoſed as before; thence to determine the Axes.

Biſect SO in v by the Perpendicular vt, cutting CF in t, and from t with the *Radius* tO deſcribe a Semicircle EOF, cutting CF in E and F; then Eg and FH drawn through S, will be the Indefinite Axes; the Extremities of which may be determined as before.

For SO being biſected in v by the Perpendicular vt, St and Ot are equal, wherefore a Circle deſcribed from t as a Center, with the *Radius* Ot, will alſo paſs through S, and therefore the Angle ESF is Right; but Gg and Hb are two Conjugate Diameters[c], and being perpendicular to each other, they are therefore the Axes[d].

[c] Cor. 2.
[d] Ellip. Art. 3.

COR. 5.

Fig. 83. N°. 9. Any two Conjugate Diameters Aa, Bb of an *Ellipſis*, and a Point p in either of them, as Aa, being given; thence to draw an Ordinate to that Diameter, and to determine its Length, without drawing the *Ellipſis*.

Find pn a mean Proportional between Ap and pa, and from A ſet off Aq on the Diameter Aa equal to pn; and having drawn AB, draw qr parallel to Bb, cutting AB in r; then from p draw pm parallel and equal to qr, and pm will be the Ordinate ſought, and m its Extremity.

For in the Similar Triangles ASB, Aqr, $\mathrm{A}q : qr :: \mathrm{AS} : \mathrm{SB}$

But $\mathrm{A}q = pn = \sqrt[2]{\mathrm{A}p \times pa}$, and $qr = pm$; therefore $\sqrt[2]{\mathrm{A}p \times pa} : pm :: \mathrm{AS} : \mathrm{SB}$.
And therefore pm is an Ordinate to the Diameter Aa[e].

[e] Ellip. Art. 13.

COR. 6.

If the Diameter Aa and its Ordinate pm were given, the Extremities of the Conjugate Diameter Bb may be determined after the ſame manner.

For if Aq be taken equal to $\sqrt[2]{\mathrm{A}p \times pa}$, and qr be drawn parallel and equal to pm, a Line Ar will cut Bb in its Extremity B, and SB and Sb are equal.

COR. 7.

Fig. 83. N°. 9. Any two Conjugate Diameters of an *Ellipſis* being given; from any Point L without the *Ellipſis*, to draw two Tangents to it, without drawing any Part of the *Ellipſis*.

Through the given Point L draw an Indefinite Diameter of the *Ellipſis*, and find its Conjugate, and likewiſe the Extremities of both[f], and let Aa and Bb be the Conjugate

[f] Method 5. and Cor. 3.

jugate Diameters thus found; then take a Point p in La, ſo that La may be Harmonically divided in L, A, p, and a[a]; or, which is the ſame[b], make Sp a third Proportional to SL and SA[c], and through p draw a double Ordinate mm to the Diameter Aa, and find its Extremities m, m[d]; then two Lines Lm, Lm, drawn from L through m and m, will be the Tangents required[e].

[a] Lem. 1.
[b] Lem. 10.
[c] Lem. 23.
[d] Cor. 5.
[e] Ellip. Art. 11.

COR. 8.

Any two Conjugate Diameters of an *Ellipſis* being given; thence to find the Points wherein any given Line cuts the *Ellipſis*, without drawing any Part of the Section.

Let the given Line be mm; draw an Indefinite Diameter Bb parallel to it, and find its Conjugate, and the Extremities of both as before[f]: then conſidering mm as a double Ordinate to the Diameter Aa in the Point p, find the Extremities m, m of this double Ordinate[g], and thoſe will be the Points deſired.

[f] Method 5. and Cor. 3.
[g] Cor. 5.

PROP. XVII. PROB. XIV.

To deſcribe a *Parabola*.

METHOD 1.

Any Diameter with a double Ordinate to it, being given; thence to deſcribe the *Parabola*. Fig. 84. N°. 1.

Let Ap be the given Diameter, and Bb its double Ordinate cutting it in s.

Produce the Diameter Ap beyond its Vertex A to O, till AO be equal to As, and having through A drawn the Tangent LM, draw OB, Ob, cutting it in L and M; divide LM or Bb in the ſame Proportion as directed for the Side of a Square circumſcribing a Circle[h], and from O draw Lines through thoſe Diviſions; and having through O drawn DD parallel to Bb, take OD, OD in that Line, each equal to the Ordinate Bs, and draw Ds, Ds, and through the Interſections of theſe with the Lines drawn from O, draw Parallels to Bb; then a Curve drawn through the Interſections of theſe Subdiviſions, correſponding to thoſe of the Subdiviſions of a Square through which a Circle paſſes, will be the *Parabola* deſired.

[h] Prob. 24. B. II.

This Method is deduced from Prob. VI. for As may be conſidered as the Image of the Semidiameter of the forming Circle which is perpendicular to the Directing Line, and O as its Vaniſhing Point[i]; and Bb as the Image of the Diameter of the forming Circle which is parallel to the Directing Line, and DO equal to Bs, as the Diſtance of the Vaniſhing Point O[k], taken as the Center of the Vaniſhing Line DD; the Originals of OB and Ob are therefore Tangents to the forming Circle, in the Extremities of its Diameter repreſented by Bb, and conſequently BLMb repreſents one half of the Square which circumſcribes the forming Circle: and the Originals of the Angles DsO, DsO, being each of 45 Degrees, the Lines Ds and Ds, which paſs through L and M and the Center s, repreſent the Diagonals of this Square; which Diagonals cut the Lines drawn from O through the Diviſions of LM or Bb, in ſuch manner, that Parallels to LM drawn through theſe Interſections, will cut the Indefinite Sides LB, Mb of this Perſpective Square, in the ſame Perſpective Proportion; and thus ſix Points c, d, e, f, g, b of the *Parabola* are found, beſides the three Points B, A, b, at firſt given; but the other Points, *viz.* the two Points on the other Side of the Center, correſponding to c and d, are out of reach, and that which ought to be formed by the other Extremity of the Diameter Ap, is infinitely diſtant. *Q. E. I.*

[i] Cor. 3. Prob. 5.
[k] Cor. 3. Prob. 6.

COR.

The Line pm, and conſequently the Points g and b, may be alſo found, by taking sp equal to sO, and drawing through p, the Parallel lm; for sp being the indeterminate Image of the Semidiameter of the forming Circle, whoſe Extremity is in the Directing Line, sp taken equal to sO repreſents a Moiety of that Semidiameter[l], and conſequently p repreſents the Point in that Diameter, through which the Parallel lm paſſes[m].

[l] Theor. 26. B. I.
[m] Schol. Meth. 1. Prob. 24. B. II.

METHOD 2.

Any Diameter AP, with one of its Ordinates PM being given; thence to draw the *Parabola*. Fig. 84. N°. 2.

Through the Vertex A of the given Diameter, draw the Tangent AL, till it meet in L, a Line ML drawn through M, parallel to AP; produce AP beyond A towards F at pleaſure, and in it take any Diſtances AF, FF, &c. and from L on the Line

LM, ſet off the like Diſtances LE, EE, *&c.* all equal to each other, or ſo as every AF may be equal to its correſponding LE; then from A to the ſeveral Points E, draw AE, AE, *&c.* and from M to the Points F draw MF, MF, *&c.* and the Interſections N of each AE with its correſponding MF, will be Points of the *Parabola.* And if from every Point N, Lines be drawn parallel to AL, and Points be taken in thoſe Lines, on the other Side of the Diameter AP, at an equal Diſtance from it with the Points N, thoſe will alſo be Points of the *Parabola,* through all which a Curve being drawn, it will be the *Parabola* deſired.

Dem. From any Point N thus found, draw NQ parallel to AL, cutting AP in Q.

Then becauſe of the Similar Triangles QAN, LEA, $QN : QA :: AL = PM : LE = AF$

Therefore $AF = \frac{PM \times QA}{QN}$

And in the Similar Triangles PMF, QNF, $PM : QN :: PA + AF : QA + AF$

And ſubſtituting the other Value of AF, $PM : QN :: PA + \frac{PM \times QA}{QN} : QA + \frac{PM \times QA}{QN}$

And multiplying the Extremes and the Meanes $PM \times QA + \frac{PM^2 \times QA}{QN} = QN \times PA + PM \times QA$

And ſubſtracting $PM \times QA$ from each ſide, reſts $\frac{PM^2 \times QA}{QN} = QN \times PA$

And multiplying by QN $PM^2 \times QA = QN^2 \times PA$

Which gives this Analogy $PM^2 : QN^2 :: PA : QA$

Let the Parameter of the Diameter AP be p, by which multiply the two laſt Terms of this Proportion, then $PM^2 : QN^2 :: PA \times p : QA \times p$

[a] Parab. Art. 11. But becauſe PM is an Ordinate to the Diameter AP $PM^2 = PA \times p$[a].

And therefore $QN^2 = QA \times p$

And conſequently QN is alſo an Ordinate to the Diameter AP, and N is therefore a Point of the *Parabola:* the ſame may be ſhewn of every other Point N. *Q.E.I.*

COR.

If only the Tangent AL, and the Parameter p of the Diameter AP were given; a Point M of the *Parabola* may be found in this manner:

From any Point L in AL, draw LE parallel to AP, and make LM a third Proportional to the Parameter p and the Line AL; and M will be a Point of the *Parabola.*

For drawing PM parallel to AL, PM will be equal to AL, and AP to LM.

If then $LM : AL :: AL : p$

It follows that $AP : PM :: PM : p$

And therefore PM is an Ordinate to the Diameter AP, and conſequently M is a Point of the *Parabola.*

METHOD 3.

Fig. 84. N°. 3. Having any Diameter AP with one of its Ordinates PM given; thence to find as many Points of the *Parabola* as may be deſired.

[b] Lem. 24. Having found the Parameter pm of the Diameter AP[b], through A draw the Tangent GL, and make AG equal to that Parameter, and from G draw GF parallel to AP, and in it take any Diſtances GF, FF, *&c.* and from A on the Tangent GL, on the oppoſite Side of G, ſet off the like Diſtances AL, LL, *&c.* all equal to each other, or ſo as that every GF may be equal to its correſponding AL; then from each of the Points F through A draw FN, FN, *&c.* and from the ſeveral Points L draw LN, LN, *&c.* parallel to AP, and the Interſections N of each FN with its correſponding LN, will be Points of the *Parabola.*

Dem. From any Point N thus found, draw NQ parallel to AL, cutting AP in Q.

Then QN will be equal to AL or GF, and AQ to LN.

Now in the Similar Triangles GAF, LAN, $AG : GF :: AL : LN$

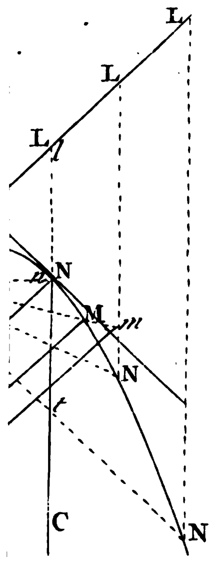
And AG being by Conſtruction equal to pm, therefore $pm : QN :: QN : AQ$

[c] Parab. Art. 11. Wherefore QN is an Ordinate to the Diameter AP, and conſequently N is a Point of the *Parabola*[c]. *Q.E.I.*

COR. 1.

The Diameter AP, the Tangent GL, and the Parameter GA being given; thence to determine the Axe and its Vertex.

Having



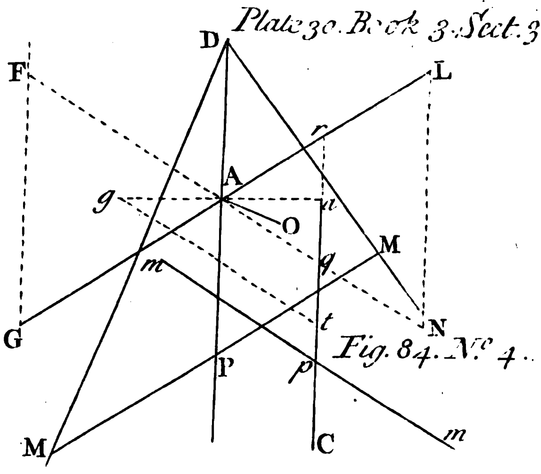


Fig. 84. N.º 4.

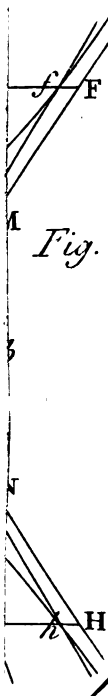


Fig. 85. N.º 1.

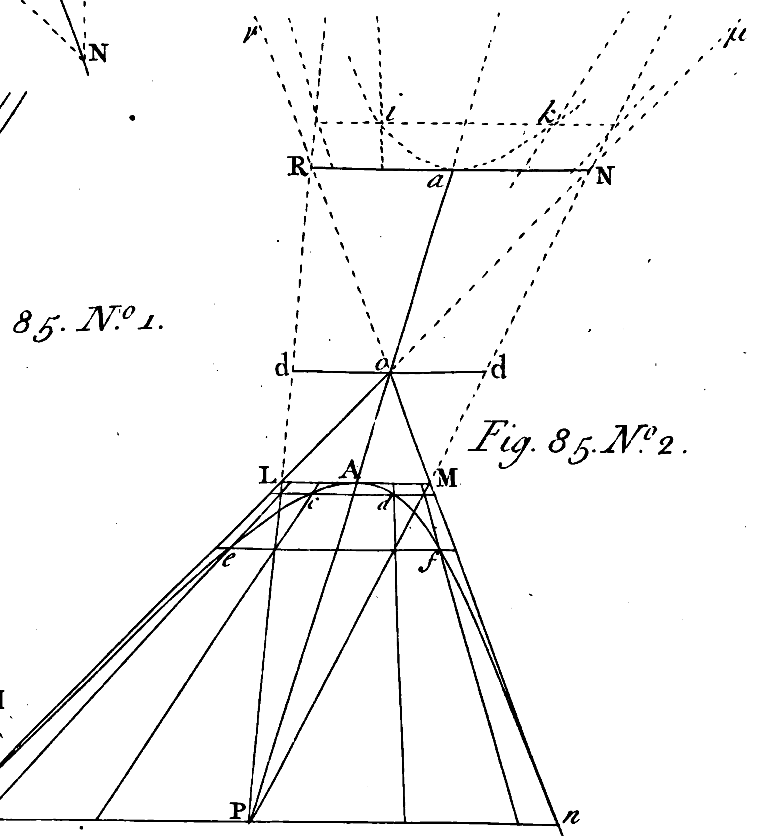


Fig. 85. N.º 2.

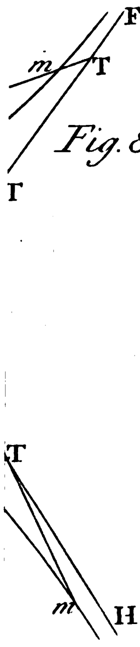


Fig. 85. N.º 3.

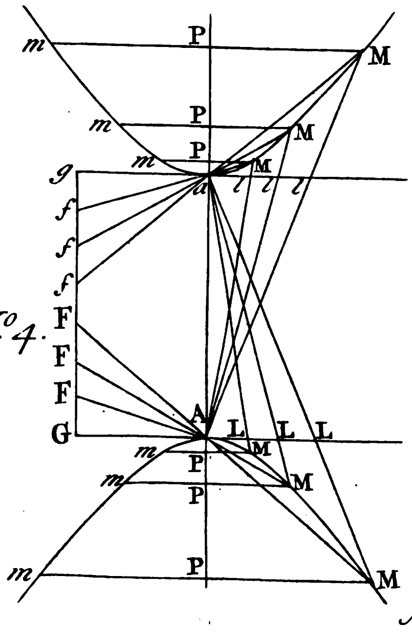


Fig. 85. N.º 4.

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Having drawn GF parallel to AP as before, through A draw fn perpendicular to AP, cutting GF in f; take Al equal to Gf and draw ln parallel to AP, cutting fn in n, biſect An in q, and through q draw rB parallel to AP, cutting AL in r, and biſect rq in a; then aB will be the Axe, and a its Vertex.

For n being a Point of the *Parabola*[a], and An being biſected in q, rB drawn through q parallel to AP is an Indefinite Diameter of the *Parabola*, to which An is a double Ordinate; and theſe two being perpendicular, rB is therefore the Indefinite Axe[b]: and AL being a Tangent to the *Parabola* in the Extremity A of the Ordinate Aq, and cutting the Axe in r, the Point a which biſects rq, is therefore the Vertex of the Axe[c].

[a] Meth. 3.

[b] Parab. Art. 3. and 4.

[c] Parab. Art. 6.

The Axe may be likewiſe found by Method 2. For if Ae be drawn perpendicular to AP cutting LE in e, take Af equal to Le, and the Interſection n, of fM with Ae, will be a Point of the *Parabola*; and An being biſected in q, rB drawn through q parallel to AP, will be the Axe, and the Point a which biſects rq, will be its Vertex. Fig. 84. N°. 2.

COR. 2.

Either of theſe Methods ſerves alſo to find a Diameter of the *Parabola* which makes any given Angle with its Ordinates.

For if from A a Line AN be drawn, making the given Angle with AP, and its Interſection N with the *Parabola* be found[d], a Diameter drawn ſo as to biſect AN, will be the Diameter ſought, whoſe Vertex may be determined as before. And as the Line AN may be made to incline either way to AP in the given Angle, ſo two different Diameters may be found that will anſwer the purpoſe, one on each ſide of the Axe, and equally diſtant from it.

[d] Meth. 2 and 3.

COR. 3.

The Diameter AP, the Tangent GL, and the Parameter GA, being given; thence to determine the Vertex, Ordinates, and Parameter of any other Indefinite Diameter lC propoſed. Fig. 84. N°. 3.

Having drawn GF parallel to AP, take in it Gf equal to Al, and draw fA, which will cut the propoſed Diameter lC in n its Vertex; take AD in the Diameter AP produced beyond its Vertex A, equal to ln, and draw Dn, which will be a Tangent to the *Parabola* in n the Vertex of the Diameter nC, and conſequently parallel to its Ordinates[e]; then from any known Point A of the *Parabola*, draw At parallel to Dn, cutting the Diameter nC in t, and At will be an Ordinate to that Diameter, whence its Parameter may be found[f].

[e] Parab. Art. 5.

[f] Lem. 24.

For it is evident by the Conſtruction, that n is a Point of the *Parabola*, and is therefore the Vertex of the Indefinite Diameter lC; and if the Ordinate nQ to the Diameter AP be drawn, ln and AQ will be equal, and therefore AD, made equal to ln, is alſo equal to AQ; wherefore Dn is a Tangent to the *Parabola* in n[g].

[g] Parab. Art. 6.

COR. 4.

Any Diameter AP, and the Tangent GL at its Vertex A, together with its Parameter GA, being given; thence to determine the Length of an Ordinate to that Diameter, meeting it in any given Point P, without drawing any part of the *Parabola*. Fig. 84. N°. 4.

From P draw PM parallel to the Tangent GL, and make PM a mean Proportional between the Parameter AG and the *Abſciſſa* AP, and PM will be the Ordinate deſired, and M its Extremity[h].

[h] Parab. Art. 11.

COR. 5.

Any Diameter of a *Parabola*, with its Parameter and the Tangent at its Vertex, being given; from any given Point D without the *Parabola*, to draw two Tangents to it, without drawing any Part of the Curve.

Through the given Point D, draw an Indefinite Diameter DP, and find its Vertex A, the Tangent GL at its Vertex, and its Parameter GA[i]; take AP on that Diameter equal to DA, and through P draw a double Ordinate MM to that Diameter, and find its Extremities M and M[k]; then DM, DM will be the Tangents deſired[l].

[i] Cor. 3.

[k] Cor. 4.

[l] Parab. Art. 6.

COR. 6.

Any Diameter AP, the Tangent GL, and its Parameter AG, being given; thence to find the *Focus* of the *Parabola*.

From A draw AO, making with the Tangent, an Angle OAL equal to the Angle GAP, made by the Tangent with the Diameter AP, and make AO equal to one fourth part of the Parameter AG, and O will be the *Focus*[a].

COR. 7.

The ſame things being given as before; thence to determine the Points m and m, wherein any Line mm, given by Poſition, cuts the *Parabola*, without drawing any part of the Curve.

Through A draw AN parallel to the propoſed Line mm, and find the Diameter aC, to which AN is a double Ordinate[b], and having thence found tg the Parameter of the Diameter aC[c], conſider the given Line mm, as a double Ordinate to that Diameter, cutting it in p, and find its Extremities m and m[d], and thoſe will be the Points required.

PROP. XVIII. PROB. XV.

To deſcribe the oppoſite *Hyperbolas*.

METHOD 1.

Any two Conjugate Diameters being given, and knowing which of them is the firſt; thence to deſcribe the oppoſite *Hyperbolas*.

Fig. 85. N°. 1. Let Aa, Bb be the given Conjugate Diameters, of which let Aa be the firſt.

Through A and a draw LM, RN parallel to Bb, which will be Tangents to the oppoſite Sections in A and a[e], and draw the Aſymptotes LN, MR[f], cutting the Tangents in L, M, R, and N: divide LM or RN in the ſame Proportion as directed for the Side of a Square circumſcribing a Circle[g], and through O the Center of the Sections, draw Lines through theſe Diviſions; and having drawn LR, MN, through the Interſections of theſe with the Lines drawn from O, draw Parallels to Bb; then through A and the Points e, c, d, and f, where the Parallels to Bb cut the Lines drawn from O beyond A, draw a Curve, and through a and the correſponding Interſections g, i, k, h, below a, draw another Curve, and theſe two will be the oppoſite *Hyperbolas* deſired.

This Method is deduced from Caſe 2. Prob. IX. and X. For Aa may be conſidered as the Complement of the Image of the perpendicular Diameter of the forming Circle, whoſe Center is ſuppoſed to be in the Directing Line; and conſequently Bb may be taken as part of the Vaniſhing Line, O as its Center, and BO or Ob as the Diſtance of the Eye; and the Aſymptotes LN, MR will then repreſent Tangents to the forming Circle, in the Extremities of its Diameter which coincides with the Directing Line: wherefore the Indefinite Figure GRNH will repreſent one Moiety of a Square circumſcribing the forming Circle, lying on one Side of the Directing Line, and ELMF will repreſent the other Moiety of that Square which lies on the other Side of that Line, and LR and MN, which paſs through the Points of Diſtance B and b, and the Angles L, R, and M, N of this Square, will repreſent its Diagonals; wherefore the Curves drawn through the Points above mentioned (which are the Points correſponding to thoſe in the Original Square, through which the forming Circle paſſes) muſt repreſent the Image of that Circle, and conſequently the oppoſite *Hyperbolas* required. And thus four Points in each *Hyperbola* are found, beſides the Points A and a at firſt given; but the two remaining Points, which ought to be formed by the Images of the Extremities of that Diameter of the forming Circle which coincides with the Directing Line, are at an infinite Diſtance. *Q. E. I.*

COR.

The Lines lm and rn, and conſequently the Points e, f and g, h, of the oppoſite Sections, may be alſo found, by taking on the Diameter Aa produced, AP, ap, each equal to OA, and through P and p drawing the Parallels lm and rn; for each half of the perpendicular Diameter of the forming Circle, being biſected by the Originals of thoſe Parallels[h], the Images of AP and ap muſt be equal to their reſpective Complements OA and Oa[i].

METHOD 2.

Fig. 85. N°. 2. Any firſt Diameter Aa of the *Hyperbolas*, together with a double Ordinate mPn to that Diameter, and the Tangents mo, no at its Extremities, being given, meeting the Diameter in o; thence to deſcribe the *Hyperbolas*.

Through A and a draw LM, RN parallel to mPn, cutting om and on in L, N, M, and R; and having through o drawn dd parallel to LM, conſider LMNR as

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[a] Parab. Art. 10.

[b] Cor. 2.

[c] Lem. 24.

[d] Cor. 4.

[e] Hyperb. Art. 1. and 2.

[f] Hyperb. Art. 10.

[g] Schol. Meth. 1 Prob. 24. B. II.

[h] Schol. Meth. 1. Prob. 24. B. II.

[i] Theor. 26. B. I.

the Complement of the Image of a Square in a Plane whoſe Vaniſhing Line is d d, and ſubdivide the two indeterminate Figures L M *m n*, and N R $\nu\mu$ in ſuch manner, that they may repreſent a Square ſubdivided in the Proportion before mentioned, lying partly on one Side and partly on the other of the Directing Line of its Plane; then two Curves drawn, one in each of theſe indeterminate Figures, through the proper Interſections of their Subdiviſions, will be the oppoſite Sections required.

This Method depends on the ſame reaſoning as the preceeding, ſave that the Center of the forming Circle is not here ſuppoſed to be in the Directing Line, but to be repreſented by P, *m* P *n* being conſidered as the Image of the Diameter of the forming Circle which is parallel to the Directing Line, and to which *m o* and *n o* are Tangents; ſo that the Figure L M *m n* repreſents one Moiety of the Square which circumſcribes the forming Circle, and being produced indefinitely below *m n*, it would repreſent ſo much more of that Square as lies before the Directing Line; and the oppoſite indeterminate Figure R N $\nu\mu$ is the Tranſprojection of the Remainder of that Square, which lies behind the Directing Line; and as P repreſents the Center of the forming Circle, P L and P M repreſent the Diagonals of the circumſcribing Square, by the help of which the Diviſions parallel to L M are obtained. *Q. E. I.*

By this Method four Points *e*, *c*, *d*, and *f* of the *Hyperbola m* A *n* are found, beſides the three given Points A, *m*, and *n*; but in the oppoſite *Hyperbola*, only two Points *i* and *k*, beſides the given Point *a*, can be conveniently had, by reaſon of the Obliquity of the Lines by which the other Points ſhould be determined; ſo that the advantage gained by this Method for the Deſcription of the *Hyperbola m* A *n*, is loſt in the other.

COR.

Here, whether A *a* be conſidered as a firſt Diameter of the *Hyperbolas*, or as the Image of the Complement of the Diameter of the forming Circle which is perpendicular to the Directing Line, the intire Line P *a* is Harmonically divided in P, A, *o*, and *a*; whence the Point *o*, and thence the Tangents *o m*, *o n* may be found, if not given, the Points P, A, and *a* being known.

For in the firſt Caſe, *o m* and *o n* being by Suppoſition Tangents to the *Hyperbola* in the Extremities *m* and *n* of the double Ordinate to the Diameter A *a*, which they meet in *o*, that Diameter is thereby Harmonically divided in P, A, *a*, and *o*[a]; and in the other Caſe, the Complement of the Original of A *a* being a Diameter of the forming Circle, which croſſes the Directing Line, and which is biſected by the Original of P, the Image P *a* is therefore Harmonically divided in P, A, *a*, and its Vaniſhing Point *o*[b].

METHOD 3.

The Aſymptotes, and any one Point of either of the *Hyperbolas*, being given; thence to find as many more Points of the Sections as may be deſired.

Let G F and E H be the Aſymptotes, and M a given Point of one of the Sections, and conſequently G S H or E S F the inward Angle of the Aſymptotes[c]. Fig. 85. N°. 3.

Through the given Point M, draw at pleaſure any Lines R T, R T, terminated by the Aſymptotes in R, T, R, T, &c. and take in each of theſe Lines from T, a Diſtance T *m*, within the Angle of the Aſymptotes, equal to R M in the ſame Line, and all the Points *m* thus found, will be Points of the *Hyperbolas*[d]; obſerving that when the given Point M is between the Points R and T, the Point *m* belongs to the ſame *Hyperbola* with M; but if R and T be both on the ſame Side of M, the Point *m* found, is in the oppoſite *Hyperbola*. *Q. E. I.*

METHOD 4.

Any firſt Diameter A *a* of the *Hyperbolas*, together with one of its Ordinates P M being given; thence to deſcribe the *Hyperbolas*. Fig. 85. N°. 4.

Through either Extremity A of the given Diameter, draw the Tangent G L, and having found the Parameter of the Diameter A *a*[e], make A G equal to it, and having drawn G F parallel to A *a*, take in it any Diſtances G F, F F, &c. and from A on the Tangent G L, on the oppoſite Side to G, ſet off the like Diſtances A L, L L, &c. all equal to each other, or ſo as that every G F may be equal to its correſponding A L; then from each F through A, draw F A, F A, &c. till they be cut in M, M, by Lines drawn from the other Extremity *a* of the Diameter A *a*, through the correſponding Points L; and from every M thus found, draw M *m*, M *m* parallel to the Tangent G L, cutting A *a* in P, and make P *m*, P *m* each equal to its correſponding P M; and the Points M, *m*, will all be Points in one of the *Hyperbolas*, by which it may be drawn: and

[a] Hyperb. Art. 26.

[b] Cor. 1. Lem. 8.

[c] Hyperb. Art. 8.

[d] Hyperb. Art. 19.

[e] Lem. 24.

and the oppoſite *Hyperbola* is found by uſing the other Extremity *a* of the Diameter A *a*, after the ſame manner as was done with the Extremity A, as appears by the Figure.

This is the ſame Method applied to the *Hyperbolas*, as was ſhewn at Method 4. for drawing the *Ellipſis*, and Method 3. for the *Parabola*, and is demonſtrated exactly in the ſame manner as that for the *Ellipſis*, which therefore needs not be repeated. *Q.E.I.*

METHOD 5.

The Aſymptotes, and any one Point of either of the *Hyperbolas*, being given; thence to find as many Points of the oppoſite Sections as may be deſired.

Fig. 85. N°. 5. Let E H, G G be the Aſymptotes, and M the given Point in one of the *Hyperbolas*.

Through M draw M L, M F parallel to the Aſymptotes E H and G G, and from M on the Line M L, take any Diſtances M L, L L, *&c.* and from the Center S, to each of the Points L, draw S L, S L, *&c.* cutting M F in F, F, *&c.* from each of which Points F, draw F *m*, F *m* parallel to E H, and from the correſponding Points L, draw L *m*, L *m* parallel to G G, and the Interſections *m*, *m*, of each F *m* with its correſponding L *m*, will be Points of one of the *Hyperbolas*; and if there be any Diſtances M *l*, *ll*, *&c.* ſet off beyond M in the Line M F, and from the Points *l* there be drawn Lines to S, cutting M L in *f*, *f*, *&c.* the Points *m* where every *f m* drawn parallel to G G, cuts its correſponding *l m* drawn parallel to E H, will alſo be Points of the ſame *Hyperbola*: and if from M through S the Center of the *Hyperbolas*, a Line M N be drawn, make S N equal to S M, and N will be a Point of the oppoſite *Hyperbola*; and by uſing the Point N in the ſame manner as the Point M was uſed, this laſt *Hyperbola* may alſo be found.

This Method is deduced from Theor. XXII. Book I. and Cor. and therefore needs no farther Demonſtration. *Q.E.I.*

METHOD 6.

Fig. 85. N°. 6. Any two Conjugate Diameters being given, and knowing which of them is the firſt; thence to find as many Points of the oppoſite *Hyperbolas* as may be neceſſary.

Let A *a* and B *b* be the given Diameters, of which let A *a* be the firſt.

From the Center S, take on the Semidiameter S A, produced at pleaſure beyond A, any Parts S E, E E, *&c.* and having drawn A B, from each of the Points E draw Parallels to it, cutting the Semidiameter B S in P, P, *&c.* and ſet off from S towards *b*, ſeveral Diſtances S *p*, *p p*, *&c.* each equal to S P, P P, *&c.* and having drawn S D perpendicular and equal to S A, through each of the Points P and *p*, draw Parallels to A *a*, and on them take on each Side of P and *p*, the Parts P M, P *m*, *p* M, *p m*, each equal to a Line E D, drawn from D to the Point E which correſponds to P, and all the Points M, will be Points of one of the *Hyperbolas*, and the Points *m* will be Points of its oppoſite.

Dem. In the Similar Triangles S B A, S P E, $SB : SP :: SA = SD : SE$

And ſquaring theſe Proportionals $SB^2 : SP^2 :: SD^2 : SE^2$

And by Addition $SB^2 + SP^2 : SD^2 + SE^2 :: SB^2 : SD^2 = SA^2$

And in the Rectangular Triangle D S E $SD^2 + SE^2 = DE^2 = PM^2$

Therefore $SB^2 + SP^2 : PM^2 :: SB^2 : SA^2$

And conſequently P M is an Ordinate to the ſecond Diameter B *b*[a], and M is therefore a Point of one of the *Hyperbolas*. The ſame may be ſhewn of every other Point M or *m* thus found. *Q.E.I.*

COR. 1.

If the Parts S E, E E, *&c.* be taken equal to each other, and from the Point E which is neareſt to S, a Line E P be drawn parallel to A B, cutting B *b* in P, then every P P will be equal to S P, by which the Points P correſponding to the Points E may be found, without drawing Parallels to A B through every Point E.

COR. 2.

Fig. 85. N°. 7. When the Conjugate Diameters A *a* and B *b* are equal, that is, when the *Hyperbolas* required are equilateral, the Conſtruction becomes more eaſy; for then having drawn S D perpendicular and equal to S B, through any Points P or *p*, on either Side of S in the Line B *b*, draw Parallels to A *a*, and on theſe Parallels, ſet off on each Side of every P or *p*, the Diſtances P M, P *m*, or *p* M, *p m*, each equal to its correſponding P D or *p* D, and the Points M, *m*, will be Points of the oppoſite *Hyperbolas*.

COR. 3.

If the given Diameters A *a*, B *b* were the Axes, the Line S D would coincide with B *b*

[a] Hyperb. Art. 23.

. No 5.

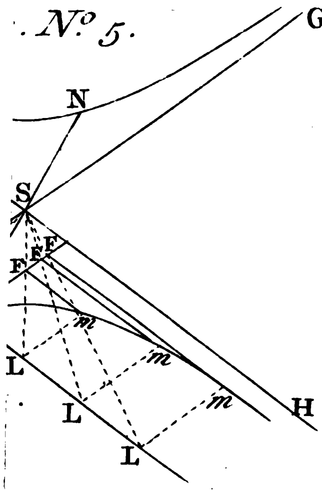
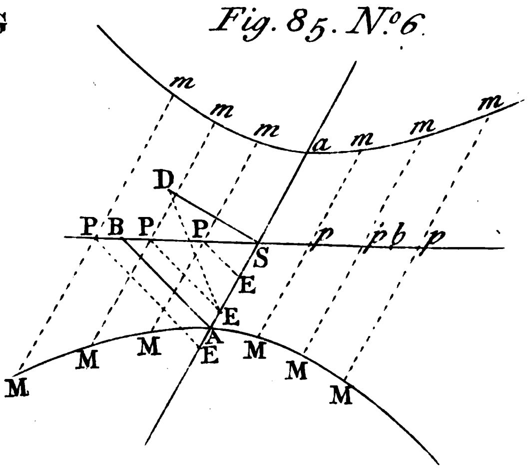


Fig. 85. No 6.



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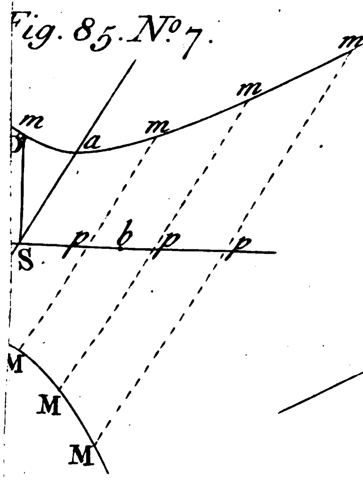


Fig. 85. No 8.

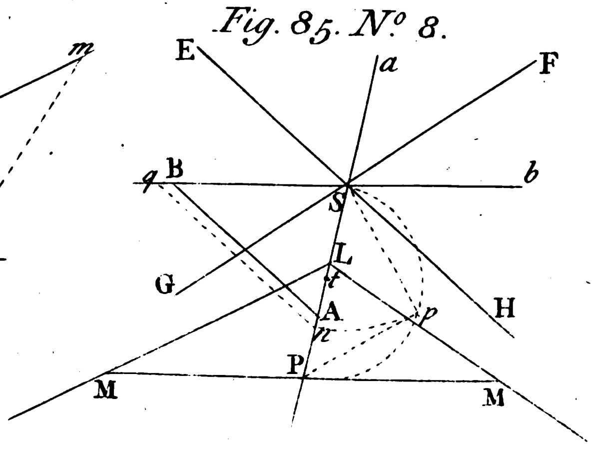
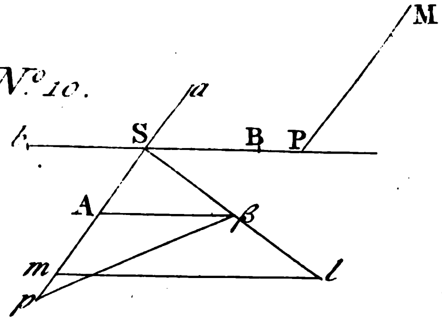


Fig. 85. No 10.



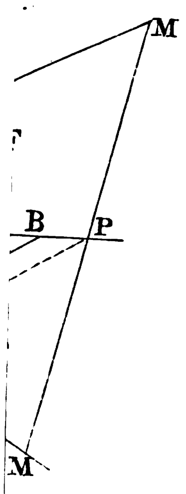
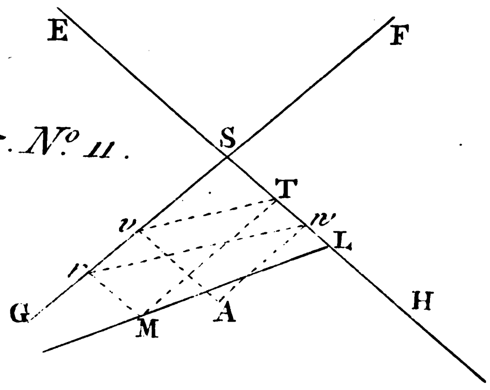


Fig. 85. No 11.



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B *b* when the Axes are unequal [a]; but if the Axes be equal, S D will coincide with A*a* [b]; but in either Caſe the Practice is the ſame as before. [a] Meth. 6. [b] Cor. 2.

C O R. 4.

Any two Conjugate Diameters of the *Hyperbolas* being given; from any given Point P in either of thoſe Diameters, to draw an Ordinate to it, and to determine its Length, without drawing any part of the Sections.

1. When the Point P is a Point in the firſt Diameter.

Let A *a*, B *b* be the given Conjugate Diameters, of which A *a* is the firſt; and in it let P be the given Point, through which an Ordinate is to be drawn. Fig. 85. N°. 8.

Take from the Center S, on the Diameter A *a*, a Diſtance S *n* equal to a mean Proportional between A P and P *a* [c], and having drawn A B, draw *n q* parallel to it, cutting B *b* in *q*; through P draw M M parallel to B *b*, and make P M, P M, each equal to S *q*, and M M will be a double Ordinate to the Diameter A *a*, paſſing through the given Point P. [c] Lem. 23.

For in the Similar Triangles S A B, S *n q*, $SA : SB :: Sn : Sq$

But by Conſtruction $Sn = \sqrt[2]{AP \times Pa}$, and $Sq = PM$.

Therefore $SA : SB :: \sqrt[2]{AP \times Pa} : PM$

and conſequently P M is an Ordinate to the Diameter A *a* [d]. [d] Hyperb. Art. 22.

The mean Proportional S *n* may be alſo found in this manner;

On S P as a Diameter, deſcribe a Semicircle S *p* P, and from P with a *Radius* equal to S A, deſcribe an Arch cutting the Semicircle in *p*, and draw S *p*, which will be equal to S *n* the mean Proportional between A P and P *a*.

For in the Rectangular Triangle S *p* P [e], $PS^2 - Pp^2 = Sp^2$ [e] 47 El. 1.

But becauſe A *a* is biſected in S [f], $AP \times Pa = PS^2 - SA^2$ [f] 6 El. 2.

And by Conſtruction $SA = Pp$, therefore $AP \times Pa = PS^2 - Pp^2 = Sp^2 = Sn^2$.

2. When the Point P is a Point in the ſecond Diameter.

Let A *a*, B *b* be the given Conjugate Diameters, of which A *a* is the firſt, and let P be a Point in the ſecond Diameter B *b*, through which an Ordinate is to be drawn. Fig. 85. N°. 9.

Having drawn A B, from P draw P E parallel to it, cutting A *a* in E; from S erect S D perpendicular and equal to the Semidiameter S A, and draw D E; then through P draw M M parallel to A *a*, and make P M, P M, each equal to D E, and M M will be a double Ordinate to the ſecond Diameter B *b*, paſſing through the given Point P [g]. [g] Meth. 6.

C O R. 5.

Having any Diameter of the *Hyperbolas*, with one of its Ordinates given; thence to find the Diameter Conjugate to it.

1. When the given Diameter is a firſt Diameter.

Let A *a* be the given firſt Diameter, and P M an Ordinate to it. Fig. 85. N°. 8.

Through the Center S draw B *b* parallel to P M, which will be the Indefinite ſecond Diameter Conjugate to A *a*; from S on the Line B *b*, take S *q* equal to P M, and on A *a* take S *n* equal to a mean Proportional between A P and P *a*, and draw *q n*, to which through A draw a Parallel A B, which will cut B *b* in B one of its Extremities, whence the other Extremity *b* is known [h]. [h] Cor. 4.

2. When the given Diameter is a ſecond Diameter.

Let B *b* be the given ſecond Diameter, and P M an Ordinate to it. Fig. 85. N°. 10.

Through the Center S, draw A *a* parallel to P M, which will be the Indefinite firſt Diameter Conjugate to B *b*. On A *a* take S *p* equal to S P, and S *m* equal to P M, and from S erect S *β* perpendicular to A *a*, and equal to S B, and draw *p β*; then take S *l* on S *β*, equal to *p β*, and draw *l m*, and through *β* draw *β* A parallel to *l m*, which will cut A *a* in A, one of its Extremities.

For in the Similar Triangles S *l m*, S *β* A, $Sl = p\beta : Sm = PM :: S\beta = SB : SA$

But in the Rectangular Triangle S *β p*, $p\beta^2 = Sp^2 + S\beta^2 = SP^2 + SB^2$

Therefore $\sqrt[2]{SP^2 + SB^2} : PM :: SB : SA$

and conſequently S A is the Semidiameter Conjugate to the Diameter B *b* [i]. [i] Hyperb. Art. 23.

C O R. 6.

Any two Conjugate Diameters of the *Hyperbolas*, or, which is the ſame, the Aſymptotes and any one Point of the *Hyperbolas*, being given; from any given Point L without the Sections, to draw two Tangents to them, without drawing any part of thoſe Curves.

1. When the given Point L is within the Angle of the Aſymptotes.

Fig. 85. N°. 8.

Let GF and EH be the Aſymptotes, and the Angle GSH their inward Angle, and let L be a Point within that Angle, from whence the Tangents are to be drawn.

Through the given Point L, draw an Indefinite firſt Diameter Aa, and find its Conjugate Bb, and the Extremities of both[a]: then on the Diameter Aa, take SP a third Proportional to SL and SA[b], on the ſame Side of S with the Point L, and through P draw MM parallel to Bb, which will be an Indefinite double Ordinate to the Diameter Aa; then find the Extremities M, M, of this double Ordinate[c], and draw LM, LM, and theſe will be the Tangents ſought[d].

[a] Meth. 1. Part third of Prob. 10.
[b] Lem. 23.
[c] Cor. 4.
[d] Hyperb. Art. 25.

2. When the given Point L is without the Angle of the Aſymptotes.

Fig. 85. N°. 9.

Through the given Point L, draw an Indefinite ſecond Diameter Bb, and find its Conjugate Aa, and the Extremities of both[e]; and on the Diameter Bb, take SP a third Proportional to SL and SB, on the contrary Side of S from the Point L, and through P draw MM parallel to Aa, which will be an Indefinite double Ordinate to the Diameter Bb; then find the Extremities M, M, of this double Ordinate[f], and draw LM, LM, which will be the Tangents deſired[g].

[e] Meth. 1. Part third of Prob. 10.
[f] Cor. 4.
[g] Hyperb. Art. 25.

3. When the given Point L is in one of the Aſymptotes.

In this Caſe only one Tangent can be drawn to the *Hyperbola*[h].

[h] Cor. 3. Part ſecond of Prob. 9.

Fig. 85. N°. 11.

Let therefore EH and GF be the Aſymptotes, and A a Point of one of the *Hyperbolas*, and let L be the given Point in the Aſymptote EH, from whence a Tangent is to be drawn to the *Hyperbola*.

From A draw Av parallel to EH, and Aw parallel to GF, cutting them in v and w, and having biſected SL in T, draw Tv, and from w draw wr parallel to Tv, cutting GF in r; then from T draw TM parallel to GF, till it cut rM, drawn parallel to EH, in M, and a Line LM will be the Tangent ſought.

For in the Similar Triangles STv, Swr, $ST : Sv = Aw :: Sw = Av : Sr = TM$
Wherefore $ST \times TM = vA \times Aw$.

And A being a Point in the *Hyperbola*, M is alſo a Point in the *Hyperbola*[i]; and becauſe ST and TL are equal, LM is therefore a Tangent to the *Hyperbola* in the Point M[k].

[i] Hyperb. Art. 27.
[k] Hyperb. Art. 17.

COR. 7.

Any two Conjugate Diameters being given; thence to determine the Points, wherein a Line, given by Poſition, cuts the *Hyperbolas*, without drawing any part of the Sections.

Fig. 85. N°. 8, 9.

Through S the Center of the *Hyperbolas*, draw a Diameter Bb or Aa, parallel to the given Line MM, and find its Conjugate Aa or Bb[l], cutting MM in P; then conſider MM as a double Ordinate to the Diameter Aa or Bb, and find its Extremities M, M[m], and thoſe will be the Points ſought.

[l] Meth. 1. Part third of Prob. 10.
[m] Cor. 4.

STEREOGRAPHY,

OR A

COMPLEAT BODY

OF

PERSPECTIVE,

In all its BRANCHES.

BOOK IV.

IN treating of the Various Methods of finding the Images of Points, Lines, and Figures, the Original Object has hitherto been ſuppoſed to lie in a known Original Plane, that is, in a Plane whoſe Vaniſhing and Interſecting Lines are given; we ſhall now proceed to the Conſideration of ſuch Points, Lines, and Figures, as do not lie in a given Plane: in order to the Management of which it is neceſſary, that the Situation of the propoſed Objects, with reſpect either to the Picture, or to ſome known Plane be given; from whence the Vaniſhing and Interſecting Points and Lines of ſuch Objects, and of the Planes in which they lie, being found, their Images may be deſcribed by the ſame Rules, as have been already given, with regard to Objects in a known Plane.

SECTION I.

Of the Seats of Points and Lines on an Original Plane.

THE Situation of a Point with reſpect to any Plane in which it doth not lie, is determined by its Seat on that Plane, and its Diſtance from its Seat.

D E F. 1.

In general, the Seat of a Point on any Plane, is where that Plane is cut by a Line perpendicular to it, drawn from the given Point.

Let GD be an Original Plane, and a an Original Point out of that Plane. If the Fig. 86. Line aA drawn from a, perpendicular to the Plane GD, cut that Plane in A, the Point A is the Seat of a on that Plane.

The Situation of a Line out of any Plane with reſpect to that Plane is determined by its Seat on that Plane, and the Angle it makes with its Seat.

D E F. 2.

In general, the Seat of a Line on any Plane, is a Line drawn in that Plane, through the Seats of any two Points of the Original Line; or the Interſection of the given Plane with another Plane perpendicular to it, paſſing through the given Line.

If ab be the Original Line, and A and B the Seats of the Points a and b on the Plane GD; then AB is the Seat of the Line ab on that Plane.

Or if through ab, a Plane aAbB be ſuppoſed to paſs perpendicular to the Plane GD; then AB, the Interſection of theſe two Planes, is the Seat of the given Line ab on the Plane GD.

The Seats of Points and Lines thus determined, are generally called the *Perpendicular Seats*, to diſtinguiſh them from another kind of Seats, which are of very convenient uſe in Practice, when the Points or Lines conſidered, are to be referred to an Original Plane, not perpendicular, but inclining to the Picture, which laſt are called the *Oblique Seats*.

D E F. 3.

The Oblique Seat of a Point on an Original Plane which inclines to the Picture, is where that Plane is cut by a Line drawn from the given Point, parallel to the Vertical Line.

Fig. 87. Let E F G H repreſent the Picture, G D an Original Plane, and *a* an Original Point out of that Plane.

The Line *a* A drawn from *a* parallel to the Vertical Line *o* P, cuts the Plane G D in A, the Oblique Seat of *a* on that Plane.

In like manner, if from *a* a Line *a α* be drawn parallel to the Line of Station K P, the Point *α*, where it cuts the Picture, is the Oblique Seat of *a* on the Picture, with reſpect to the Plane G D: and if another Plane D C be ſuppoſed parallel to the Picture, the Point a, where *a α* cuts that Plane, may be alſo called the Oblique Seat of *a* on that Plane, with reſpect to the Plane G D.

If the Original Plane G D be perpendicular to the Picture, the Seats A and *α* of the Point *a* on the Original Plane and the Picture thus found, will be the ſame with its perpendicular Seats on thoſe Planes; ſeeing that in this Caſe, the Vertical Line *o* P is perpendicular to the Original Plane, and the Line of Station K P is perpendicular to the Picture [a].

[a] Theor. 9. B. I.

D E F. 4.

The Oblique Seat of a Line on any Plane, is a Line drawn in that Plane, through the Oblique Seats of any two Points of the Original Line.

D E F. 5.

The Line *a* A, which meaſures the Diſtance of the Original Point *a* from its Seat on the Plane G D, is called the *Support* of that Point on the Original Plane, either Perpendicular or Oblique, according as A is the Perpendicular or Oblique Seat of the given Point; and after the ſame manner, *a α* is the Support of *a* on the Picture.

C O R.

The Seat of any Point on the Picture, is the Interſecting Point of its Support on the Picture.

Thus *α* is the Interſecting Point of the Line *a α*.

D E F. 6.

A Plane paſſing through any Original Line and its Seat on any Plane, whether Perpendicular or Oblique, is called the *Plane of the Seat* of that Line.

Thus the Plane *a* A *b* B, is the Plane of the Seat of *a b* on the Plane G D, and the Plane *a α b β* is the Plane of the Seat of *a b* on the Picture.

C O R.

The Seat of any Line on the Picture, is the Interſecting Line of the Plane of the Seat of the Original Line on the Picture.

Thus *α β* is the Interſecting Line of the Plane *a α b β*.

PROP. I. THEOR. I.

The Image of the Oblique Support of any Point on an Original Plane which inclines to the Picture, is parallel to the Vertical Line of that Plane, or coincides with it.

Dem. Becauſe *a* A is by Conſtruction parallel to the Vertical Line *o* P [b], and conſequently to the Picture; the Image of *a* A is therefore parallel to its Original [c], whence it is alſo parallel to the Vertical Line *o* P [d], or muſt coincide with it. *Q. E. D.*

[b] Def. 3.
[c] Theor. 2. B. I.
[d] 9 El. 11.

PROP. II. THEOR. II.

The Vaniſhing and Interſecting Lines of the Plane of the Oblique Seat

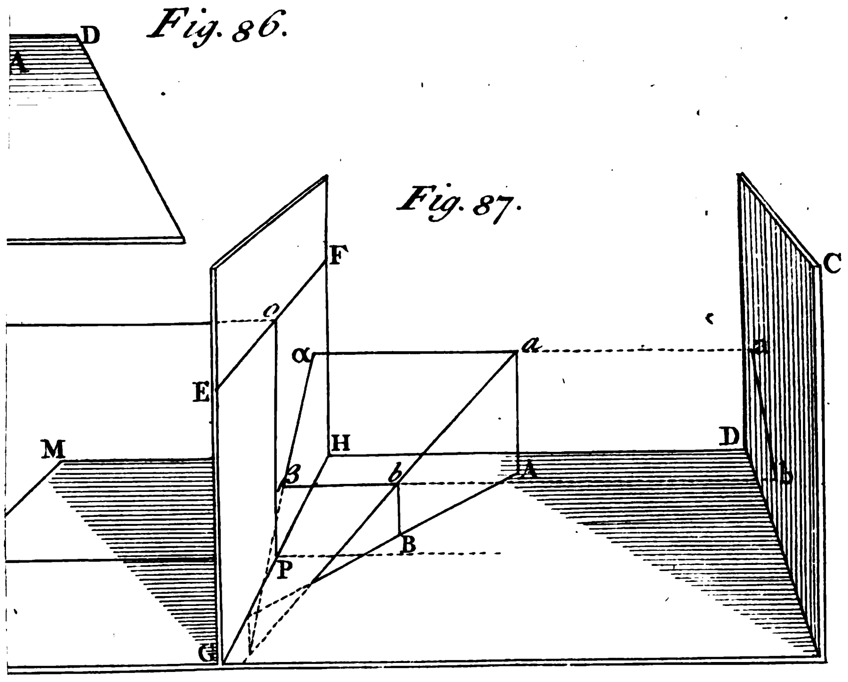

Fig. 86.

Fig. 87.

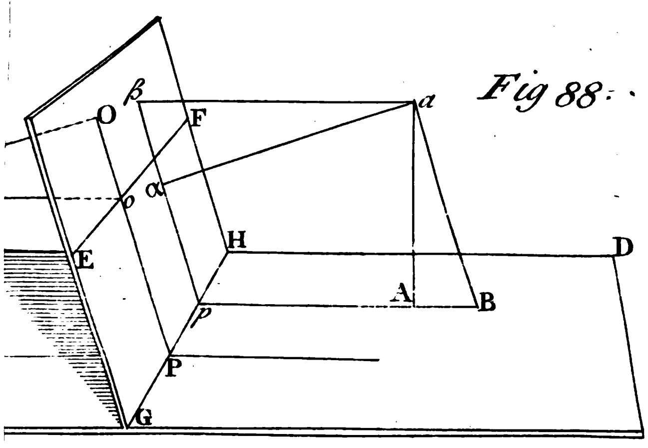

Fig 88.

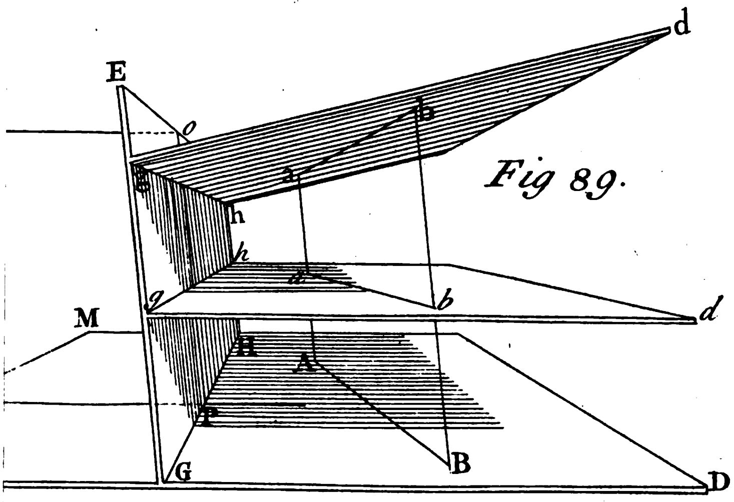

Fig 89.

J. Mynde sc.

Seat of any Line on an Original Plane, are parallel to the Vertical Line of that Original Plane, or elſe coincide with it.

Dem. Becauſe the Plane $aAbB$ paſſes through the Lines aA and bB, which are parallel to the Picture, the Images of aA and bB are parallel to the Vaniſhing and Interſecting Lines of the Plane $aAbB$ [a]; but the Images of aA and bB are parallel to oP, the Vertical Line of the Original Plane GD [b], therefore the Vaniſhing and Interſecting Lines of $aAbB$, the Plane of the Oblique Seat of the Original Line ab, are alſo parallel to oP, if they do not coincide with it, which they would do, if AB were in the Line of Station KP. *Q. E. D.*

C O R. 1.

The Directing Line of the Plane of the Oblique Seat of any Line on an Original Plane, is parallel to the Eye's Director of that Original Plane, or elſe coincides with it.

Becauſe the Vertical Line and the Eye's Director, which relate to the ſame Original Plane, are parallel [c].

C O R. 2.

If the Original Line be parallel to the Picture, the Plane of its Oblique Seat on any Original Plane which cuts the Picture, will alſo be parallel to the Picture, and hath no Vaniſhing, Interſecting, or Directing Lines [d].

PROP. III. THEOR. III.

If an Original Plane GD cut the Picture in GH, and any Point a be given out of that Plane; then the Perpendicular and Oblique Seats α, β, A, B of that Point, both on the Picture and Original Plane, are all in a Plane $a\beta pB$, parallel to $IoKP$, the Vertical Plane of the Original Plane. Fig. 88.

Dem. Becauſe the Supports $a\alpha$ and aA are perpendicular to the Picture and Original Plane [e], a Plane paſſing through thoſe two Lines is perpendicular to the Picture and Original Plane [f], and conſequently parallel to the Vertical Plane [g]: again, the Supports aB and $a\beta$ being reſpectively parallel to oP and KP [h], which are Lines in the Vertical Plane $IoKP$ [i], therefore a Plane paſſing through aB and $a\beta$ is alſo parallel to the Vertical Plane [k]: but through the ſame Point a, there cannot paſs two different Planes parallel to the ſame Plane, therefore the Points a, A, B, β, and α, are all in the ſame Plane, parallel to the Vertical Plane $IoKP$. *Q. E. D.*

C O R.

The Line AB, which joins the Seats A and B of the Point a on the Original Plane, is parallel to KP the Line of Station [l], and its Vaniſhing Point is in o, the Center of the Vaniſhing Line EF [m]; and the Line $\alpha\beta$ which joins the Seats of the Point a on the Picture, is parallel to the Vertical Line oP; unleſs the Point a be in the Vertical Plane, in which Caſe, AB and $\alpha\beta$ will coincide with KP and oP.

PROP. IV. THEOR. IV.

If an Original Plane βapB, be parallel to the Vertical Plane $IoKP$ of another Original Plane GD; then the Perpendicular and Oblique Seats of any Point or Line in the Plane βapB on the Plane GD, will fall in pB, the common Interſection of thoſe two Planes; and the Perpendicular and Oblique Seats of any Point or Line in the Plane βapB on the Picture, will be in βp the Interſecting Line of that Plane. Fig. 88.

Dem. It was ſhewn in the preceeding Theorem, that the Perpendicular and Oblique Supports of the Point a on the Picture and Original Plane, were all in a Plane βapB, parallel to the Vertical Plane of the Plane GD; but the Support of any other Point in the Plane βapB, muſt be parallel to the correſponding Support of the Point a [n], and is therefore in the ſame Plane, and conſequently muſt fall in the common Interſection of that Plane, either with the Plane GD, or with the Picture; the firſt of which is the Line pB, and the other the Line $p\beta$; and if the Supports of all Points in the

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[a] Theor. 2. and Cor. 1. Theor. 15. B. I.

[b] Prop. 1.

[c] Cor. 3. Def. 15. B. I.

[d] Theor. 3. B. I.

[e] Def. 1.

[f] 18 El. 11.

[g] Cor. Def. 11. B. I.

[h] Def. 3.

[i] Def. 13, 15. B. I.

[k] 15 El. 11.

[l] 16 El. 11.

[m] Cor. 2. Theor. 11. B. I.

[n] 9 El. 11.

Plane βapB, fall in the Line pB or βp, the Seats of all Lines in that Plane, which always paſs through the Seats of their Points[a], muſt alſo be in the ſame Line pB or βp. *Q. E. D.*

C O R.

It is evident, that if the Plane βapB coincide with the Vertical Plane $IoKP$, the Perpendicular and Oblique Seats of all Points or Lines in that Plane on the Plane GD, and on the Picture, will fall in KP, and Po, the Line of Station and Vertical Line of the Plane GD.

PROP. V. THEOR. V.

Fig. 89. If two Planes ghd, GHD be parallel, and an Original Line ab in the Plane ghd, with its Oblique Seat AB on the Plane GHD, be given; then if the Seat AB, be taken as an Original Line in this laſt Plane, its Oblique Seat on the Plane ghd, will be ab, the Line firſt given.

Dem. Becauſe the Oblique Seat of ab on the Plane GHD, is determined by the Oblique Seats A and B, of the Points a and b on that Plane[b], the Supports aA and bB of which, are parallel to the Vertical Line oP of the Plane GHD[c], and the ſame Line oP being alſo the Vertical Line of the Plane ghd, theſe two Planes being parallel[d], therefore the ſame Lines Aa and Bb, are the Oblique Supports of the Points A and B on the Plane ghd; wherefore a and b are the Seats of A and B, and conſequently ab is the Oblique Seat of AB on the Plane ghd. *Q. E. D.*

C O R.

The ſame is true, when the given Planes g h d, GHD are not parallel, but only have parallel Vaniſhing Lines; the ſame Line oP being ſtill their common Vertical Line[e], to which the Supports a A and b B are parallel.

PROP. VI. THEOR. VI.

If two or more Lines be parallel to each other, the Planes of their Seats of the ſame kind, on any given Plane, will be parallel, if they do not coincide.

Dem. For the Supports, either Oblique or Perpendicular, of all the Points in any one of thoſe Lines, being, by Conſtruction, parallel to the like Supports of all the Points in any other of them, as being either parallel to the Vertical Line of the given Plane[f], or perpendicular to that Plane[g]; and the Lines themſelves being, by Suppoſition, alſo parallel; the Planes which reſpectively paſs through each Line and the correſponding Supports of its Points, muſt be parallel to each other[h]. *Q. E. D.*

C O R.

Hence the correſponding Seats of all parallel Lines on any given Plane are parallel[i], that is, the Oblique Seats to the Oblique, and the Perpendicular to the Perpendicular; and conſequently the Angles made by the propoſed Parallels with their reſpective Seats are equal[k].

Of the Generation and Properties of Vaniſhing Points and Lines.

PROP. VII. THEOR. VII.

If from the Eye a Perpendicular be drawn to any Original Plane, cutting it in a Point; the Image of that Point will form a Point in the Picture, which will be the Vaniſhing Point of all Lines whatſoever, which are perpendicular to the Original Plane.

Fig. 90. *Dem.* Let LMD be the Original Plane, EFGH the Picture, IK the Eye's Director, and O the Center of the Picture.

From I draw IS perpendicular to the Original Plane, which muſt cut the Line of Station KP in ſome Point S, the Vertical Plane $IoKP$ being perpendicular to the

Original

[a] Def. 2 and 4.

[b] Def. 4.

[c] Def. 5, and 3.

[d] Cor. 2. Theor. 13. B. I.

[e] Cor. 2. Theor. 14. B. I.

[f] Def. 3.

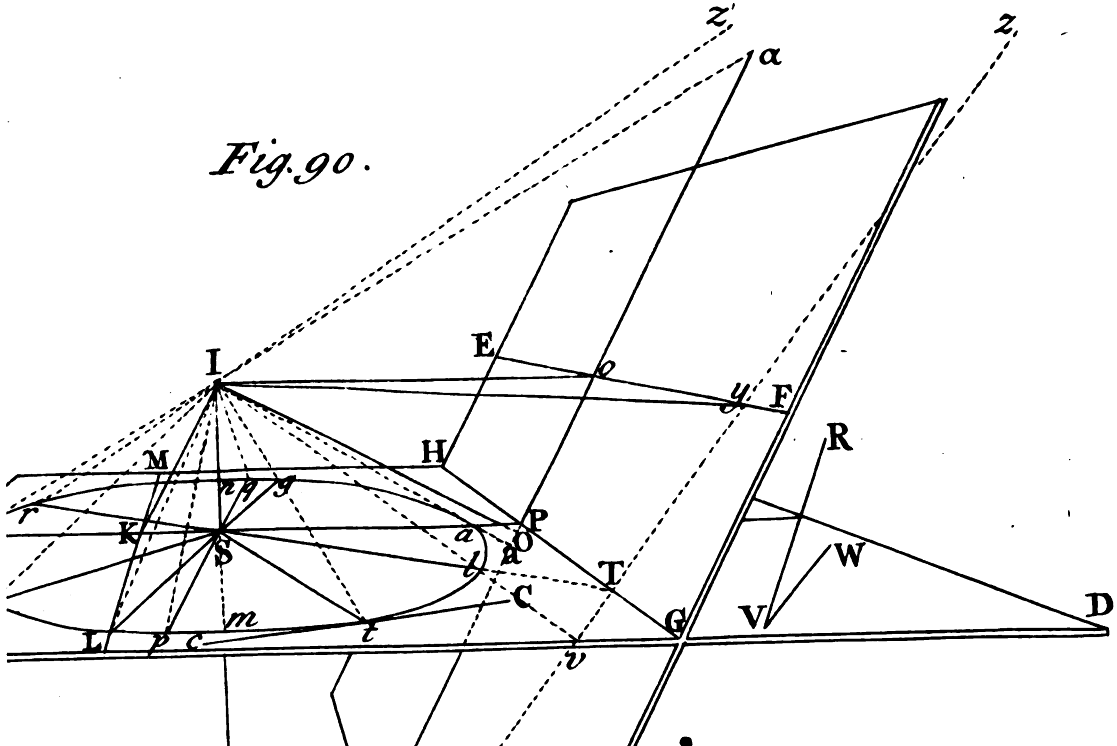
[g] Def. 1.

[h] 15 El. 11.

[i] 16 El. 11.

[k] 10 El. 11.

Fig. 90.



g.91.

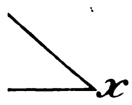
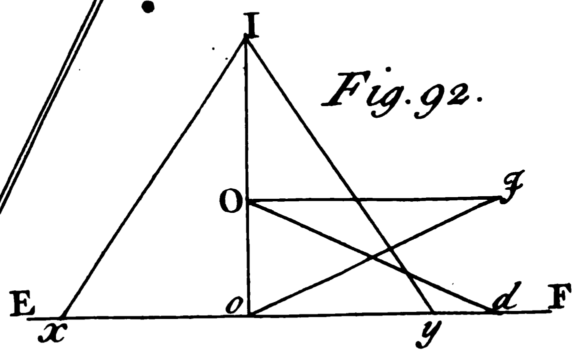


Fig. 92.



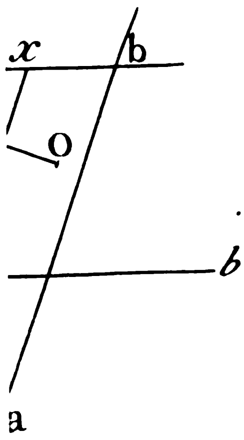
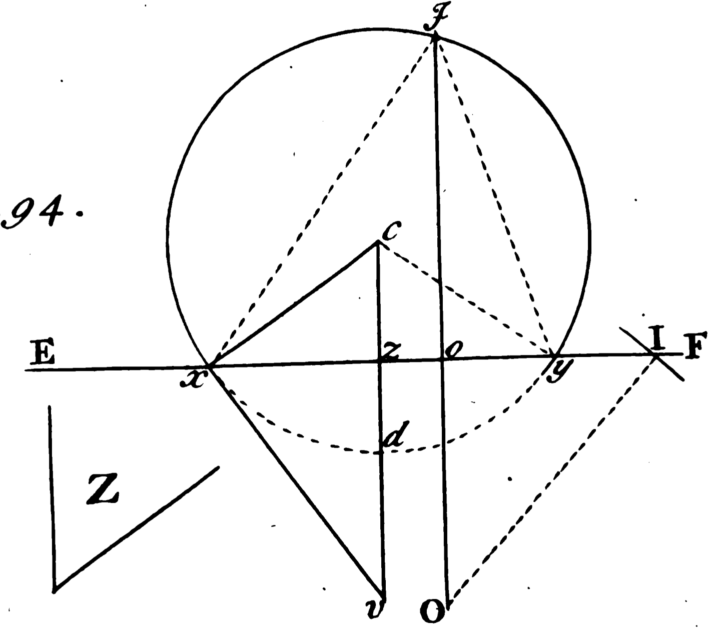


Fig. 94.



5 N°1.

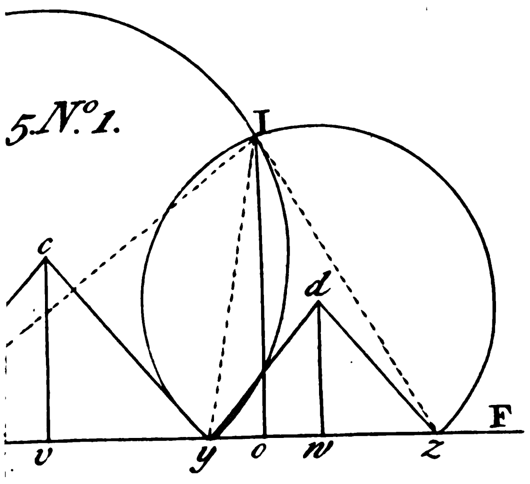
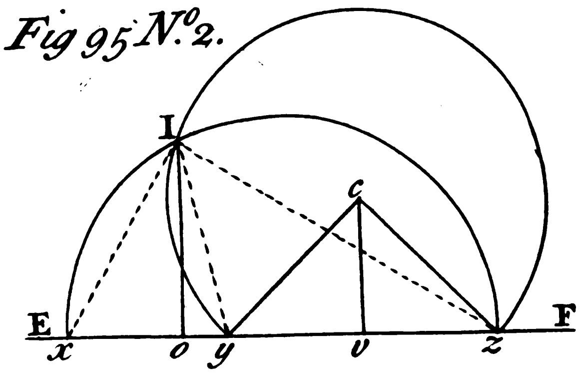


Fig 95 N°2.



J. Mynde sc.

Original Plane[a], and if I S be produced till it cut the Picture in x, that Point muſt fall ſomewhere in the Vertical Line o P, I S and o P being both Lines in the Vertical Plane; but I S being a Line paſſing through the Eye, x the Image of S is alſo the Vaniſhing Point of that Line[b], and conſequently of all Lines parallel to I S[c]; and I S being perpendicular to the Plane L M D, all Lines parallel to I S, are alſo perpendicular to that Plane[d]; wherefore x is the Vaniſhing Point of all Lines whatſoever, which are perpendicular to the Plane L M D. *Q. E. D.*

[a] 38 El. 11.
[b] Theor. 18. B. I.
[c] Theor. 5. B. I.
[d] 8 El. 11.

C O R. 1.

The Radial Io of the Original Plane, is perpendicular to Ix the Radial of the Vaniſhing Point x.

For Ix is perpendicular to K P, to which Io is parallel[e].

[e] Cor. 3. Def. 15. B. I.

C O R. 2.

If the Picture and Original Plane be perpendicular, the Point of Station K then coinciding with S, I S and oP will be parallel, and the Vaniſhing Point x will be infinitely diſtant; in which Caſe, all Perpendiculars to the Plane L M D are parallel to the Picture and to oP, and have no Vaniſhing Point[f].

[f] Theor. 1. B. I.

C O R. 3.

If the Picture and Original Plane be parallel, a Line drawn from the Eye perpendicular to the Original Plane, will alſo be perpendicular to the Picture[g], and will therefore coincide with the Eye's Axe; and the Center of the Picture then becomes the Vaniſhing Point of Perpendiculars to the Original Plane.

[g] 14 El. 11.

C O R. 4.

The Vaniſhing Point of Perpendiculars to any Original Plane, is the Vaniſhing Point of Perpendiculars to all Planes parallel to that Plane.

For all Lines which are perpendicular to one Plane, are perpendicular to all Planes which are parallel to that Plane[h].

[h] 14 El. 11.

C O R. 5.

No Lines whatſoever can be perpendicular to the Plane L M D, but ſuch only whoſe Vaniſhing Point is x.

Becauſe no other Lines can be parallel to I S[i].

[i] Cor. 1. Theor. 5. B. I.

P R O P. VIII. T H E O R. VIII.

If through S any Line S T be drawn in the Plane L M D; the Image of that Line will form in the Picture, a Vaniſhing Line of Planes perpendicular to the Plane L M D, which Vaniſhing Line will paſs through x.

Fig. 90.

Dem. If a Plane be imagined to paſs through I S and S T, this Plane muſt cut the Picture in xT the Image of S T, which muſt therefore paſs through x the Image of S; but the Plane I S T paſſing through the Eye, xT the Image of S T, is alſo the Vaniſhing Line of that Plane[k], and conſequently of all Planes parallel to it[l]; now I S being perpendicular to the Original Plane, the Plane I S T which paſſes through I S, is alſo perpendicular to that Plane[m]; and all Planes parallel to the Plane I S T, being therefore perpendicular to the Original Plane, xT is therefore a Vaniſhing Line of Planes perpendicular to the Plane L M D, and conſequently to all other Planes, to which the Plane L M D is parallel: the ſame may be ſhewn of any other Line in the Plane L M D paſſing through S. *Q. E. D.*

[k] Theor. 19. B. I.
[l] Theor. 13. B. I.
[m] 18 El. 11.

C O R.

No Planes can be perpendicular to the Plane L M D, but ſuch only whoſe Vaniſhing Lines paſs through x.

Becauſe if x be not in the Vaniſhing Line of any Plane, no Line in that Plane can be parallel to thoſe whoſe Vaniſhing Point is x[n]; wherefore no Line in that Plane can, nor conſequently can the Plane itſelf, be perpendicular to the Plane L M D.

[n] Theor. 5. and 10. B. I.

P R O P. IX. T H E O R. IX.

If from S as a Center, with any *Radius* S A, a Circle Anam be deſcribed on the Plane LMD, and from I to either Extremity A, of the Diameter A*a*, a Line IA be drawn, inclining to the Plane LMD in the Angle IAS, equal to any Angle Z; the Image of this Circle will be the place of the Vaniſhing Points of all Lines whatſoever, which incline to the Plane LMD in an Angle equal to the Angle Z.

Dem. Imagine a Cone to be formed on the Circle A*nam* as a Baſe, with the Vertex I; then IS the Axe of this Cone, being perpendicnlar to its Baſe, all its Sides incline to the Plane of the Baſe in the ſame Angle IAS[a] equal to Z; and all Lines whatſoever which incline to the Plane LMD in an Angle equal to Z, muſt be parallel to one or other of the Sides of this Cone, and muſt therefore have the ſame Vaniſhing Point with that Side[b]; but the Vaniſhing Points of the Sides of this Cone are the Images of the ſeveral Points of the Circular Baſe, through which they reſpectively paſs, ſeeing all the Sides of the Cone paſs through the Eye[c]; therefore the Vaniſhing Points of all the Sides of this Cone, and conſequently of all Lines parallel to thoſe Sides, muſt fall in the Image of the Circular Baſe. *Q. E. D.*

COR. 1.

Any Line RV, inclining to the Plane LMD in an Angle equal to Z, together with VW its perpendicular Seat on that Plane, being given; thence to find the Side of the Cone IA*ma*, which is parallel to the given Line RV.

Through S draw SN parallel to VW, cutting the Circle in N, and IN will be the Side of the Cone required.

For the Planes RVW, INS are parallel, being both of them perpendicular to the Plane LMD, and cutting that Plane in VW and NS which are parallel; and the Angles RVW, INS being equal, the Lines RV and IN are therefore parallel[d].

COR. 2.

If the Angle IAS, or Z, be leſs than IKS, the Angle of Inclination of the Picture to the Original Plane, the Image of the Circle A*nam* will be two oppoſite *Hyperbolas*; if the Angle Z be equal to IKS, the Image will be a *Parabola*; and if the Angle Z be greater than IKS, the Image will be an *Ellipſis* or a Circle.

For in the firſt Caſe, the Circle A*nam* will cut the Directing Line LM; in the ſecond Caſe, it will touch that Line; and in the laſt Caſe, it will fall all on the ſame Side of LM[e].

COR. 3.

In all theſe Caſes, the Image a*a* of the Diameter A*a* of the forming Circle, which coincides with the Line of Station, will coincide with the Vertical Line *o*P, and will be one of the Axes of the Section produced by the Image of the Circle A*nam*; its Center S being in the Line of Station[f].

COR. 4.

The Image of the Circle A*nam*, is alſo the Place of the Vaniſhing Points of all Lines whatſoever which incline to the Line IS, in an Angle equal to the Complement of the Angle Z.

For the Angle SIA is the Complement of the Angle IAS to a Right Angle.

PROP. X. THEOR. X.

Fig. 90. If any Vaniſhing Line *xy* of Planes perpendicular to the Plane LMD, be formed by a Diameter *rl* of the Circle A*nam*[g], which cuts the Directing Line LM; the Radials of the Vaniſhing Points *v* and *z*, formed by the Extremities *l* and *r* of that Diameter, will make with I*y* the Radial of that Diameter, Angles equal to the Angle Z.

Dem. Produce I*l* and I*r*, till they cut *xy* in *v* and *z*, then *v* and *z* will be the Vaniſhing Points formed by *l* and *r*, and I*v* and I*z* will be their Radials; and becauſe I*y*, the Radial of the Diameter *rl*, is parallel to *rl*[h], the Angles *z*I*y*, I*r*S are equal, as are alſo the Angles *y*I*v*, I*l*S[i]; but the Angles I*r*S, I*l*S are each equal to the Angle Z, and therefore the Angles *z*I*y*, *y*I*v* are alſo each equal to Z. *Q. E. D.*

COR.

[a] Con.Sec.Art. 2. B. III.

[b] Theor. 5. B. I.

[c] Theor. 18. B. I.

[d] Lem. 1. B. I.

[e] Con.Sec.Art. 15, 12, and 9. B. III.

[f] Caſe 2. Prob. 1 and 5, and Caſe 3. Prob. 9. B. III.

[g] Prop. 8.

[h] Def. 17. B. I.

[i] 29 El. 1.

COR. 1.

If a Vaniſhing Line be formed by a Diameter pq of the Circle A*nam*, lying wholly on the ſame Side of the Directing Line; then the Radials of the Vaniſhing Points formed by the Extremities p and q of that Diameter, will make with Ix the Radial of the Vaniſhing Point x, Angles equal to the Complement of Z to a Right Angle.

For Ip and Iq produced to the Picture, are the Radials of the Vaniſhing Points formed by p and q, and the Angles pIS, qIS, are equal to the Complements of IpS, IqS, or Z, to a Right Angle.

COR. 2.

If a Vaniſhing Line be formed by a Diameter Lg of the Circle A*nam*, having one of its Extremities L in the Directing Line; the Point L can form no Vaniſhing Point; and the Line IL which ſhould form that Vaniſhing Point, becomes the Director of Lg, and is therefore parallel to the Vaniſhing Line formed by its Image[a]: nevertheleſs the Angles LIS, gIS, made by the Radial Ix, with the Director IL, and the Radial Ig of the Vaniſhing Point formed by g, are ſtill the Complements of Z to a Right Angle; and the Angles made by the Radial of the Diameter Lg with the ſame Lines IL and Ig, are equal to the Angle Z, they being equal to the Angles ILS, IgS, ſeeing Lg and its Radial are parallel; and laſtly, the Vaniſhing Point formed by g, will biſect the Diſtance between x and the Vaniſhing Point of Lg, in regard that Lg being biſected in S, the Image of Sg is equal to its Complement[b].

[a] Cor. 1. Def. 18. B. I.

[b] Theor. 26. B. I.

PROP. XI. THEOR. XI.

If through any Point t of the Circle A*nam*, a Tangent Cc be drawn; the Image of that Tangent will form a Vaniſhing Line of Planes, inclining to the Plane LMD, in an Angle equal to Z; which Vaniſhing Line will alſo be a Tangent to the Section, produced by the Image of the Circle.

Dem. For if a Plane be imagined to paſs through It and the Tangent Cc, this Plane will touch the Cone in the Line It, which will be perpendicular to Cc, and a *Radius* St being drawn, St will alſo be perpendicular to Cc; wherefore the Angle ItS will be the Angle of Inclination of the Plane ICc to the Plane LMD[c], which Angle is equal to Z; but the Image of Cc is the Vaniſhing Line of the Plane ICc[d], and of all Planes parallel to it[e]; all which Planes incline to the Plane LMD, in an Angle equal to Z; this Vaniſhing Line therefore is a Vaniſhing Line of Planes inclining to the Plane LMD in the given Angle; and the Line Cc being a Tangent to the Circle in t, its Image muſt be a Tangent to the Image of the Circle in the Image of t. The ſame may be ſhewn of any other Tangent to the Circle A*nam*. *Q. E. D.*

[c] Def 19. B. I.

[d] Theor. 19. B. I.

[e] Theor. 13. B. I.

COR.

No Planes can incline to the Plane LMD in an Angle equal to Z, but ſuch only, whoſe Vaniſhing Lines are Tangents to the Image of the Circle A*nam*.

For if the Line in the Plane LMD, which forms the Vaniſhing Line, be not a Tangent to the Circle A*nam*, a Plane paſſing through that Line and the Point I, cannot incline to the Plane LMD in the Angle propoſed.

PROP. XII. PROB. I.

The Center and Diſtance of the Picture, and any Vaniſhing Point being given; thence to find the Diſtance of that Vaniſhing Point.

Let O be the Center of the Picture, and x the given Vaniſhing Point. Fig. 91.

Draw Ox, and from O erect OI perpendicular to it, making OI equal to the Diſtance of the Picture, then Ix will be the Diſtance of the Vaniſhing Point x.

Dem. This is evident, if Ox be conſidered as a Vaniſhing Line, paſſing through O the Center of the Picture, in which Line, x is a Vaniſhing Point; for then OI will be the Radial of the Vaniſhing Line, and conſequently Ix is the Radial of the Vaniſhing Point x, the ſame with its Diſtance. *Q. E. I.*

COR.

The Diſtance Ix of any Vaniſhing Point x, is the Hypotenuſe of a Right Angled

Triangle, of which IO the Diſtance of the Picture is one Side, and Ox the Diſtance between the Center of the Picture and the given Vaniſhing Point, is the other; wherefore the Square of Ix is always equal to the Squares of IO and Ox[a].

[a] 47 El. 1.

PROP. XIII. PROB. II.

The Center and Diſtance of the Picture, and any Vaniſhing Line, not paſſing through the Center, being given; thence to find the Center and Diſtance of that Vaniſhing Line.

Fig. 92. Let O be the Center of the Picture, and EF the given Vaniſhing Line.

Through O draw Oo perpendicular to EF cutting it in o, and take od in the Line EF equal to the Diſtance of the Picture, and draw Od; then o is the Center and Od the Diſtance of the given Vaniſhing Line, and making oI equal to Od, oI will be the Radial of that Vaniſhing Line.

Dem. From O draw $O\mathcal{J}$ parallel to EF, and equal to od the Diſtance of the Picture, and draw $\mathcal{J}o$; it is then evident, that $\mathcal{J}o$ and Od will be equal, but $\mathcal{J}o$ is the Diſtance of the Vaniſhing Point o[b], and o being the Center of the Vaniſhing Line EF[c], Oo being its Vertical Line[d], $\mathcal{J}o$, or its equal Od, is therefore the Diſtance of that Vaniſhing Line, and conſequently oI taken equal to $\mathcal{J}o$ or Od, is its Radial. *Q. E. I.*

[b] Prop. 12.
[c] Cor. Def. 13. B. I.
[d] Cor. 2. Def. 15. B. I.

COR.

The Angle $\mathcal{J}oO$, or its equal dOo, is the Angle of Inclination of the Planes, whoſe Vaniſhing Line is EF, to the Picture.

For $\mathcal{J}oO$ is the Angle which the Radial of the Vaniſhing Line EF makes with its Vertical Line Oo[e].

[e] Theor. 9. B. I.

PROP. XIV. PROB. III.

Fig. 92. The Center O and Diſtance $O\mathcal{J}$ of the Picture, and any two Vaniſhing Points x and y being given; thence to determine the Angle made by the Originals of any two Lines, in the ſame Plane, which have x and y for their Vaniſhing Points.

Through x and y draw the Vaniſhing Line xy, and having drawn its Vertical Line OI[f], make oI equal to the Diſtance of its Center o[g], and from I draw Ix and Iy, and the Angle xIy will be the Angle ſought[h]. *Q. E. I.*

[f] Cor. 2. Def. 15, B. I.
[g] Prop. 13.
[h] Cor. 3. Theor. 11. B. I.

DEF. 7.

The Angle xIy, made by the Radials of any two Vaniſhing Points x and y, is ſometimes called the *Angle ſubtended* by thoſe Vaniſhing Points, or by xy, or thoſe Points are ſaid to *ſubtend* ſuch an Angle; and if that Angle be Right, then thoſe Vaniſhing Points are ſaid to be perpendicular to each other.

PROP. XV. PROB. IV.

Fig. 93. The Center and Diſtance of the Picture, and the Indefinite Image xy of an Original Line, and its Vaniſhing Point x being given; thence to find the Image y, of a Point in that Line, from whence a Line drawn to the Eye, ſhall make an Angle with the Original Line, equal to any Angle propoſed.

Conſider the given Line xy, as the Vaniſhing Line of a Plane paſſing through the Eye and that Line[i], and find Io the Radial of that Vaniſhing Line[k], and having drawn Ix the Radial of the Original Line, draw Iy, making the Angle xIy equal to the Angle propoſed, and y will be the Image of the Point deſired.

Dem. Let $Ixab$ repreſent the Plane which paſſes through the Eye and the Original Line ab, and xy the Interſection of that Plane with the Picture; then the Image of ab, and of every Point of that Line, muſt be in xy, which is alſo the Vaniſhing Line of that Plane[l]. Now the Original Line ab, being parallel to its Radial Ix, the Line Iy which cuts thoſe Parallels, makes the alternate Angles xIa, Iam equal; and y being the Image of a, y therefore repreſents a Point in the Original Line, from

[i] Theor. 19. B. I.
[k] Prop. 13.
[l] Theor. 19. B. I.

whence a Line yI drawn to the Eye, makes with the Original Line ab, an Angle Iam equal to the Angle xIy, which was made equal to the Angle propoſed. *Q. E. I.*

C O R.

If the Original of xy be parallel to the Picture, the Point y may be found, by uſing xy as a Vaniſhing Line, and finding its Radial Io as before; for then a Line Iy, drawn ſo as to make the Angle Iyx equal to the Angle propoſed, will give the Point y required.

For if the Original Line ab be parallel to the Picture, and conſequently to xy the Interſection of the Picture with the Plane which paſſes through the Eye and the Original Line; it is evident, the Angles Iyx, Iab, are equal, and conſequently that y the Image of a, repreſents a Point of the Original Line, from whence a Line yI drawn to the Eye, makes an Angle Iab with that Line, equal to the Angle Iyx, which is the Angle propoſed.

L E M. I.

On a given determinate Line xy, to deſcribe a Segment of a Circle, which ſhall contain an Angle equal to a given Angle Z. Fig. 94.

1. When the propoſed Angle Z is Acute.

Biſect xy in z by the Perpendicular cz, and draw xc, making the Angle xoz equal to the propoſed Angle Z, and from c as a Center, with the *Radius* cx or cy, deſcribe the Segment of a Circle xJy, on the ſame Side of xy with the Center c; and xJy will be the Segment required.

Dem. Draw cy, and from x and y draw any two Lines xJ, yJ, meeting in any Point J of the Segment xJy.

Then becauſe the Triangles xzc, yzc, are Similar and equal, the Angles xcz, zcy, are equal, wherefore xcy is the double of the Angle xcz; but xcy is an Angle at the Center, and xJy an Angle at the Circumference of the Circle xJy; therefore xcy is the double of the Angle xJy[a], which laſt is therefore equal to the Angle xcz, which was made equal to Z the Angle propoſed, and conſequently the Segment xJy is the Segment required[b].

2. When the propoſed Angle is Right.

In this Caſe it is evident, the Segment required is a Semicircle, of which xy is the Diameter and z the Center[c].

3. When the propoſed Angle is Obtuſe.

Having biſected xy by the Perpendicular cz as before, draw xc, making the Angle xcz equal to the Complement to two Rights of the Angle Z, and from c as a Center, with the *Radius* cx, deſcribe the Segment xdy, on the contrary Side of xy from the Center c, and xdy will be the Segment deſired.

For the Segment xJy containing an Angle equal to xcz, the Segment xdy, which is the Complement of the Circle, contains an Angle equal to the Complement to two Rights of the Angle xcz[d], and conſequently equal to the Angle propoſed. *Q. E. I.*

C O R.

If xv be drawn, ſo as to make with xy an Angle vxy equal to Z, a Line xc drawn perpendicular to xv will cut cz in c the Center of the Segment required[e].

For if cz be produced till it cut xv in v, the Triangles xvz, zxc, will be Similar[f], and therefore the Angles xcz, zxv, will be equal.

PROP. XVI. PROB. V.

The Center of the Picture O, and any Vaniſhing Line EF being given, and in that Line two Vaniſhing Points x and y, ſubtending a known Angle Z; thence to find the Center and Diſtance of the given Vaniſhing Line, and alſo the Diſtance of the Picture. Fig. 94.

On xy deſcribe a Segment xJy of a Circle containing the given Angle Z[g], and from O draw Oo perpendicular to EF, cutting it in o, and the Segment xJy in J; and from O as a Center, with a *Radius* equal to oJ, deſcribe an Arch cutting EF in I; then o will be the Center, and oJ the Diſtance of the Vaniſhing Line EF, and oI will be equal to the Diſtance of the Picture.

Dem. Becauſe Oo is the Vertical Line of the Vaniſhing Line EF[h], the Radials of the Vaniſhing Points x and y muſt meet ſomewhere in that Line, and make together

an

[a] 20 El. 3.

[b] 21 El. 3.

[c] 31 El. 3.

[d] 22 El. 3.

[e] 33 El. 3.

[f] 8 El. 6.

[g] Lem. 1.

[h] Cor. 2. Def. 15. B. I.

an Angle equal to the given Angle Z; wherefore $x\mathcal{J}$ and $y\mathcal{J}$ which meet Oo in $\mathcal{J}$, and make together the given Angle, are the Radials of the Vaniſhing Points x and y, and conſequently o is the Center, and $o\mathcal{J}$ the Radial or Diſtance of the Vaniſhing Line EF; and if $o\mathcal{J}$ be the Diſtance of that Vaniſhing Line, it is evident oI is equal to the Diſtance of the Picture[a]. *Q. E. I.*

[a] Prop. 13.

PROP. XVII. PROB. VI.

Fig. 95. N°. 1.

Any Vaniſhing Line EF, and in it three Points x, y, and z being given, and the Angles ſubtended by thoſe Vaniſhing Points being known; thence to find the Center and Diſtance of that Vaniſhing Line, when neither the Center nor Diſtance of the Picture are given.

On xy deſcribe a Segment of a Circle xIy, containing an Angle equal to that ſubtended by the Vaniſhing Points x and y, and on yz deſcribe a Segment yIz, containing an Angle equal to that ſubtended by y and z[b], and from I the Interſection of theſe two Segments, let fall Io perpendicular on EF cutting it in o; then o will be the Center, and Io the Radial of the Vaniſhing Line EF.

[b] Lem. 1.

Dem. Becauſe the Radials of the Vaniſhing Points x and y muſt make together an Angle equal to the Angle ſubtended by x and y, thoſe Radials muſt meet ſomewhere in the Segment xIy which contains that Angle; and for the ſame reaſon, the Radials of the Vaniſhing Points y and z muſt meet ſomewhere in the Segment yIz; and in regard theſe Radials muſt all meet in ſome one Point which repreſents the Eye, they muſt therefore meet in I, the Interſection of the Segments xIy and yIz; the Point I therefore repreſents the Place of the Eye, and Io is therefore the Radial, and o the Center of the Vaniſhing Line EF. *Q. E. I.*

COR.

If on xz, a Segment were drawn, containing an Angle equal to that ſubtended by x and z, that Segment would cut the other two Segments in their common Interſection I; ſeeing the Angle xIz, made by the Radials Ix and Iz, muſt fall in the ſame Point I. If therefore the Segment xIz be drawn, either of the other two Segments, as yIz, will be ſufficient to determine the Point I.

Fig. 95. N°. 2.

PROP. XVIII. PROB. VII.

Fig. 96. N°. 1.

Any Trapezium ACDB being given; thence to find the Poſition of a Vaniſhing Line, with reſpect to which, the given Trapezium ſhall repreſent a Parallelogram.

Produce the oppoſite Sides CD and AB, which, if they be not parallel, will meet in ſome Point z; likewiſe produce AC and BD, which, if not parallel, will meet in ſome other Point x; then EF drawn through x and z, will be the Vaniſhing Line of a Plane, in which the Trapezium ACDB repreſents a Parallelogram.

Dem. For by reaſon of the Vaniſhing Points z and x, the Sides CD, AB, repreſent parallel Lines, as do alſo the Sides AC and BD. *Q. E. I.*

COR. 1.

Fig. 96. N°. 2.

If the oppoſite Sides CD and AB be parallel, their Vaniſhing Point being then infinitely diſtant, the Vaniſhing Line EF muſt be drawn through x, the Point of Concourſe of AC and BD, parallel to AB[c]; but if AC and BD be alſo parallel, then the Plane required hath no Vaniſhing Line, and muſt therefore be parallel to the Picture[d], and the Image ACDB is then Similar to its Original[e].

[c] Cor. 2. Lem. 22. B. III.

[d] Theor. 3. B. I.

[e] Cor. 3. Theor. 23. B. I.

COR. 2.

Fig. 96. N°. 1.

If either of the Vaniſhing Points, as z, ſhould be out of reach, the Vaniſhing Point y or v, of either of the Diagonals AD or BC, and thence the Vaniſhing Line EF, may be found in this manner.

Produce AD or BC to y or v, until Ay be Harmonically divided in A, S, D, and y, or Bv in B, S, C, and v; and EF drawn through x and either of the Points y or v, will be the Vaniſhing Line deſired.

For the Diagonals AD and BC of the Trapezium ACDB, are Harmonically divided

vided in A, S, D, and B, S, C, and their reſpective Interſections y and v, with the Line xz [a].

COR. 3.

Fig. 96. N°. 1.

If both the Vaniſhing Points x and z be out of reach, the Line EF may be found by the help of the Vaniſhing Point y, of the Diagonal AD, determined as in the preceeding Corollary; by drawing through y, a Line tending to the ſame inacceſſible Point z, with the Sides AB and CD, or to the ſame Point x, with the Sides AC and BD [b].

SCHOL.

By this Corollary, the Vaniſhing Line of a Plane may be found, which paſſes through two inacceſſible Vaniſhing Points, having the Images of two Lines, tending to each of thoſe Points, given.

Thus if Dz and Bz be given, tending to an inacceſſible Point z, and Dx, Cx, tending to another inacceſſible Point x; produce the given Lines, till, by their mutual Interſections, they form a Trapezium ACDB, and having drawn the Diagonals AD, BC, find the Vaniſhing Point y, of either of them AD, as is moſt convenient [c], and thereby the Vaniſhing Line required may be found as above directed.

COR. 4.

If either of the Diagonals BC, be biſected by the other in S, a Line drawn through y, x, or z, parallel to BC, will be the Vaniſhing Line ſought [d].

COR. 5.

Fig. 96. N°. 2.

If any two oppoſite Sides AB and CD be parallel, and the Vaniſhing Point x, of the contrary Sides AC and BD, be out of reach; a Line EF drawn through v, the Vaniſhing Point of the Diagonal BC, parallel to AB, will be the Vaniſhing Line required [e].

For the Vaniſhing Line ſought muſt paſs through v [f], and muſt be parallel to AB and CD [g].

SCHOL.

By this Corollary, if two Lines AC, BD, be given, tending to an inacceſſible Point x, a third Line EF tending to the ſame Point may be found, which ſhall be parallel to any propoſed Line X, not parallel to either of the two Lines firſt given.

For having drawn through AC and BD, any two Lines AB, CD, parallel to the Line X, forming with them a Trapezium ACDB, draw the Diagonals AD, BC; and having found the Vaniſhing Point v of the Diagonal BC [h], EF drawn through v, parallel to the Line X, will be the Line required.

If the Line X be parallel to AC or BD, the Line ſought will coincide with AC or BD, there being then no other Line which can anſwer the Condition required.

If Bx and Gx were the given Lines, inclining to each other in ſuch manner, that the required Line EF, parallel to the Line X, would fall between them; through either of the given Lines Bx, draw AB, CD, parallel to the Line X, and having taken any Point A, in either of theſe Parallels AB, through A draw a Line tending to x [i], cutting CD in C, whereby a Trapezium ACDB will be got, by the help of which the Point v, and thence EF may be found as before.

COR. 6.

Fig. 96. N°. 1.

Any Subdiviſions of the Figure ACDB, repreſenting Diviſions by Lines parallel to its Sides, may be found, by drawing through A, a Line nr parallel to EF, cutting DC and DB produced, in n and r; for then An and Ar being divided in the Proportion intended to be repreſented by the Diviſions of AC and AB, Lines drawn from thoſe Diviſions to z and x, will divide the Figure ACDB in the manner deſired [k].

The ſame may be done by a Line lm, drawn through D parallel to EF, and divided in the like Proportion; or it may be done by the Lines lm and nr thus divided, without the Points x and z.

COR. 7.

If by reaſon of the great Diſtance of all the Vaniſhing Points, no two of them could be had, to determine the Vaniſhing Line EF; yet the Lines lm and nr, and thence the Subdiviſions of the Figure ACDB may be obtained, by the help of a Parallel to EF, found in the following manner.

Take in either of the Sides AB, which contain the neareſt Inward Angle CAB

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[a] Lem. 22. B. III.

[b] Prob. 18. B. II.

[c] Cor. 2.

[d] Cor. 1. Lem. 22. B. III.

[e] Cor. 2. Lem. 22. B. III.

[f] Cor. 2.

[g] Cor. 1.

[h] Cor. 2.

[i] Prob. 18. B. II.

[k] Caſe 3. Prob. 20. B. II.

of the Figure, any Point p within reach, from whence draw pc parallel to the oppoſite Side CD, cutting AC in c, and through c draw cb parallel to the Diagonal CB which ſubtends the Angle CAB, cutting AB in b; from b draw bq parallel to BD, cutting AC in q; then a Line qp will be parallel to EF, whereby lm and nr may be found.

For in the Similar Triangles Acp, ACz, $Ac : AC :: Ap : Az$
And in the Similar Triangles Abq, ABx, $Ab : AB :: Aq : Ax$
But in the Similar Triangles Acb, ACB, $Ac : AC :: Ab : AB$
Therefore $Ap : Az :: Aq : Ax$

wherefore the Triangles Aqp, Axz, are Similar, and conſequently qp is parallel to xz or EF[a].

[a] 2 El. 6.

It is evident from the Conſtruction, that the Figures Acdb and ACDB are Similar, and that cp and bq interſect in d, a Point of the Diagonal AD; ſo that bq may be found by drawing it through d, without drawing cb.

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Hence, if any irregular quadrilateral Piece of Ground were propoſed to be divided into Walks, Allies, Rows of Trees, *&c.* ſo as to appear to anſwer the moſt regularly to each other as the Ground could admit of; the Sides and either of the Diagonals being exactly meaſured, and the Plan laid down by a Scale on Paper, the Lines, correſponding to lm and nr, may be found by the Method laſt propoſed, let the Interſections of the Sides be ever ſo far diſtant, and by this means the proportional Diviſions of the Sides may be determined.

PROP. XIX. PROB. VIII.

Fig. 96. N°. 1.

The ſame Things being ſuppoſed as in the laſt Propoſition; thence to find the Center and Diſtance of the Vaniſhing Line EF, requiſite to make the Figure ACDB repreſent a Square.

On xz, the Diſtance between the Vaniſhing Points of the Sides of the Figure, as a Diameter, deſcribe a Semicircle xIz, and on vy, the Diſtance between the Vaniſhing Points of the Diagonals, deſcribe another Semicircle vIy, cutting the Semicircle xIz in I, from whence let fall Io perpendicular on EF, cutting it in o; then o will be the Center, and Io the Diſtance of the Vaniſhing Line EF, required.

Dem. For by reaſon of the Semicircle xIz, the Angle xIz is Right, therefore the Angles CAB, CDB of the Figure ACDB repreſent Right Angles; and becauſe of the Semicircle vIy, the Angle vIy being Right, the Angle CSD, made by the Diagonals AD and BC, alſo repreſents a Right Angle; and conſequently the Figure ACDB repreſents a Square, in a Plane whoſe Vaniſhing Line is EF, o its Center, and Io its Diſtance. *Q. E. I.*

COR. 1.

If the Figure ACDB be required to repreſent a Rectangular Parallelogram, whoſe Diagonals make together an inward Angle CSD, repreſenting any Angle propoſed; the Semicircle xIz muſt be drawn as before, but inſtead of drawing a Semicircle on vy as a Diameter, a Segment of a Circle muſt be drawn on vy, capable of containing the Angle propoſed to be repreſented by CSD; and the Interſection of this Segment with the Semicircle xIz, will give the Place of the Eye, from whence a Perpendicular drawn to the Vaniſhing Line EF, will cut it in its Center, and determine its Diſtance.

COR. 2.

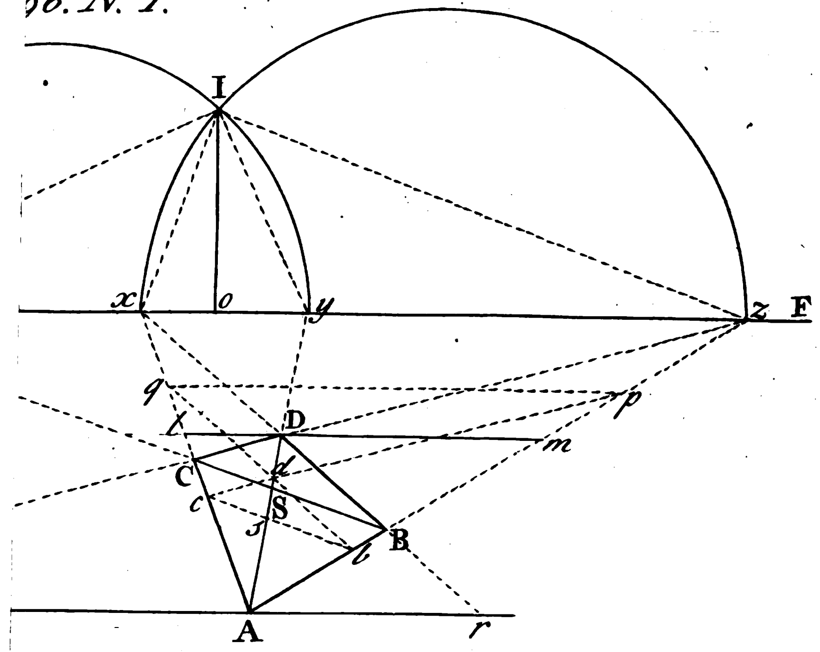
If the Figure ACDB were required to repreſent an Oblique Angled Parallelogram, having its inward Angle repreſented by CAB or CDB of any given Bigneſs, and the inward Angle of the Diagonals alſo of a given Size; there muſt be drawn on xz and vy Segments of Circles, capable reſpectively of containing the required Angles, and their Interſection will determine the Place of the Eye.

COR. 3.

If either of the Sides AB of the given Figure, be parallel to EF, and the Angle CAB be intended to repreſent a Right Angle; the Vaniſhing Point x of the other Sides AC and BD, will be the Center of the Vaniſhing Line[b], and a Line drawn from x, perpendicular to EF, will cut the Segment vIy in the Place of the Eye.

[b] Prob. 3. B. II.

In



96. N°1.

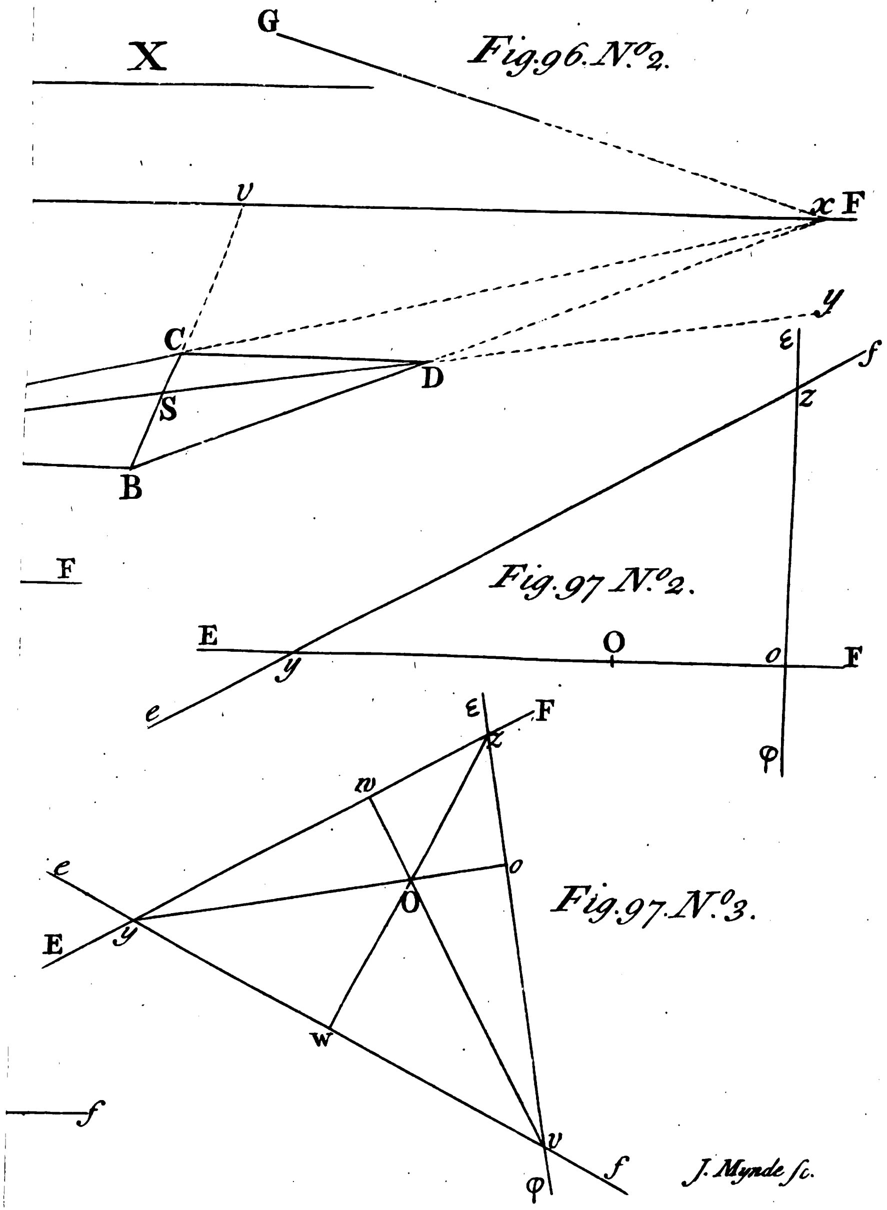


Fig.96.N°2.

Fig.97 N°2.

Fig.97.N°3.

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In like manner, if either of the Diagonals BC, be parallel to EF, and the Angle CSD be to repreſent a Right Angle; the Vaniſhing Point y of the other Diagonal AD, will be the Center of the Vaniſhing Line, and a Perpendicular drawn from thence to EF, will cut the Segment xIz in the Place of the Eye.

COR. 4.

If the Side AB of the Figure be parallel to EF, and the Angle CAB be intended to repreſent an Oblique Angle; a Line muſt be drawn from x, making with EF an Angle equal to the Angle propoſed, and inclining to EF the contrary way, to which the Originals of CA and AB are ſuppoſed to incline[a], and the Line thus drawn from x, will cut the Segment vIy in the Place of the Eye.

The ſame is to be underſtood of the Diagonals, when either of them is parallel to EF.

DEF. 8.

If a Vaniſhing Line EF be given; then, by the *Planes* EF are meant all Planes in general, which have EF for their Vaniſhing Line; and as amongſt theſe, there muſt always be one which paſſes through the Eye, and is the Vaniſhing Plane of all the reſt; if this particular Plane be conſidered as an Original Plane, it is called the *Plane* EF, in the ſingular Number, that Line being the whole Image of that Plane[b].

DEF. 9.

If any Vaniſhing Point x be given; then, by the *Lines* x, are meant all Lines in general, which have x for their Vaniſhing Point; and as amongſt theſe, there muſt be one which paſſes through the Eye, and is the Radial of all the reſt; if this laſt be conſidered as an Original Line, it is called the *Line* x, the Point x being the whole Image of that Line[c].

PROP. XX. PROB. IX.

The Center and Diſtance of the Picture, and a Vaniſhing Line EF being given; thence to find the Vaniſhing Point of Lines perpendicular to the Planes EF.

Fig. 97. N°. 1.

Through O the Center of the Picture, draw the Vertical Line xb, perpendicular to EF, cutting it in o, and IO parallel to EF, and equal to the Diſtance of the Picture; and having drawn Io, draw Ix perpendicular to it, which will cut xo in x the Vaniſhing Point ſought.

Dem. Becauſe O is the Center, and IO the Diſtance of the Picture; if the Triangle oIx be turned upon the Line xo, until it become perpendicular to the Plane of the Picture, then I, will come into the Place of the Eye, and the Triangle xIo, in this Poſition, will coincide with the Vertical Plane, and Io will be the Radial of the Vaniſhing Line EF; wherefore Ix, drawn perpendicular to Io, muſt cut the Vertical Line xo in x, the Vaniſhing Point of Lines perpendicular to all Planes whatſoever, whoſe Vaniſhing Line is EF[d]. *Q. E. I.*

COR. 1.

If EF paſs through O the Center of the Picture, that is, if the Planes EF be perpendicular to the Picture[e]; the Lines Io and IO coinciding, Ix will be parallel to xo, and the Point x will therefore be infinitely diſtant in the Line Oo[f]: if the Planes EF be parallel to the Picture, that is, if EF be infinitely diſtant[g]; the Point o being then at an infinite Diſtance from O, the Lines Io and Oo will be parallel, wherefore Ix will coincide with IO, and x with O[h].

COR. 2.

The Vaniſhing Point of Lines perpendicular to any Plane, is in the Vertical Line of that Plane[i].

COR. 3.

All Vaniſhing Lines which paſs through x, are Vaniſhing Lines of Planes perpendicular to the Planes EF; and no Planes can be perpendicular to the Planes EF, but ſuch only, whoſe Vaniſhing Lines either paſs through x[k], or are parallel to Oo, when x is infinitely diſtant[l].

COR. 4.

The Vaniſhing Point x is perpendicular to every Point in the Vaniſhing Line EF[m].

For

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[a] Caſe 2. Prob. 3. B. II.

[b] Theor. 17. B. I.

[c] Theor. 8. B. I.

[d] Prop. 7. and Cor. 1, 4.

[e] Cor. 1. Theor. 9. B. I.

[f] Cor. 2. Prop. 7.

[g] Theor. 3 B. I.

[h] Cor. 3. Prop. 7.

[i] Prop. 7.

[k] Prop. 8. and Cor.

[l] Cor. 1.

[m] Def. 7.

For if any Line, whoſe Vaniſhing Point is x, cut the Planes EF in any Point, that Line will be perpendicular to every Line in the Planes EF, which paſſes through that Point[a]; but from any ſuch Point, Lines may be drawn in the Planes EF, to any Vaniſhing Point in EF; wherefore the Point x is perpendicular to every Point in EF[b].

[a] Def. 3 El. 11.

[b] Cor. 3. Theor. 11. B. I.

COR. 5.

If two Planes be perpendicular to each other, but neither of them perpendicular to the Picture, their Vaniſhing Lines cannot be perpendicular.

Becauſe no Vaniſhing Line drawn through x, can be perpendicular to EF, ſave only $x\,o$, which paſſes through O the Center of the Picture, and whoſe Planes are therefore perpendicular to the Picture, as well as to the Planes EF[c].

[c] Cor. 2. Theor. 9. B. I.

PROP. XXI. PROB. X.

Fig. 97. N°. 1.

The Center and Diſtance of the Picture, and a Vaniſhing Point x being given; thence to find the Vaniſhing Line of Planes perpendicular to the Lines x[d].

[d] Def. 9.

From x, through O the Center of the Picture, draw $x\,O$, and having drawn IO perpendicular to it, and equal to the Diſtance of the Picture, draw Ix, and Io, perpendicular to it, cutting $x\,O$ in o, through which draw EF perpendicular to $x\,o$, and EF will be the Vaniſhing Line deſired.

Dem. This is evident, for, by the Conſtruction, x is the Vaniſhing Point of Perpendiculars to the Planes EF[e], and conſequently the Planes EF are perpendicular to all Lines whoſe Vaniſhing Point is x. *Q. E. I.*

[e] Prop. 20.

COR. 1.

If x were in the Center of the Picture, the Vaniſhing Line EF would be infinitely diſtant: if x were infinitely diſtant, that is, if the Lines propoſed, were parallel to the Picture, and the Image of one of them were given, the Vaniſhing Line EF would paſs through the Center of the Picture[f], and be perpendicular to the given Image; the Planes in this laſt Caſe required, being perpendicular to the Picture, as well as to the Original Line; which Line, as well as its Image[g], muſt therefore be parallel to the Vertical Line of thoſe Planes[h], and conſequently perpendicular to their Vaniſhing Line.

[f] Cor. 1. Prop. 20.

[g] Theor. 2. B. I.

[h] Cor. 1. Theor. 9. B. I.

COR. 2.

No other Vaniſhing Line beſides EF, can be a Vaniſhing Line of *Planes perpendicular* to the Lines x.

Becauſe the Lines x cannot be perpendicular to any Planes, but to ſuch as are parallel between themſelves, and which have therefore all the ſame Vaniſhing Line EF[i].

[i] Theor. 13. B. I.

COR. 3.

The Line EF is the Vaniſhing Line of Planes, perpendicular to all Planes whatſoever, whoſe Vaniſhing Lines paſs through x[k].

[k] Cor. 3. Prop. 20.

PROP. XXII. PROB. XI.

The Center and Diſtance of the Picture, and any two Vaniſhing Lines being given; thence to find the Vaniſhing Line of Planes perpendicular to thoſe whoſe Vaniſhing Lines are given.

Fig. 97. N°. 2, 3.

Let O be the Center of the Picture, and EF and ef the given Vaniſhing Lines, Interſecting in y.

Find $\varepsilon\varphi$, the Vaniſhing Line of Planes perpendicular to the Lines y[l], and that will be the Vaniſhing Line required.

[l] Prop. 21.

Dem. Becauſe y is the Vaniſhing Point of the common Interſections of the Planes EF and ef[m], the Planes $\varepsilon\varphi$ being perpendicular to thoſe Interſections, are therefore perpendicular to the Planes EF and ef[n]. *Q. E. I.*

[m] Theor. 16. B. I.

[n] 18 and 19 El. 11.

COR. 1.

Fig. 97. N°. 1.

If the given Vaniſhing Lines EF and ef be parallel, then $\varepsilon\varphi$, drawn through O the Center of the Picture, perpendicular to EF or ef, will be their common Vertical Line[o], and is therefore the Vaniſhing Line of Planes perpendicular to the Planes EF and ef[p].

[o] Cor. 2. Theor. 14. B. I.

[p] Cor. Def. 11. B. I.

COR. 2.

If two Planes be perpendicular, and neither of their Vaniſhing Lines paſs through the Center of the Picture; then the Vaniſhing Point of Lines perpendicular to either of thoſe Planes, will be in the Interſection of the Vertical Line of that Plane, with the Vaniſhing Line of the other.

Let the Planes E F and *ef* be perpendicular; then the Vaniſhing Line E F muſt paſs through the Vaniſhing Point of Perpendiculars to the Planes *ef*[a]; but this Vaniſhing Point is alſo in w*z*, the Vertical Line of the Planes *ef*[b], and is therefore in *z*, the Interſection of w*z* with E F; and for the ſame reaſon, the Vaniſhing Point of Perpendiculars to the Planes E F, is in *v*, the Interſection of the Vertical Line *wv* of the Planes E F with the Vaniſhing Line *ef*. Fig. 97. N°. 3.

[a] Cor. 3. Prop. 20.

[b] Cor. 2. Prop. 20.

Thus alſo, if the Vaniſhing Lines E F and *ef* be parallel, and their Planes perpendicular; the Vaniſhing Points of Perpendiculars to the Planes E F and *ef*, will be at *x* and *o*, the Interſections of *ef* and E F with *xo* their common Vertical Line. Fig. 97. N°. 1.

COR. 3.

If E F and εφ be the given Vaniſhing Lines, and either of them εφ, paſs through the Center of the Picture; then *xo* the Vertical Line of the Planes E F, coinciding with εφ[c], the Point *x* cannot be determined by their Interſection, but is found in εφ as already ſhewn[d]; and the Vertical Line I O of the Planes εφ, being parallel to E F, their Interſection is infinitely diſtant, ſo that the Perpendiculars to the Planes εφ have no Vaniſhing Point, but are parallel to the Picture. Fig. 97. N°. 1.

[c] Cor. 3. Theor. 16. B. I.

[d] Prop. 20.

COR. 4.

If three Planes be perpendicular to each other, and neither of them perpendicular to the Picture; the Vertical Line of any one of thoſe Planes will cut the Vaniſhing Lines of the other two Planes in their common Interſection.

Let the Planes E F, *ef*, and εφ be perpendicular to each other. Then becauſe the Planes εφ and E F are perpendicular, the Vaniſhing Point *y* of Perpendiculars to the Planes εφ is in E F[e]; and becauſe the Planes εφ and *ef* are perpendicular, the Point *y* is in *ef*; but the Point *y* is alſo in *oy*, the Vertical Line of the Planes εφ[f], the Vertical Line *oy* therefore paſſes through the Interſection of E F with *ef*. Fig. 97. N°. 3.

[e] Cor. 3. Prop. 20.

[f] Cor. 2. Prop. 20.

After the ſame manner it may be ſhewn, that the Vertical Lines *wv*, w*z*, of the Planes E F and *ef*, paſs through *v* and *z* the Interſections of *ef* and E F with εφ.

COR. 5.

If three Planes be perpendicular to each other, and neither of them parallel to the Picture, and the Vaniſhing Line of one of them εφ, paſs through the Center of the Picture; the other two Vaniſhing Lines E F and *ef* will be parallel to each other, and perpendicular to εφ. Fig. 97. N°. 1.

For the Planes εφ being perpendicular to the Picture, the Vaniſhing Lines of all Planes perpendicular to the Planes εφ, not parallel to the Picture, are perpendicular to the Vaniſhing Line εφ[g], and conſequently parallel to each other.

[g] Cor. 3. Theor. 16. B. I.

Alſo if two of the Vaniſhing Lines E F and *ef* be parallel, the third εφ muſt paſs through the Center of the Picture, and be perpendicular to them[h].

[h] Cor. 1.

COR. 6.

If three Planes be perpendicular to each other, and the Vaniſhing Lines of two of thoſe Planes paſs through the Center of the Picture, the third Plane will be parallel to the Picture, and hath no Vaniſhing Line.

For this laſt Plane muſt be perpendicular to the common Interſection of the other two Planes[i], which Interſection is perpendicular to the Picture[k].

[i] 19 El. 11.

[k] Cor. 1. Theor. 16. B. I.

COR. 7.

The Vaniſhing Line of Planes perpendicular to any Number of other Planes, muſt paſs through the Vaniſhing Points of Perpendiculars to each of thoſe Planes.

Becauſe the Vaniſhing Point of Perpendiculars to any Plane, muſt be in the Vaniſhing Line of Planes perpendicular to that Plane[l].

[l] Cor. 3. Prop. 20.

PROP. XXIII. PROB. XII.

The Center and Diſtance of the Picture, and any two Vaniſhing Lines being given; thence to find the Angle of Inclination of the Planes of thoſe Vaniſhing Lines to each other,

Fig. 98. N°. 1, 2. Let O be the Center, and OI the Diſtance of the Picture, and let EF and ef be the given Vaniſhing Lines interſecting in y.

Find $\varepsilon\varphi$, the Vaniſhing Line of Planes perpendicular to the Planes EF and ef[a], cutting EF and ef in v and z; and having found Io, the Radial of the Vaniſhing Line $\varepsilon\varphi$[b], draw Iv and Iz, and the Angle zIv will be the Angle of Inclination of the Planes EF to the Planes ef.

Dem. Becauſe the Planes $\varepsilon\varphi$ are perpendicular to the Planes EF and ef, the Interſections of the Planes $\varepsilon\varphi$ with the Planes EF and ef, determine the Angle of Inclination of thoſe Planes to each other[c]; but z and v are the Vaniſhing Points of thoſe Interſections, therefore the Angle zIv, ſubtended by z and v, is the Angle of Inclination ſought. *Q.E.I.*

[a] Prop. 22.

[b] Prop. 13.

[c] Def. 19. B.I.

COR. 1.

Fig. 98. N°. 3. If the Interſection of the given Vaniſhing Lines be in o the Center of either of them EF, and that Center be not the Center of the Picture; then having found x, the Vaniſhing Point of Perpendiculars to the Planes EF[d], through x draw $\varepsilon\varphi$ parallel to EF, cutting ef in z, and having found Ix, the Radial of the Vaniſhing Line $\varepsilon\varphi$[e], draw Iz, and the Angle Izx will be the Angle of Inclination of the Planes EF and ef.

For $\varepsilon\varphi$ being the Vaniſhing Line of Planes perpendicular to the Vaniſhing Point o[f], it is the Vaniſhing Line of Planes perpendicular to the Planes EF and ef[g]: but the Interſections of the Planes $\varepsilon\varphi$ with the Planes EF, are Lines in the Planes $\varepsilon\varphi$ parallel to the Picture, and conſequently to $\varepsilon\varphi$[h]; and the Interſections of the Planes $\varepsilon\varphi$ with the Planes ef, being Lines in the Planes $\varepsilon\varphi$, which have z for their Vaniſhing Point, the Angle Izx is therefore the Angle made by the Interſections of the Planes $\varepsilon\varphi$ with the Planes EF and ef[i], and is conſequently the Angle of Inclination of the Planes EF and ef to each other.

[d] Prop. 20.

[e] Prop. 13.

[f] Prop. 21.

[g] Prop. 22.

[h] Theor. 15. and Cor. 1. B. I.

[i] Theor. 11. B. I.

COR. 2.

If the given Vaniſhing Lines croſs in the Center of the Picture, the Angle they make together is the ſame with the Angle of Inclination of their Planes[k].

[k] Cor. 1. Theor. 16. B. I,

COR. 3.

Fig. 98. N°. 4. If the given Vaniſhing Lines EF and ef be parallel, the Vaniſhing Line $\varepsilon\varphi$ of Planes perpendicular to them, is their common Vertical Line[l]; and therefore the Radials Io and Iv, give vIo, the Angle of Inclination of thoſe Planes.

[l] Cor. 1. Prop. 22.

COR. 4.

Fig. 98. N°. 5. If the Vaniſhing Lines EF and ef, ſhould be ſo ſituated, that $\varepsilon\varphi$, the Vaniſhing Line of Planes perpendicular to their common Interſection y, ſhould be parallel to one of them, as ef; having found the Radial Iw of the Planes $\varepsilon\varphi$, and drawn Iv, draw In parallel to ef, and vIn will be the Angle of Inclination of the Planes EF and ef: or if ef and $\varepsilon\varphi$ be not parallel, but meet in a Point at an inacceſſible Diſtance, the Line In being drawn tending to that Point[m], will determine vIn the Angle deſired.

[m] Prob. 18. B. II.

COR. 5.

Fig. 98. N°. 5. If the Vaniſhing Lines EF and yz ſhould incline in ſuch manner, that their Interſections z and v with the Planes $\varepsilon\varphi$, ſhould ſubtend an Obtuſe Angle zIv; produce either of the Radials Iv beyond I to m, and mIz the Complement to two Rights of the Angle zIv, will be the Angle of Inclination ſought[n].

[n] Def. 19. B.I.

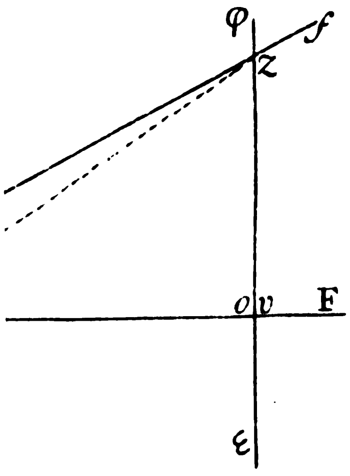
PROP. XXIV. PROB. XIII.

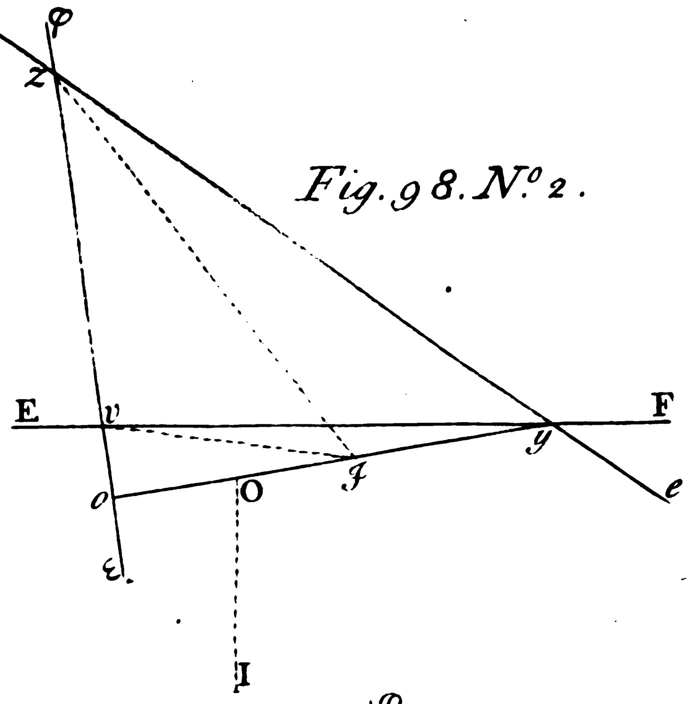
Fig. 99. N°. 1, 2. The Center O and Diſtance OI of the Picture, and any two Vaniſhing Lines EF and ef, of Planes perpendicular to each other, being given; thence to find the Vaniſhing Points of Lines in the Planes ef, which incline to the Planes EF in any given Angle Z.

CASE 1.

Fig. 99. N°. 1. When the Vaniſhing Line EF paſſes through O the Center of the Picture, and of conſequence is perpendicular to the Vaniſhing Line ef[o].

[o] Cor. 3. Theor. 16. B. I.





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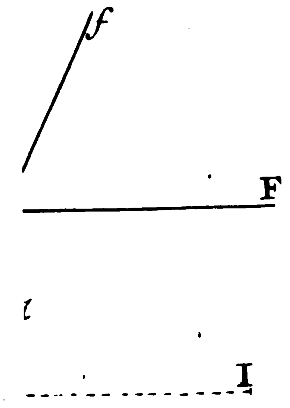

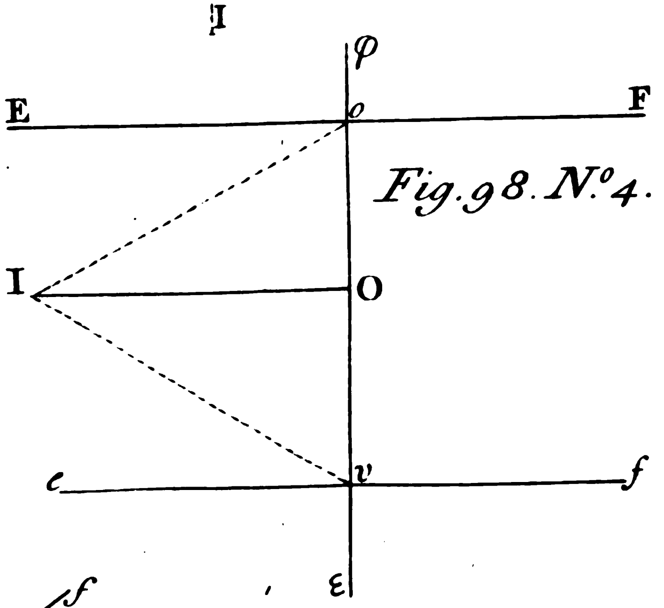

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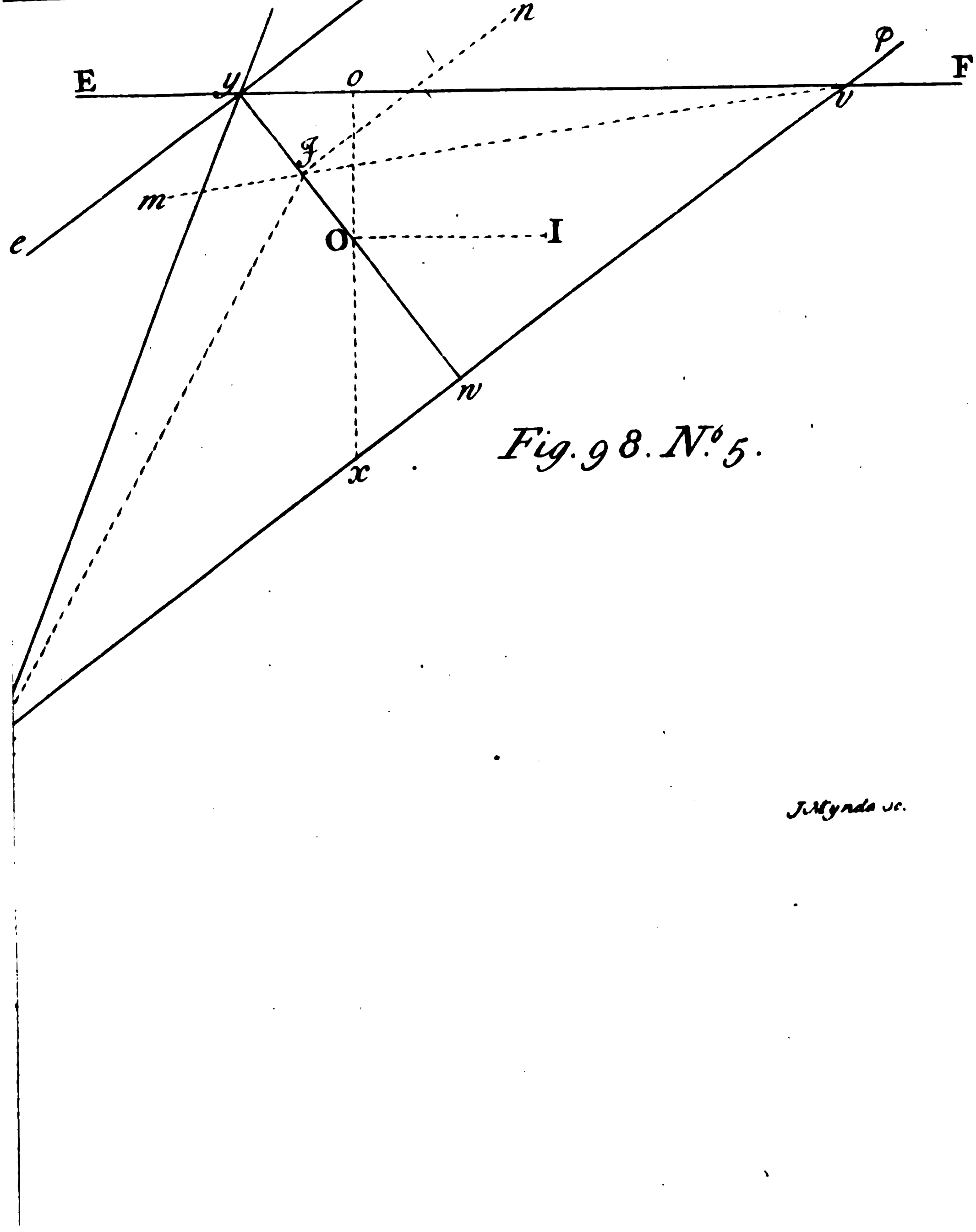

Fig. 98. No. 5.

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METHOD 1.

Find the Radial Iy of the Vaniſhing Line ef[a], and draw Iv, Iz, making with Iy the Angles vIy, yIz, each equal to the given Angle Z; and the Points v and z, where they cut ef, will be the Vaniſhing Points deſired.

Dem. For ef being a Vaniſhing Line of Planes perpendicular to the Planes EF, y is the Vaniſhing Point of the Perpendicular Seats of all Lines in the Planes ef on the Planes EF[b]; and therefore the Angle ſubtended by y, and any other Vaniſhing Point v or z in ef, is the Angle of Inclination of the Lines, whoſe Vaniſhing Points are v or z, to their Perpendicular Seats on the Planes EF, and conſequently to the Planes themſelves[c]; and this Angle being taken equal to the Angle Z, the Points v and z are therefore rightly determined: the Lines whoſe Vaniſhing Point is v, inclining upwards, and thoſe whoſe Vaniſhing Point is z, inclining downwards to the Planes EF, in the Angle propoſed. *Q. E. I.*

METHOD 2.

Having ſet off the Diſtance OI at d in the Line EF, draw dA, da, making the Angles AdO, Oda, each equal to the Angle Z, and cutting the Vertical Line OI in A and a; then draw the Radial Iy, and biſect the Angle OIy by the Line Iu, cutting EF in u, and Lines drawn from a and A through u, will cut ef in the ſame Points z and v as before.

Dem. Becauſe the Planes EF are perpendicular to the Picture, the Eye's Director is perpendicular to the Original Planes[d]; and conſequently the Foot of the Eye's Director or Point of Station, is the Center of the Circle in the Original Plane, whoſe Image forms the Place of the Vaniſhing Points of all Lines which incline to the Planes EF in the given Angle[e]; and the Angles OdA, Oda, being each made equal to the propoſed Angle Z, the Points A and a are the Images of the Extremities of the perpendicular Diameter of the forming Circle[f]; and in regard the Center of that Circle is in the Directing Line, the Images of all other Diameters of the forming Circle will be parallel to Aa[g], and therefore the Vaniſhing Line ef repreſents an Indefinite Diameter of that Circle, the Images of the Extremities of which, are therefore Points in the Image of the Circle; but the Angle OIy, being biſected by Iu, the Lines au, Au cutting ef in z and v, thereby determine the Images of the Extremities of the Diameter of the forming Circle repreſented by ef[h]; wherefore v and z are Points in the Image of the forming Circle, and conſequently Vaniſhing Points of Lines in the Planes ef inclining to the Planes EF in the Angle propoſed[i]. *Q. E. I.*

COR.

When the Vaniſhing Line EF paſſes through O the Center of the Picture; the Place of the Vaniſhing Points of Lines inclining to the Planes EF in any given Angle Z, muſt be two oppoſite *Hyperbolas*[k]; of which O is the Center, Aa the Tranſverſe Axe, and dd, equal to the double of OI, is the Conjugate Axe[l].

CASE 2.

When the Vaniſhing Line EF doth not paſs through the Center of the Picture.

Let O be the Center, and OI the Diſtance of the Picture, and let EF and ef be the given Vaniſhing Lines, the laſt of which muſt paſs through x, the Vaniſhing Point of Perpendiculars to the Planes EF[m]. Fig. 99. N°. 2.

METHOD 1.

Having found the Radial iw of the Planes ef[n], draw iy, and alſo iv, iz, making with iy, the Angles viy, yiz, each equal to the propoſed Angle Z; and the Points v and z will be the Vaniſhing Points required.

Dem. For the Planes EF and ef being perpendicular, y is the Vaniſhing Point of the Perpendicular Seats of all Lines in the Planes ef on the Planes EF; wherefore the Vaniſhing Points v and z which ſubtend with y Angles equal to the propoſed Angle Z, are the Vaniſhing Points ſought. *Q. E. I.*

METHOD 2.

Having drawn the Vertical Line xo of the Planes EF, and taken IO perpendicular to it, and equal to the Diſtance of the Picture, draw Io, and alſo IA and Ia, making with Io the Angles AIo, oIa, each equal to Z, and cutting xo in A and a; take

[a] Prop. 13.

[b] Theor. 16. B. I. and 38 El. 11.

[c] Def. 20. B. I.

[d] Cor. 1. Theor. 9. B. I.

[e] Prop. 9.

[f] Prop. 10.

[g] Cor. 4. Theor. 12. B. I.

[h] Cor. 2. Meth. 2. Prob. 24. B. II.

[i] Prop. 9.

[k] Cor. 2. Prop. 9.

[l] Caſe 2 and 3, Prob. 10. B. III.

[m] Cor. 3. Prop. 20.

[n] Prop. 13.

take oJ in the Vertical Line xo, equal to Io the Diſtance of the Vaniſhing Line EF, and draw Jy, then biſect the Angle oJy by the Line Ju, cutting EF in u, and Lines drawn from A and a through u, will cut ef in v and z the Vaniſhing Points deſired.

Dem. Becauſe x is the Vaniſhing Point of Perpendiculars to the Planes EF, it repreſents the Center of the Circle in the Original Plane, whoſe Image forms the Place of the Vaniſhing Points of all Lines which incline to the Planes EF in the propoſed Angle[a]; and the Angles AIo, oIa, being each made equal to that Angle, the Points A and a are the Images of the Extremities of the perpendicular Diameter of the forming Circle[b]; and in regard that x is the Image of its Center, all Lines drawn through x, repreſent Indefinite Diameters of that Circle, the Extremities of which are Points in the Image of the Circle; wherefore the Vaniſhing Line ef, being conſidered as the Indefinite Image of a Diameter of the forming Circle, its Extremities v and z found by the Method here propoſed, are the Vaniſhing Points required. *Q. E. I.*

COR. 1.

Fig. 99. N°. 2.

When the Angle aIo, or Z, is leſs than the Angle IoO, which is the Angle of Inclination of the Planes EF to the Picture[c]; the Place of the Vaniſhing Points of Lines, inclining to the Planes EF in the propoſed Angle, muſt be two oppoſite *Hyperbolas*[d], of which Aa, terminated by the Vaniſhing Points which ſubtend the propoſed Angle with o[e], will be the Tranſverſe Axe; which being biſected in S, a Line DD drawn through S, parallel to EF, will be the Indefinite ſecond Axe; the Length of which is found, by making each Moiety SD, in the ſame Proportion to Io the Diſtance of the Vaniſhing Line EF, as the Semitranſverſe Axe SA, is to a mean Proportional between Ao and oa, the Segments of that Axe by the Vaniſhing Line EF[f].

The ſecond Axe DD is in this Caſe eaſily found, by deſcribing on Aa as a Diameter, a Semicircle Aga, cutting EF in g, and from d the Point of Diſtance of o, drawing dq parallel to gS, which will be equal to SD.

For in the Similar Triangles dqo, gSo, $dq : do = Io :: gS = SA : go = \sqrt[2]{Ao \times oa}$[g]. And conſequently dq is equal to SD.

COR. 2.

Fig. 99. N°. 3.

When the Angle aIo, or Z, is equal to the Angle IoO; the Place of the Vaniſhing Points is a *Parabola*[h]. of which Ax is the Axe, and A the Vertex[i]; x repreſenting the Center of the forming Circle, and xA the determinate Image of the perpendicular Semidiameter of that Circle; the other Extremity of which Diameter being in the Point of Station, its Image is infinitely diſtant[k]; as is the Interſection of Ia with ox, which Lines are parallel, the alternate Angles aIo, IoO being by Suppoſition equal; and the Line Pp, drawn through x parallel to EF, repreſents the Diameter of the forming Circle, which is parallel to the Directing Line; and each Moiety xP, xp, being made equal to Io, gives its Extremities P and p, and Pp is therefore a double Ordinate to the Axe Ax[l]; whence the intire *Parabola* may be determined[m].

In this Caſe, the Points v and z, in any given Vaniſhing Line ef which paſſes through x, may be found, by ſetting off the Diſtance yJ of the Vaniſhing Point y, on the Line EF at i, and from i drawing ip, iP, which will cut ef in v and z, if the laſt of theſe Points be within reach.

For Pp repreſenting a Diameter of the forming Circle, parallel to the Directing Line, and ef repreſenting an Indefinite Diameter of the ſame Circle; the Extremities of this laſt Diameter are rightly determined by the Method here propoſed[n].

The Points v and z may be alſo determined, by biſecting the Angle oJy, by the Line Ju, and uſing the Point u as before[o]; only obſerving, that as here, one of the Extremities of the perpendicular Diameter of the forming Circle, is the Directing Point of Ax; the Line uv which ought to meet Ax in that Point, muſt be drawn parallel to it[p], whereby the Point v will be found, and uA will give z if within reach.

And here, one Extremity of the Diameter of the forming Circle repreſented by xA, being in the Directing Line, the Point A biſects xo, the Angles aIo, oIA, are equal to the Angle Z, and the Angles AIx, xIa are equal to its Complement[q].

COR. 3.

Fig. 99. N°. 4.

When the Angle aIo, or Z, is larger than IoO; the Place of the Vaniſhing Points is an *Ellipſis* or a Circle[r]; and the propoſed Angle being ſet off on each Side of Io, the Line Ia will meet xo in a Point a, beyond x, and IA will meet it in A between x and o;

then

[a] Prop. 9.

[b] Prop. 10.

[c] Theor. 9. B. I.

[d] Cor. 2. Prop. 9.

[e] Prop. 10.

[f] Cor. 1 and 2. Part firſt of Prob. 10. B. III. and Caſe 3. of that Prob.

[g] 13 El. 6.

[h] Cor. 2. Prop. 9.

[i] Caſe 2. Prob. 6. B. III.

[k] Cor. 3. Prob. 5. B. III.

[l] Cor. 3. Prob. 6. B. III.

[m] Caſe 2. Prob. 6 and Prop. 17. B. III.

[n] Cor. 4. Meth. 2. Prob. 24. B. II.

[o] Meth. 2. Caſe 2. of this Prop.

[p] Cor. 4. Theor. 12. B. I.

[q] Cor. 2. Prop. 10.

[r] Cor. 2. Prop. 9.

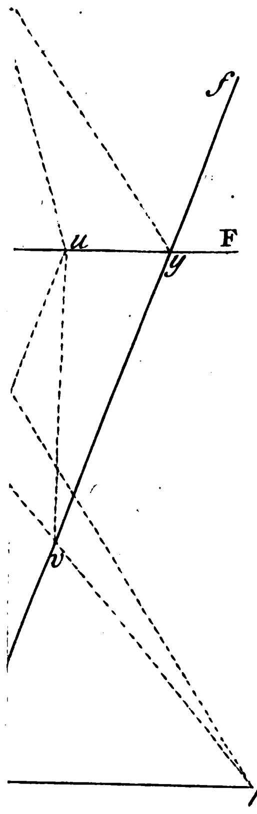


Fig.99.N°.4.

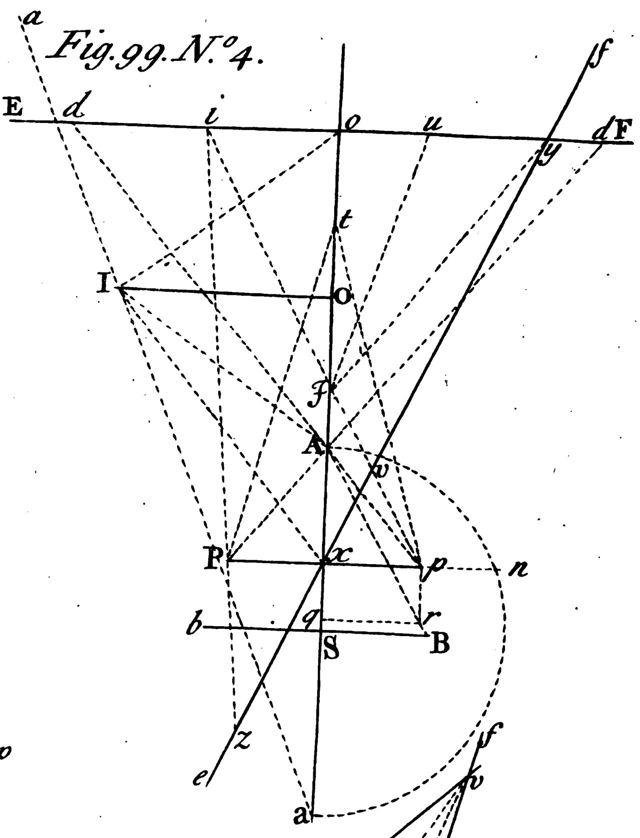
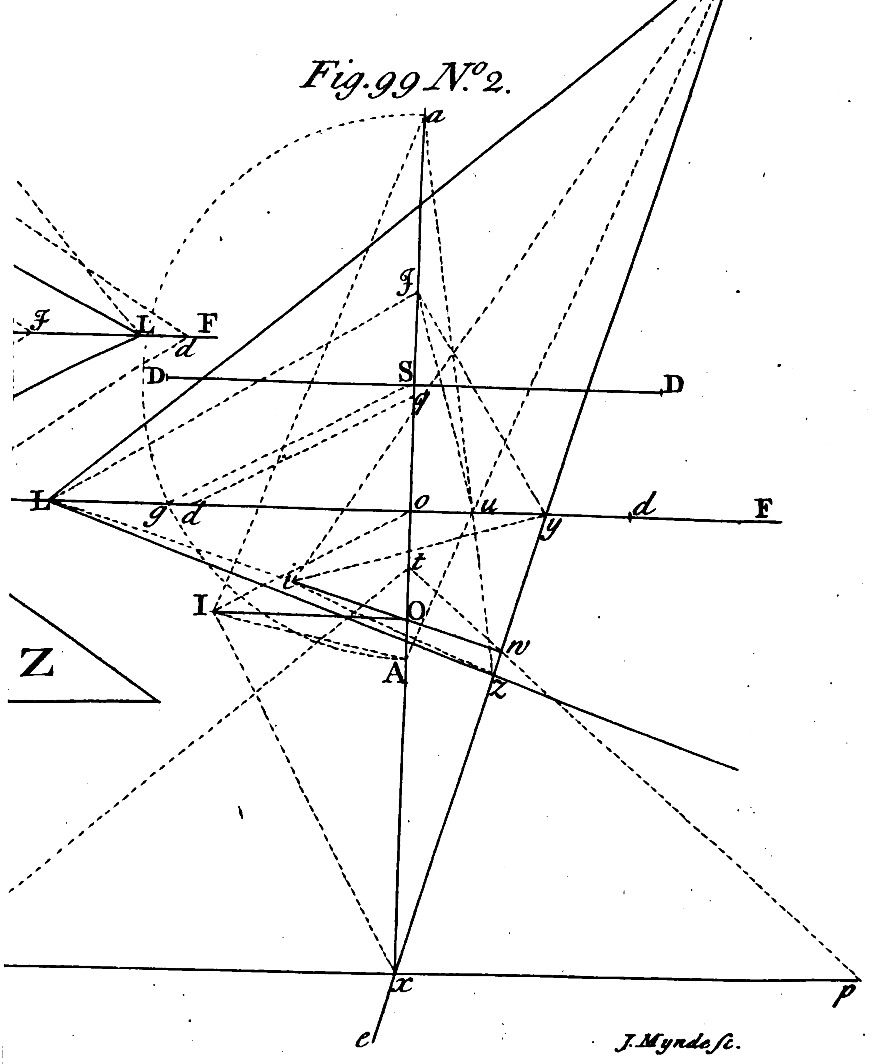


Fig.99.N°.2.



J. Mynde sc.

then A a will be one of the Axes of the generated Curve, it repreſenting the perpendicular Diameter of the forming Circle[a]; and if through x, Pp be drawn parallel to E F and terminated in P and p, by dA, dA, drawn through A and the Points of Diſtance d, d, of the Vaniſhing Point o; Pp will repreſent the Diameter of the forming Circle which is parallel to the Directing Line, and conſequently a double Ordinate to the Axe A a; and if A a be biſected in S, that will be the Center of the Curve, whence the Conjugate Axe B b may be found as in the Figure[b].

The Extremities v and z of any Indefinite Diameter ef are found, either by the Point i and the Extremities P and p of the Diameter Pp, or by the Point u and the Extremities of the Diameter A a, as in the laſt Corollary.

And here, A a repreſenting a Diameter of the forming Circle, which lies wholly on the ſame Side of the Directing Line, the Angles a I x, x I A, are each equal to the Complement of the Angle Z[c].

C O R. 4.

Fig. 99. N°. 2, 3, 4.

If from x a Diſtance xt be ſet off on xo, equal to I x the Radial of the Vaniſhing Point x; the Radials drawn from t to P and p, will make with Pp, the Angles t P x, tpx, each equal to the Angle a I o, or Z.

For Pp repreſenting a Diameter of the forming Circle parallel to the Directing Line, the Angles P tx, xtp, are each equal to the Complement of the Angle Z[d], wherefore t P x, tpx, the Complements of theſe Angles, are each equal to Z.

PROP. XXV. PROB. XIV.

Fig. 99. N°. 1, 2.

The Center and Diſtance of the Picture, and any Vaniſhing Line E F being given; from any given Point L in that Line, to draw two Vaniſhing Lines of Planes, which incline to the Planes E F, in any propoſed Angle Z.

C A S E 1.

When the given Vaniſhing Line E F paſſes through O the Center of the Picture.

Fig. 99. N°. 1.

Draw the Radial I L, and perpendicular to it draw I y, cutting E F in y, and having through y drawn ef perpendicular to E F, find the Points v and z in the Line ef, which ſubtend with y, Angles equal to the propoſed Angle Z[e], and from L draw L v, L z, and theſe will be the Vaniſhing Lines ſought.

Dem. Becauſe the Vaniſhing Points L and y are perpendicular, the Originals of L v and L z are perpendicular to the Original of ef[f]; and the Points v and z repreſenting the Extremities of a Diameter of the Circle in the Original Plane which forms the Vaniſhing Points ſought[g], L v and L z paſſing through thoſe Points, repreſent Tangents to the forming Circle[h], and are therefore Tangents to its Image from the Point L; and conſequently L v and L z are Vaniſhing Lines of Planes inclining to the Planes E F in the propoſed Angle Z[i].

Or thus: The Planes ef being perpendicular to the Planes L v and L y[k], the Interſections of the Planes L v and L y with the Planes ef, determine their Angle of Inclination[l]; but v and y are the Vaniſhing Points of theſe Interſections[m], and they ſubtending an Angle v J y equal to the propoſed Angle Z, the Planes L v and L y therefore incline to each other in that Angle.

The ſame Demonſtration ſerves alſo for the Vaniſhing Line L z. *Q. E. I.*

C O R. 1.

If the propoſed Point L, were in O the Center of the Picture, the Vaniſhing Lines ſought muſt make with E F, Angles equal to Z[n], that is, they muſt be drawn through O parallel to dA and da; theſe Vaniſhing Lines are therefore the Aſymptotes of the *Hyperbolas* produced by the Image of the forming Circle[o], which Lines are conſidered as Tangents to the oppoſite *Hyperbolas* at an infinite Diſtance[p], the Point y in this Caſe being alſo infinitely diſtant.

C O R. 2.

If the propoſed Point L were imagined to be at an infinite Diſtance in the Line E F, that is, if the Vaniſhing Lines ſought were required to be parallel to E F, they muſt be drawn through A and a.

For Lines drawn through A and a, the Vertices of the oppoſite Sections, parallel to the ſecond Axe dd, are Tangents to the Sections in A and a[q].

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[a] Caſe 2. Prob. 1. B. III.

[b] Cor. 6. Meth. 5. Prop. 16. B. III.

[c] Cor. 1. Prop. 10.

[d] Cor. 1. Prop. 10.

[e] Caſe 1. Prop. 24.

[f] Cor. 3. Theor. 11. B. I.

[g] Meth. 2. Caſe 1. Prop. 24.

[h] 16 El. 3.

[i] Prop. 11.

[k] Prop. 22.

[l] Def. 19. B. I.

[m] Theor. 16. B. I.

[n] Cor. 1. Theor. 16. B. I.

[o] Hyperb. Art. 6. B. III.

[p] Hyperb. Art. 5. B. III.

[q] Hyperb. Art. 1. B. III.

COR. 3.

No Vaniſhing Line of Planes which incline to the Planes EF in the Angle Z, can make with EF a greater Angle than the Angle Z.

For wherever the Point L is taken in EF, Ly the Hypotenuſe of the Right Angled Triangle LIy, will be larger than the Side Iy, which is the Diſtance of the Vaniſhing Point y, and which being ſet off at $\mathcal{Y}$ in the Line EF, determines $v\mathcal{Y}y$ the Angle ſubtended by v and y, equal to the Angle AdO, or Z; which is therefore larger than the Angle vLy, except only when L coincides with O, in which Caſe thoſe Angles are equal[a], the Lines Iy and Ly being both infinite.

COR. 4.

If the Angle Z be Right, there can but one Vaniſhing Line be drawn, which will anſwer the Problem, and that muſt in this Caſe paſs through L perpendicular to EF[b].

CASE 2.

When the given Vaniſhing Line EF doth not paſs through O the Center of the Picture.

Fig. 99. N°. 2. From the given Point L through O, draw LO, and from x, the Vaniſhing Point of Perpendiculars to the Planes EF, draw xv perpendicular to LO cutting it in w, and the Vaniſhing Line EF in y; find the Points v and z in the Line xv, which ſubtend with y the propoſed Angle Z[c], and Lv and Lz will be the Vaniſhing Lines required.

Dem. Becauſe xv is a Vaniſhing Line of Planes perpendicular to the Planes EF[d], and Lw is its Vertical Line, the Vaniſhing Point of Perpendiculars to the Planes xv is therefore in L, the Interſection of Lw with EF[e], the Vaniſhing Points L and y are therefore perpendicular[f]: the reſt of the Demonſtration is exactly the ſame as in the firſt Caſe of this Problem. *Q. E. I.*

COR. 1.

If the Point L be in o the Center of the Vaniſhing Line EF, the Extremities P and p of the Image of the Diameter of the forming Circle which paſſes through x parallel to EF muſt be found[g]; and then Lines drawn from P and p through o, will be the Vaniſhing Lines deſired.

For the Vaniſhing Points o and x being perpendicular, the Originals of oP and op are perpendicular to the Original of Pp, and therefore repreſent Tangents to the forming Circle in the Extremities of its Diameter repreſented by Pp.

COR. 2.

If the Vaniſhing Lines ſought were required to be parallel to EF, they muſt paſs through A and a[h]; but if either of the Points A or a be infinitely diſtant, the Planes which ought to be determined by it, are parallel to the Picture.

COR. 3.

If from x to L a Line xL be drawn, then Lv, Ly, Lz, and Lx, will be Harmonical Lines.

For the Vaniſhing Points x and y being perpendicular[i], and the Radials of z and v ſubtending equal Angles with the Radial of y, the Line xv is Harmonically divided in v, y, z, and x[k], and conſequently Lv, Ly, Lz, and Lx are Harmonical Lines[l].

COR. 4.

If the Point L be out of reach, yet if its Indefinite Radial be known, the required Vaniſhing Lines may be found: for $\mathcal{Y}y$ drawn perpendicular to the Indefinite Radial $\mathcal{Y}$L, gives the Point y, whence the Line ef and the Points v and z are found as before; and two Lines drawn through v and z tending to L[m], will be the Vaniſhing Lines deſired.

COR. 5.

If the Angle Z be Right, there can but one Vaniſhing Line be drawn through the propoſed Point L, to anſwer the Problem, which Line muſt alſo paſs through the Point x[n].

PROP. XXVI. PROB. XV.

Fig. 100. N°. 1. The Center and Diſtance of the Picture, and a Vaniſhing Line EF being

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[a] Cor. 1.

[b] Cor. 3. Theor. 16. B. I.

[c] Caſe 2. Prop. 24.

[d] Cor. 3. Prop. 20.

[e] Cor. 2. Prop. 22.

[f] Cor. 4. Prop. 20.

[g] Cor. 4. Meth. 2. Prob. 24. B. II.

[h] Cor. 2. Caſe 1. of this Prop.

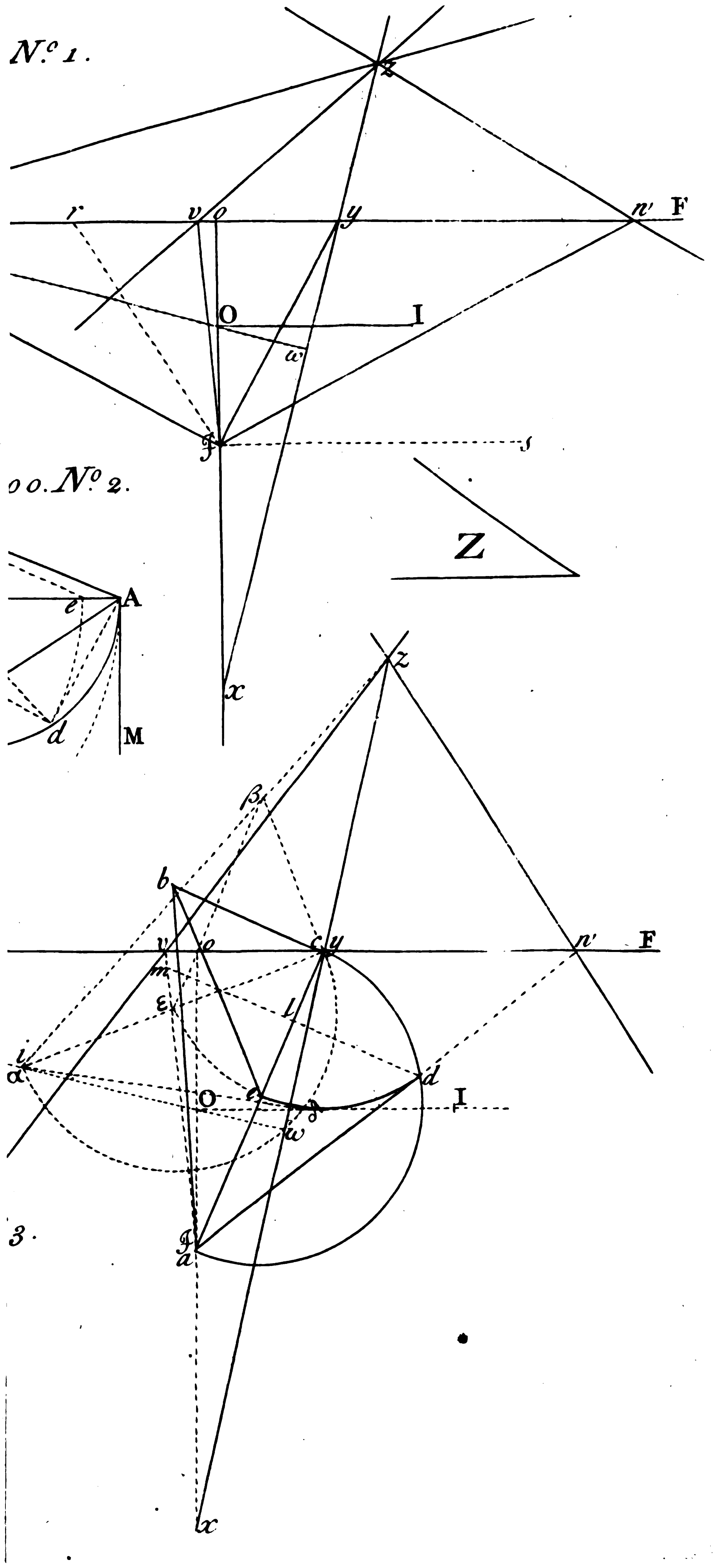
[i] Cor. 4. Prop. 20.

[k] Cor. Lem. 7. B. III.

[l] Def. 2. B. III.

[m] Prob. 18. B. II.

[n] Cor. 3. Prop. 20.


N.º 1.
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x

being given; through any Vaniſhing Point z, out of that Line, to draw two Vaniſhing Lines of Planes, which incline to the Planes EF, in any given Angle Z.

METHOD 1.

Through z draw a Vaniſhing Line zx of Planes perpendicular to the Planes EF[a], cutting EF in y, and having found the Angle ſubtended by z and y[b], which is the Angle of Inclination of the Lines z to their Perpendicular Seats on the Planes EF[c], any where a-part, draw a Rectangular Triangle ABC, having its Angle BAC equal to the Angle ſubtended by zy, and from B draw BE cutting AC in E, making the Angle BEC equal to Z the propoſed Angle of Inclination of the required Planes to the Planes EF; on AC, as a Diameter, deſcribe the Semicircle ADC, and from C as a Center, with the *Radius* CE, deſcribe the Arch ED, cutting the Semicircle in D, and draw AD: this being done, find in the Vaniſhing Line EF, two Vaniſhing Points v and w, each ſubtending with y an Angle equal to CAD[d], and zv and zw, drawn through z and the Points v and w, will be the Vaniſhing Lines deſired.

Fig. 100. N°. 2.

Dem. Imagine the Plane of the Triangle ABC to be turned up on the Line AC, until it become perpendicular to the Plane of the Semicircle ADC, then BC will be perpendicular to the Plane ADC, and AC will be the Perpendicular Seat of AB on that Plane; and if in this Situation of BC, the Plane of the Triangle BCE be turned on the Line BC, until the Point E cut the Semicircle in D, the Point E in turning round the Center C, deſcribing the Arch ED in the Plane ADC, while the Angle BEC remains unaltered, the Triangle BEC will come into the Poſition BDC; and becauſe of the Semicircle ADC, AD and DC being perpendicular, the Plane of the Triangle BEC, in the Poſition BDC, will be perpendicular to AD, and conſequently the Angle repreſented by BDC, that is, BEC, will be the Angle of Inclination of the Plane of the Semicircle ADC, to a Plane paſſing through AB and AD[e], and CAD will be the Angle, which AD, the Interſection of the Planes BAD and ADC, makes with AC, the Perpendicular Seat of AB on the Plane ADC; but BAC is by Conſtruction the Angle which the Lines z make with their Perpendicular Seats on the Planes EF, and BEC is the Angle of Inclination of the required Planes to the Planes EF, conſequently CAD is the Angle which the Interſections of the required Planes with the Planes EF make with the Perpendicular Seats of the Lines z on thoſe Planes; and y being the Vaniſhing Point of thoſe Seats, the Points v and w, which ſubtend with y Angles equal to CAD, are therefore the Vaniſhing Points of the Interſections of the required Planes with the Planes EF, and conſequently vz and wz are the Vaniſhing Lines of Planes paſſing through the Lines z, and inclining to the Planes EF in the Angle required. *Q. E. I.*

SCHOL.

Inſtead of drawing the ſeparate Figure N°. 2. the ſame Figure may be drawn on the Radial Jy of the Point y in the Vaniſhing Line EF, by making it ſerve as the Diameter of the Semicircle, and placing the Extremity correſponding to A at J.

Fig. 100. N°. 3.

Thus, having found the Angle yiz ſubtended by z and y, and drawn the Radial Jy of the Point y in the Vaniſhing Line EF, on Jy as a Diameter deſcribe a Semicircle Jdy, and likewiſe a Rectangular Triangle Jyb, having its Angle bJy equal to yiz, and having drawn be, ſo as to make the Angle bey equal to the propoſed Angle of Inclination of the required Planes, from the Center y, with the *Radius* ye, deſcribe an Arch cutting the Semicircle Jdy in d; then Jd will give the Point w in EF, through which one of the required Vaniſhing Lines wz paſſes; and dl being drawn perpendicular to Jy, and produced to m, making lm equal to ld, Jm will determine the Point v, through which the other required Vaniſhing Line vz is drawn; it being evident, that by this Conſtruction, the Angles yJd, yJm are equal.

And here, the Letters a, b, c, d, and e, correſpond to A, B, C, D, and E in the ſeparate Figure.

Note, *It is not neceſſary that the whole Radial Jy ſhould be taken as the Diameter of the Semicircle, but any convenient part of it may be uſed, ſo as the Extremity correſponding to* A *be at J.*

COR. 1.

If the Angles BEC and BAC be equal, that is, if the Inclination of the required Planes to the Planes EF, be equal to the Inclination of the Lines z to their Seats y; the Planes

Fig. 100. N°. 1, 2.

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[a] Prop. 20. and Cor. 3.

[b] Prop. 14.

[c] Prop. 24.

[d] Prob. 3. B. II.

[e] Def. 19. B. I.

Planes of their Seats zy, muſt be perpendicular to the Planes required, as well as to the Planes EF[a]; and therefore the Vaniſhing Line of the required Planes muſt paſs through the Vaniſhing Point of Perpendiculars to the Planes zy[b]: if therefore t the Vaniſhing Point of Perpendiculars to the Planes zy be found[c], which Point muſt be ſomewhere in EF[d], the Vaniſhing Line of the Planes required muſt paſs through t and z; and as only one ſuch Line can be drawn, tz is the only Vaniſhing Line that in this Caſe can anſwer the Problem.

SCHOL.

When the Angles BAC and BEC are equal, the Points A and E coincide, and the Point E, moving round C as a Center, can only touch the Semicircle ADC in A, ſo that D alſo coincides with A; and CD therefore coinciding with CA, AD drawn perpendicular to CD, in this Poſition, muſt coincide with the Tangent AM, which is perpendicular to AC; wherefore the Angle CAD being in this Caſe Right, a Point muſt be found in EF ſubtending with y a Right Angle, which Point will be the ſame with the Point t found by the Corollary, ſeeing that Point is perpendicular to every Point in zy[e], and conſequently to y.

COR. 2.

When the Angle BEC is greater than BAC, but leſs than a Right, then two Vaniſhing Lines may always be found, which will anſwer the Problem; ſeeing the Angle CAD may be ſet off on either Side of Jy the Radial of the Vaniſhing Point y: but if the Angle BEC be Right, then E and conſequently D coinciding with C, zy is itſelf the only Vaniſhing Line that anſwers the Problem, it being a Vaniſhing Line of Planes perpendicular to the Planes EF paſſing through z.

Laſtly, if the Angle BEC be leſs than BAC, the Problem becomes impoſſible; in regard that the Angle of Inclination of two Planes cannot be leſs than the Angle made by the Interſections of thoſe Planes with any other Planes perpendicular to either of them[f].

In this laſt Caſe, CE being larger than CA, the Point E revolving round C, can never meet the Semicircle ADC in any Point.

COR. 3.

If the Vaniſhing Point v fall nearer to y than to t, the other Vaniſhing Point w will fall beyond y; if v be nearer to t than to y, w will fall beyond t; and if v biſect ty, the Radial which ſhould produce the other Vaniſhing Point w, will be parallel to EF; in which laſt Caſe the Point w being infinitely diſtant, the Vaniſhing Line, which ſhould paſs through that Point and z, muſt be drawn through z parallel to EF.

For the Radials Jv and Jw making equal Angles with Jy, to which the Radial Jt is perpendicular, the Lines Jt, Jv, Jy, and Jw, are Harmonical Lines[g]; if then EF cut them all four, it will be Harmonically divided by them in t, v, y, and w[h]; and the middle Part of a Line thus divided being leſs than either of the Extremes[i]; if yv be leſs than vt, w muſt fall beyond y; and if vt be leſs than vy, w muſt fall beyond t; but if v biſect ty, EF can cut but three of the Harmonicals, and muſt therefore be parallel to the fourth[k].

COR. 4.

If the Angle ACd be equal to the Angle oJy ſubtended by o and y, then one of the Vaniſhing Lines found will be parallel to EF.

For in the Rectangular Triangles AdC, oJy, the Angles ACd, oJy, being by Suppoſition equal, the Angles CAd, Jyo muſt alſo be equal; if then the Angle rJy be made equal to CAd, it will alſo be equal to Jyo; wherefore the Triangle rJy will have its two Sides ry, rJ equal[l], and conſequently a Semicircle drawn from r as a Center, with the *Radius* ry, will alſo paſs through J; but becauſe yJt is a Right Angle, that Semicircle muſt alſo paſs through t[m], therefore the Point r biſects ty; and if the Angle rJy, or CAd be ſet off on the other Side of Jy by the Line Js, the Angle sJy being equal to rJy, is therefore alſo equal to ryJ, and conſequently ry or EF and Js are parallel, and therefore the Vaniſhing Line determined by Js, muſt paſs through z parallel to EF[n].

Here, the Angle of Inclination of the required Planes to the Planes EF *is* BeC.

COR, 5.

If any two Vaniſhing Lines EF and vz be given, cutting each other in v; the Angle

of

[a] Def. 19. B. I.

[b] Cor. 3. Prop. 26.

[c] Prop. 20.

[d] Cor. 2. Prop. 22.

[e] Cor. 4. Prop. 20.

[f] Cor. 1. Def. 19. B. I.

[g] Lem. 5. B. III.

[h] Lem. 8. B. III.

[i] Cor. 1. Lem. 1. B. III.

[k] Lem. 7. B. III.

[l] 6 El. 1.

[m] 31 El. 3.

[n] Cor. 3.

of Inclination of their Planes may be found by this Method, when the way propoſed at Prop. XXIII. is inconvenient.

Draw any Vaniſhing Line zx of Planes perpendicular to either of the given Planes, as EF, cutting both the given Vaniſhing Lines in y and z, and having found the Angles ſubtended by yz and yv, draw a Rectangular Triangle ABC, having its Angle BAC equal to that ſubtended by yz; and having on the Diameter AC drawn the Semicircle ADC, draw AD cutting it in D, making the Angle CAD equal to that ſubtended by yv; then on the Center C, with the *Radius* CD, deſcribe the Arch DE cutting AC in E, and a Line BE gives BEC, the Angle of Inclination of the Planes EF and vz.

COR. 6.

If a Vaniſhing Line EF be given, and it be required through any Point v in EF, to draw two Vaniſhing Lines of Planes inclining in any given Angle to the Planes EF; it may be done by this Method, when the way propoſed at Prop. XXV. is inconvenient, which it will generally be, when the Point v falls near the Center of the Picture.

Draw any Vaniſhing Line zx of Planes perpendicular to the Planes EF, cutting EF in y, at any Diſtance from v; and having found the Angle ſubtended by v and y, any where a-part draw a Semicircle ADC, with any Diameter AC, and in it draw AD, making the Angle CAD equal to the Angle ſubtended by vy; and having drawn CB perpendicular to AC, from C as a Center, with the *Radius* CD, draw the Arch DE, cutting AC in E, and from E draw EB, making BEC equal to the Angle of Inclination propoſed, and from B, where EB cuts CB, draw BA; then on each Side of y, in the Line zx, find a Vaniſhing Point ſubtending with y an Angle equal to BAC, and Lines drawn through thoſe Points and the Point v, will be the Vaniſhing Lines deſired,

SCHOL.

Fig. 100. N°. 3.

In Corol. 5. the Radial $\mathcal{J}y$ of the Point y, in the Vaniſhing Line EF, may be taken as the Diameter of the Semicircle of the ſeparate Figure, placing the Extremity correſponding to A at $\mathcal{J}$, as already obſerved[a]; but in Corol. 6. it will be more convenient to uſe the Radial iy of the Point y, in the Vaniſhing Line xy, for the Diameter, as in the Figure, where the Letters α, β, c, δ, ε, repreſent A, B, C, D, and E, in the ſeparate Figure.

METHOD 2.

This Problem may be likewiſe ſolved, by finding the Tangents from the given Point z, to the Curve produced by the Image of the Circle in the Original Plane EF, which forms the Vaniſhing Points of the Angle of Inclination propoſed: and as it has been ſhewn how to determine the Curve formed by thoſe Vaniſhing Points, whatever the given Angle of Inclination be[b], this Problem is reduced to the ſame as the third, ſeventh, and eleventh Problems of Book III. where the Point from whence the Tangents are to be drawn, is ſuppoſed to lie in a Diameter of the forming Circle; of which Problems the preſent is only one Caſe, *viz.* when the Point from whence the Tangents are to be drawn, is ſuppoſed to be a Vaniſhing Point in the Plane EF, and conſequently when the Tangents from thence paſs through the Extremities of a Diameter of the forming Circle[c].

This Problem, as well as the preceeding, may alſo be ſolved from the Conſideration of the Properties of the produced Curve itſelf, whichever of the Conick Sections it be; as has been ſhewn at Cor. 7. Prop. XVI. Cor. 5. Prop. XVII, and Cor. 6. Prop. XVIII. Book, III. *Q. E. I.*

PROP. XXVII. PROB. XVI.

The Center and Diſtance of the Picture, and a Vaniſhing Line EF being given; thence to find two Vaniſhing Lines of Planes which incline to the Planes EF in any given Angle Z, and which Vaniſhing Lines may make with EF any Angle propoſed.

METHOD 1.

CASE 1.

When the given Vaniſhing Line EF doth not paſs through the Center of the Picture. Fig. 101. N°. 1.

Draw a Vaniſhing Line xy, of Planes perpendicular to the Planes EF[d], cutting EF in y, and making with it an Angle xyF, equal to the Angle which the Vaniſh-

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[a] Schol. Meth. 1.

[b] Cor. 1, 2, 3. Prop. 24.

[c] Cor. 1. Prob. 3. B. III.

[d] Cor. 3. Prop. 20.

ing Lines ſought are required to make with EF; and having found the Radial iu of the Vaniſhing Line xy[a], draw iy; then y being the Vaniſhing Point of the Perpendicular Seats of all Lines whatſoever in the Planes xy on the Planes EF[b], it is the Vaniſhing Point of the Seats of all Lines in the Planes xy which are parallel to the Picture, and conſequently to their Interſecting Lines; and the Angle iyu is the Angle made by thoſe Interſecting Lines, or their Parallels, with their Perpendicular Seats on the Planes EF[c]. This being premiſed, any where a-part draw a Rectangular Triangle ABC, having its Angle BAC equal to the Angle iyu, and from B draw BE, making the Angle BEC equal to the Angle of Inclination of the required Planes to the Planes EF; and having on the Diameter AC deſcribed a Semicircle ADC, from C as a Center with the *Radius* CE, deſcribe the Arch ED, cutting the Semicircle in D, and draw AD; then in the Vaniſhing Line EF, find two Points v and w, each ſubtending with y an Angle equal to CAD, and ef and ef drawn through v and w parallel to xy will be the Vaniſhing Lines required.

Dem. Becauſe BAC (equal to iyu) is the Angle made by the Interſecting Lines of the Planes xy, or Parallels to them, with their Perpendicular Seats on the Planes EF[d], if ADC be conſidered as the Original of one of the Planes EF, and the Plane BAC be ſuppoſed perpendicular to it, BA may be taken to repreſent a Parallel to the Interſecting Lines of the Planes xy, having AC for its Perpendicular Seat on the Plane ADC; and then CAD will be the Angle which AC makes with AD, the Interſection of the Plane CAD with a Plane BAD paſſing through BA, and inclining to the Plane CAD in the propoſed Angle BEC, or Z[e]. If then y be the Vaniſhing Point of AC, the Points v and w, which ſubtend with y, Angles equal to CAD, will be the Vaniſhing Points of the Interſections of the Planes repreſented by BAD with the Planes EF[f]; and theſe Planes, by Conſtruction, paſſing through Parallels to the Interſecting Lines of the Planes xy, to which their Vaniſhing Lines are therefore alſo parallel[g], ef and ef drawn through v and w parallel to xy, are the Vaniſhing Lines of Planes inclining to the Planes EF in the given Angle BEC or Z, and which make with EF, Angles equal to xyF, the other Angle propoſed. *Q. E. I.*

SCHOL.

Here, as in the preceeding Problem, the ſeparate Figure may be brought into the Picture, by uſing $\mathcal{J}y$, the Radial of the Point y in the Vaniſhing Line EF, as the Diameter of the Semicircle ADC, placing the Extremity correſponding to A at $\mathcal{J}$.

COR. 1.

If the Angles BEC and BAC be equal, that is, if the Inclination of the required Planes to the Planes EF, be equal to the Inclination of the Interſecting Lines of the Planes zy to their Seats on the Planes EF; then there is only one Vaniſhing Line which will anſwer the Problem, namely, a Line ef drawn parallel to xy, through t a Vaniſhing Point in EF, perpendicular to the Vaniſhing Point y[h].

COR. 2.

When the Angle BEC is greater than BAC, but leſs than a Right, then two Vaniſhing Lines may always be found which will anſwer the Problem[i]; except only when the Point v biſects ty, which happens, when the Angle ACD is equal to the Angle $o\mathcal{J}y$[k]; in which Caſe, the Radial which ſhould determine the other Point w, becomes parallel to EF[l], and therefore the Point w, which ſhould mark the Interſection of EF with the other Vaniſhing Line required, or the Vaniſhing Point of the Interſections of the required Planes with the Planes EF, is infinitely diſtant; wherefore the Planes, to which that Vaniſhing Line ſhould belong, cutting the Planes EF in Lines parallel to EF, and conſequently to the Picture, and alſo paſſing through Lines parallel to the Picture and to xy, thoſe Planes are therefore parallel to the Picture, and have no Vaniſhing Line.

If the Angle BEC be Right, then xy is itſelf the only Vaniſhing Line which anſwers the Problem[m]; ſeeing it is a Vaniſhing Line of Planes perpendicular to the Planes EF, and making with EF the Angle xyF required.

But this is to be underſtood, when the Inclination of the required Vaniſhing Lines to EF is determined one particular way; for it is evident, that if the required Vaniſhing Lines were propoſed to incline the contrary way to EF in the ſame Angle, another Line may be drawn from x, cutting EF on the other Side of its Center o, and making an Angle with EF the contrary way, equal to xyF; which Line being uſed in

all

[a] Prop. 13.

[b] Theor. 16. B. I. and 38 El. 11.

[c] Theor. 11. B. I.

Fig. 101. N°. 2.

[d] Theor. 11. B. I.

[e] Meth. 1. Prop. 26.

[f] Cor. 2. Meth. 1. Prop. 26.

[g] Cor. 1. Theor. 15. B. I.

[h] Cor. 1. Prop. 26. and Schol.

[i] Cor. 2. Prop. 26.

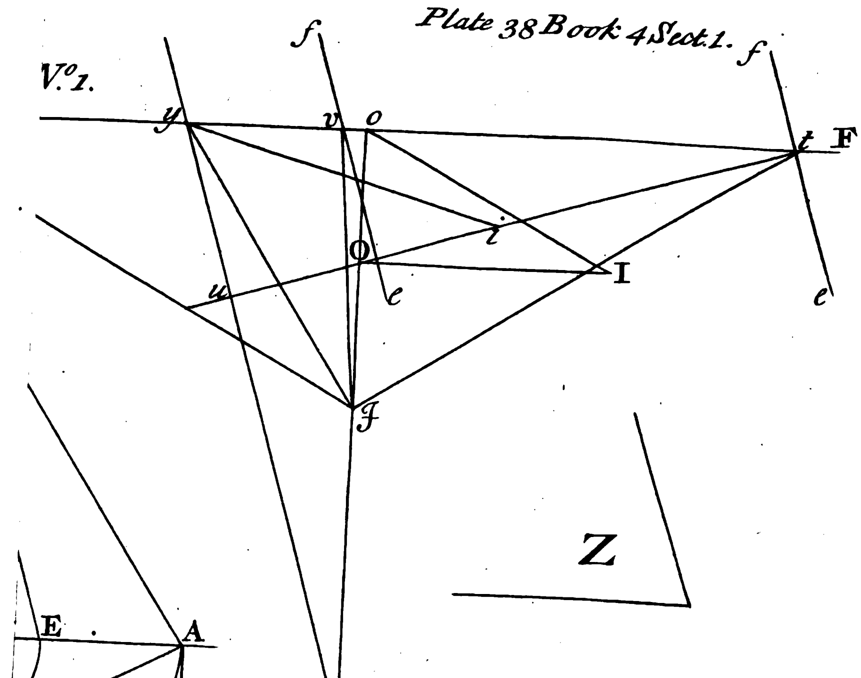
[k] Cor. 4. Prop. 26.

[l] Cor. 3. Prop. 26.

[m] Cor. 2. Prop. 26.

Plate 38 Book 4 Sect. 1.

N°1.



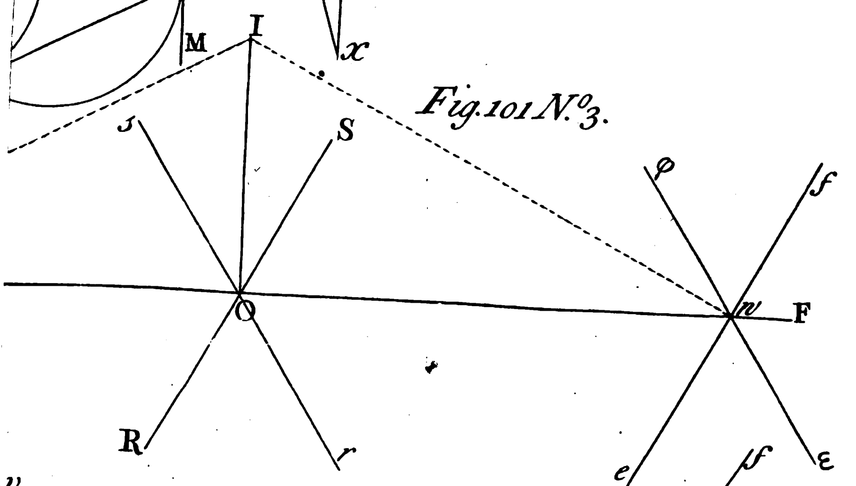
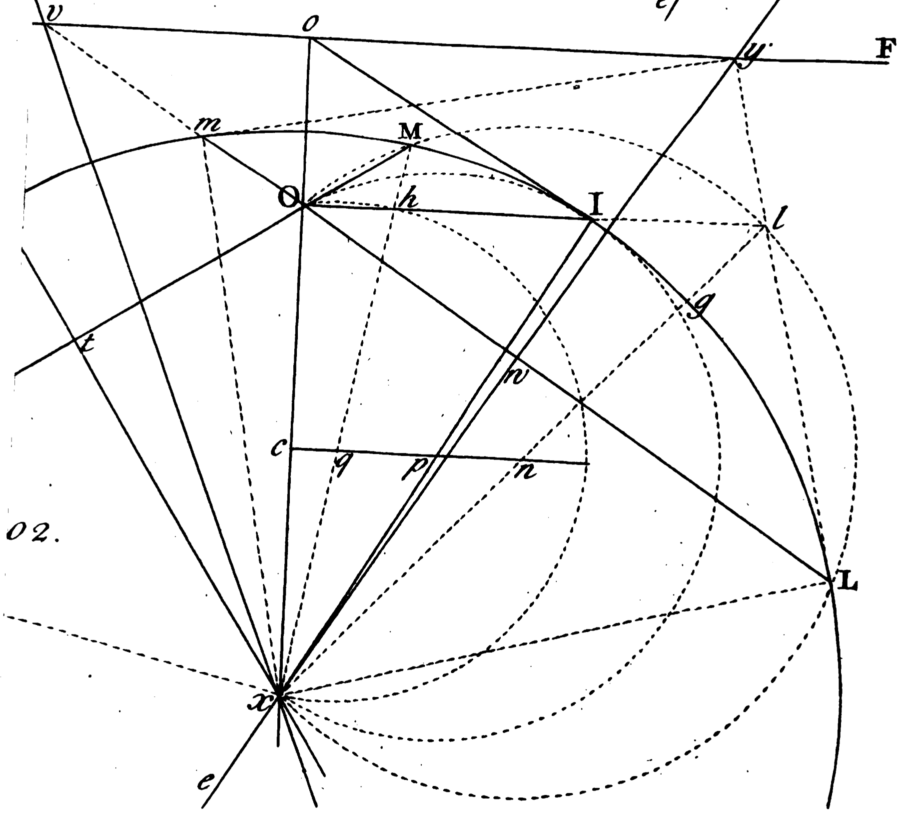


Fig. 101 N°3.



02.

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all reſpects as the Line xy, other Vaniſhing Lines parallel to it may be thereby found, which will alſo anſwer the Problem.

Laſtly, if the Angle BEC be leſs than BAC, the Problem becomes impoſſible[a].

COR. 3.

When the Point v biſects ty, the Angle BEC muſt be equal to IoO the Angle of Inclination of the Planes EF to the Picture[b]; for one ſet of Planes then determined, being parallel to the Picture[c], they muſt incline to the Planes EF in the ſame Angle as the Picture doth; but thoſe Planes incline to the Planes EF in the Angle BEC, wherefore this Angle is equal to the Angle IoO: and hence, as by enlarging the Angle BEC, the Angle CAD and conſequently yJv is leſſened, and *vice verſa*; if the Angle BEC be greater than IoO, the Point v will fall nearer to y than to t, wherefore w will fall beyond y; and if the Angle BEC be leſs than IoO, v will fall nearer to t than to y, and w will therefore fall beyond t[d].

COR. 4.

If the Vaniſhing Lines ſought were required to be perpendicular to EF, the Angle BEC muſt be greater than the Angle of Inclination of the Planes EF to the Picture; for then xy coming into the Poſition xo, the Angle IoO, which is the Angle of Inclination of the Planes EF to the Picture, is alſo the Angle of Inclination of the Interſecting Lines of the Planes xo to their Perpendicular Seats on the Planes EF, which Angle is repreſented by BAC in the ſeparate Figure, and which cannot be greater than the Angle BEC; and if the Angles BAC and BEC be equal, the Angle DAC becomes Right, and a Radial perpendicular to Jo being parallel to EF, the Planes whoſe Vaniſhing Line ſhould be determined by the Interſection of that Radial with EF, are parallel to the Picture, their Vaniſhing Line being infinitely diſtant[e].

COR. 5.

If two Vaniſhing Lines EF and ef (neither of them paſſing through the Center of the Picture) be given, cutting each other in v; the Angle of Inclination of their Planes may be alſo found by this Method.

For ef being given, its Parallel xy is thence found, and conſequently the Angles iyu and yJv; wherefore the Rectangular Triangle ABC being drawn, with its Angle BAC equal to iyu, and the Semicircle ADC being alſo drawn, and the Angle CAD made equal to yJv, the Arch DE gives the Point E, and conſequently BEC the Angle of Inclination of the Planes EF and ef.

CASE. 2.

When the Vaniſhing Line EF paſſes through O the Center of the Picture.

Fig. 101. N°. 3.

Through O draw RS, making with EF, the Angle SOF equal to the Angle which the Vaniſhing Lines ſought are required to make with EF; then becauſe the Planes EF and RS are perpendicular to the Picture, the Perpendicular Seats of the Interſecting Lines of the Planes RS, or their Parallels, on the Planes EF, are the Interſecting Lines of the Planes EF, or Lines in thoſe Planes parallel to them; wherefore the Angle made by the Interſecting Lines of the Planes RS, or their Parallels, with their Perpendicular Seats on the Planes EF, is equal to SOF the Angle made by their Vaniſhing Lines.

Fig. 101. N°. 2.

Having therefore made the Rectangular Triangle ABC, with its Angle BAC, equal to SOF, and the Angle BEC equal to the propoſed Inclination of the Planes, find the Angle CAD as before; then BA repreſenting a Parallel to the Interſecting Lines of the Planes RS, and its Seat AC on the Plane CDA therefore repreſenting a Parallel to the Interſecting Lines of the Planes EF, CAD is the Angle which the Interſections of the Planes required with the Planes EF, make with the Interſecting Lines of theſe laſt Planes[f]; therefore two Radials Iv and Iw being drawn, making with EF, on each Side of O, the Angles IvO, IwO, each equal to CAD, the Lines ef, ef, drawn through v and w, parallel to RS, will be the Vaniſhing Lines required[g]. *Q. E. I.*

SCHOL.

Fig. 101. N°. 3.

Here alſo the ſeparate Figure may be brought into the Picture, by uſing IO the Radial of O in the Vaniſhing Line EF, as the Diameter of the Semicircle ADC; but then the Extremity correſponding to C muſt be placed at O, the Angles OIv, OIw being here equal to ACD, the Complement of CAD to a Right Angle: Or, if through I a Parallel to EF be drawn (which then repreſents the Radial of the Seats of the In-

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[a] Cor. 2. Prop. 26.

[b] Theor. 9. B. I.

[c] Cor. 2.

[d] Cor. 3. Prop. 26.

[e] Cor. 2.

[f] Caſe 1.

[g] Theor. 11. B. I.

terſecting Lines of the Planes RS, as Jy doth that of the Planes xy in the former Caſe) then the Parallel thus drawn, being uſed as the Line CA in the ſeparate Figure, and the Point correſponding to A being placed at I, the Line Iv will correſpond to AD.

COR. 1.

If the Line RS were made to incline the contrary way to EF in the ſame Angle, as rs; it is evident this will make no Alteration in the Angle CAD, and therefore the Points v and w remaining the ſame as before, the Lines $\varepsilon\varphi$, $\varepsilon\varphi$, drawn through v and w parallel to rs, will alſo anſwer the Problem.

COR. 2.

If the Angles BEC and BAC be equal, the Line AD being then perpendicular to AC, as coming into the Poſition AM, the Points v and w will coincide with O; ſo that O the Center of the Picture, will be the only Point in EF, through which the required Vaniſhing Lines can paſs; which agrees with what was formerly ſhewn[a], that all Vaniſhing Lines which paſs through the Center of the Picture, make the ſame Angles with each other as their Planes do.

COR. 3.

If the Angle BAC be Right, the Angle BEC muſt be ſo too, ſeeing the Angle BEC cannot be leſs than BAC[b]; which alſo agrees with what has been already ſhewn[c], that when a Vaniſhing Line EF paſſes through the Center of the Picture, the Vaniſhing Lines of all Planes perpendicular to the Planes EF, are perpendicular to the Vaniſhing Line EF.

COR. 4.

If two Vaniſhing Lines EF and ef (one of which EF paſſes through O the Center of the Picture) be given, cutting each other in v; the Angle of Inclination of their Planes may be alſo found by this Method.

For a Rectangular Triangle ABC being made, with its Angle BAC equal to the Angle fvO, and the Semicircle CDA being drawn; draw AD making the Angle CAD equal to IvO, and the Point E being thence found, will give BEC the Angle of Inclination of the Planes propoſed.

METHOD 2.

This Problem may alſo be ſolved, from the Conſideration of the Properties of the Curves produced by the Image of the Circle which forms the Vaniſhing Points of the Angle of Inclination of the required Planes to the Planes EF.

For drawing any Line through EF, parallel to the Vaniſhing Lines required, find a Diameter of the produced Section, whoſe Ordinates may be parallel to that Line, as ſhewn at Cor. 8. Prop. XVI. Cor. 7. Prop. XVII. and Cor. 7. Prop. XVIII. Book III. obſerving that when the Curves produced are oppoſite *Hyperbolas*, the Diameter found muſt be a firſt Diameter; for then the Tangents at the Extremities of this Diameter, being parallel to its Ordinates, will be the Vaniſhing Lines required[d].

And as two Tangents may be drawn through the Extremities of any Diameter of a Circle or *Ellipſis*, or of a firſt Diameter of the *Hyperbolas*, which will be parallel to each other[e]; when the Section produced is any of theſe, there will be two Vaniſhing Lines found which will anſwer the Problem: but when the Section produced, is a *Parabola* (which it will be, when the Angle of Inclination of the required Planes to the Planes EF, is equal to the Inclination of theſe laſt to the Picture[f]) there can only one Vaniſhing Line be found that will ſatisfy the Problem; ſeeing only one Tangent can be drawn to a *Parabola* through the Vertex of any Diameter, the other Extremity of that Diameter being infinitely diſtant[g]; ſo that the Planes, which ſhould have the Tangent, which paſſes through this infinitely diſtant Point, for their Vaniſhing Line, muſt therefore be parallel to the Picture. *Q. E. I.*

COR.

Hence, when the Place of the Vaniſhing Points is two oppoſite *Hyperbolas*[h], no Vaniſhing Line of Planes, which incline to the Planes EF in the propoſed Angle, can make with EF a greater Angle than that which the Aſymptotes make with the ſecond Axe[i]; when the Place of the Vaniſhing Points is a *Parabola*, the Vaniſhing Lines required may incline to the Line EF in any Angle leſs than a Right; but it cannot be a Right Angle, in regard that all Perpendiculars to EF are parallel to the Axe of

[a] Cor. 1. Theor. 16. B. I.

[b] Cor. 2. Prop. 26.

[c] Cor. 3. Theor. 16. B. I.

[d] Prop. 11. and Cor.

[e] Ellip. Art. 4. and Hyperb. Art. 1. B. III.

[f] Cor. 2. Meth. 2. Caſe 2. Prop. 24.

[g] Parab. Art. 3 and 5. and Schol. Prob. 7. B. III.

[h] Cor. 1. Meth. 2. Caſe 2. Prop. 24.

[i] Cor. 4. Part ſecond of Prob. 9. B. III.

of the *Parabola*, and are therefore Diameters of that Curve, and not Tangents[a]; but when the Place of the Vaniſhing Points is an *Ellipſis*[b] or a Circle, the Vaniſhing Lines ſought may incline to the Line E F in any Angle whatſoever.

PROP. XXVIII. THEOR. XII.

Fig. 102.

If through *x* the Vaniſhing Point of Perpendiculars to any Planes E F, any Vaniſhing Line *ef* be drawn, cutting E F in *y*; the Radial *w* L or *wm*, of that Vaniſhing Line, will terminate in the Circumference of a Circle R M I L, whoſe Center is *x*; and *Radius x* I the Diſtance of the Vaniſhing Point *x*.

Let *w* L be the Radial of the Vaniſhing Line *ef*, and draw *x* L.

Dem. Becauſe *w* L is the Radial of the Vaniſhing Line *ef*[c], $wL^2 = OI^2 + Ow^2$
And in the Rectangular Triangle *xw*O $xw^2 = xO^2 - Ow^2$
And adding theſe together $wL^2 + xw^2 = OI^2 + xO^2$
But in the Rectangular Triangle *x*IO $xI^2 = OI^2 + xO^2$
Therefore $xI^2 = wL^2 + xw^2$
Laſtly in the Rectangular Triangle *x*L*w* $xL^2 = wL^2 + xw^2$
Therefore $xL = xI.$

And conſequently L is a Point in the Circumference of a Circle, of which *x* is the Center, and *x* I the *Radius*; and *w m* being equal to *w* L, *m* is alſo a Point in the Circumference of the ſame Circle, *ef* and *m* L being by Conſtruction perpendicular[d]. *Q. E. D.*

COR. 1.

The Vertical Line *m* L of the Planes *ef*, is the Chord of the Tangents to the Circle RMIL from the Point *y*; and L*y* or *my*, the Radial of the Vaniſhing Point *y*, is a Tangent to the Circle RMIL from *y*, wherever the Point *y* be taken in the Line E F.

For the Vaniſhing Points *x* and *y* being perpendicular[e], their Radials L*x*, L*y* are perpendicular, and L*x* being a *Radius* of the Circle RMIL, L*y* is a Tangent to that Circle in L: the ſame may be ſhewn of the Line *ym*, wherefore *m* L is the Chord of the Tangents to the Circle RMIL from the Point *y*.

COR. 2.

The Line *m* L produced to E F, cuts it in *v* the Vaniſhing Point of Perpendiculars to the Planes *ef*[f]; wherefore the Vaniſhing Points *v* and *y* are alſo perpendicular[g]; and a Line drawn through *x* and *v* is the Vaniſhing Line of Planes perpendicular to the Vaniſhing Point *y*; this laſt Point being perpendicular to the Vaniſhing Points *x* and *v*, through which the Vaniſhing Lines *xv* paſſes[h].

COR. 3.

The Angle L*yw* is the Angle made by the Interſections of the Planes *ef*, with the Planes E F, and with the Picture.

For the Interſections of the Planes *ef* with the Picture, that is, the Interſecting Lines of the Planes *ef*, are parallel to *ef*[i]; and the Interſections of the Planes *ef* with the Planes E F, are Lines in the Planes *ef*, which have *y* for their Vaniſhing Point; wherefore theſe laſt Interſections make with the Interſecting Lines of the Planes *ef*, an Angle equal to L*yw*[k].

PROP. XXIX. PROB. XVII.

Fig. 102.

The Center and Diſtance of the Picture, and a Vaniſhing Line E F, not paſſing through that Center, being given; thence to find a Vaniſhing Line of Planes perpendicular to the Planes E F, the Interſections of which with the Planes E F, and with the Picture, may make a given Angle Z.

Having found *x* the Vaniſhing Point of Perpendiculars to the Planes E F[l], from *x* as a Center, with the *Radius x* I, deſcribe the Circle RMIL; and on O*x*, the Diſtance between *x* and the Center of the Picture, deſcribe a Segment of a Circle, which will contain the given Angle Z[m]:

[a] Cor. 2. Meth. 2. Caſe 2. Prop. 24.

[b] Cor. 3. Meth. 2. Caſe 2. Prop. 24.

[c] Cor. Prop. 12.

[d] 3 El. 3.

[e] Cor. 4. Prop. 20.

[f] Cor. 2. Prop. 22.

[g] Cor. 4. Prop. 20.

[h] 4 El. 11.

[i] Cor. 1. Def. 10. B. I.

[k] Theor. 11. B. I.

[l] Prop. 20.

[m] Lem. 1. or 33 El. 3.

Now this Segment muſt either cut the Circle RMIL in two Points, or it will touch it in one Point, or fall wholly within the Circle.

1. Let the Segment $OMLx$ cut the Circle RMIL in two Points M and L.

Through O and the Points M and L draw OM, OL, and from x draw xy perpendicular to OL, and xz perpendicular to OM; and either of the Lines xy or xz will anſwer the Problem.

Dem. Becauſe of the Vaniſhing Point x, the Line xy is a Vaniſhing Line of Planes perpendicular to the Planes EF[a], and wL is its Radial[b]: And becauſe L is a Point of the Segment of the Circle $OMLx$, the Angle OLx is therefore equal to the Angle Z; but the Rectangular Triangles Lwx, ywL being ſimilar[c], the Angles wLx, wyL are equal, wherefore wyL is equal to the Angle Z; and wyL being the Angle made by the Interſections of the Planes xy with the Planes EF, and with the Picture[d], xy is therefore a Vaniſhing Line of Planes perpendicular to the Planes EF, the Interſections of which with the Planes EF, and with the Picture, make together an Angle equal to Z, the Angle propoſed. The ſame may be ſhewn of the Vaniſhing Line xz, the Angle OMx equal to the Angle Z, being equal to the Angle tRx, which is equal to the Angle Rzt. *Q. E. I.*

COR. 1.

If the Segment $OMLx$ had been deſcribed on the other Side of Ox, it would have cut the Circle RMIL in the ſame Points R and m, where it is cut by OM and OL; whence the ſame Lines xy and xz would have been found as before.

For the Angles ORx, OLx being each equal to the Angle Z, and the Angle xmL being alſo equal to OLx, mL being biſected in w; it is evident the Points m and R muſt fall in the Segment of a Circle capable of that Angle, and inſiſting on the Chord xO, and that therefore this Segment muſt cut the Circle RMIL in R and m.

COR. 2.

The Points z and y are equally diſtant from o the Center of the Vaniſhing Line EF.

For the Rectangular Triangles xtM, xwL, having their Angles at M and L, and their Hypotenuſes xM and xL equal, are therefore ſimilar and equal, wherefore their Sides xw, xt are equal; and the Rectangular Triangles xwO, xtO, having their Sides xw, xt equal, and the ſame Hypotenuſe xO, are therefore alſo ſimilar and equal; wherefore the Angles wxO, txO, and conſequently zo and oy, are equal.

COR. 3.

If the Angle Z be leſs than IoO the Angle of Inclination of the Planes EF to the Picture; the Segment $OMLx$ which contains that Angle, will cut the Circle RMIL in two Points M and L.

Biſect Ox in c, and draw cn perpendicular to it, cutting Ix the *Radius* of the Circle RMIL in p; then xO being biſected in c, Ix will alſo be biſected in p, and the Angle xpc, which is equal to the Angle xIO, will therefore be equal to IoO the Angle of Inclination of the Planes EF to the Picture[e].

Now the Center n of the Segment of the Circle which paſſes through O and x, and contains the Angle Z, being determined by the Interſection of cp with a Line xn, drawn ſo as to make the Angle xnc equal to that Angle[f]; if the Angle Z or xnc be leſs than xpc or IoO, the Point n muſt fall beyond p, and conſequently xn the *Radius* of that Segment, will be greater than xp the Moiety of the *Radius* xI or xg of the Circle RMIL; wherefore xn being produced to l, until nl and nx be equal, that is, until it cut OI produced in l, the Point l muſt fall without the Circle, and therefore the Segment deſcribed from the Center n, with the *Radius* nx, paſſing through l, a Point without the Circle RMIL, muſt cut that Circle in two Points.

2. When the Angle Z is equal to the Angle IoO, the Segment of the Circle which contains that Angle, will touch the Circle RMIL; and there is then only one Vaniſhing Line that can anſwer the Problem, *viz.* the Vertical Line xo.

For when the Angle Z is equal to IoO, xp drawn ſo as to make the Angle xpc equal to it, will coincide with xI, and the Point p biſecting that Line, the Segment drawn from the Center p with the *Radius* px, muſt alſo paſs through I, and touch the Circle RMIL in that Point; and the Line drawn from O through I, coincides with the Radial IO, to which the Vaniſhing Line xo is perpendicular.

3. When the Angle Z is greater than IoO, the Angle cqx being greater than

cpx,

[a] Cor. 3. Prop. 20.

[b] Prop. 28.

[c] 8 El. 6.

[d] Cor. 3. Prop. 28.

[e] 8 El. 6.

[f] Lem. 1.

cpx, xq muſt be leſs than xp, and qb being made equal to qx, the Diameter xb of the Segment will be leſs than xM the *Radius* of the Circle RMIL; and therefore the Segment drawn from the Center q, with the *Radius* qx, paſſing alſo through b, will fall wholly within the Circle, and the Problem then becomes impoſſible; there being no Vaniſhing Line which can anſwer the Conditions propoſed.

GENERAL COROLLARY.

No Vaniſhing Point being peculiar to any one Original Line, but common to all Lines which are parallel to each other, and have the ſame particular Direction; when a Vaniſhing Point is given, no particular Original Line is thereby determined, but only in general, the Direction which all Original Lines to which that Vaniſhing Point belongs, have with regard to the Picture; after the ſame manner, when a Vaniſhing Line is given, no particular Original Plane is determined, but only in general, the Inclination and Poſition which all Original Planes to which that Vaniſhing Line belongs, have with reſpect to the Picture; ſo that the Conſideration of Vaniſhing Points and Lines doth not regard any particular Original Line or Plane only, but includes all Lines and Planes whatſoever, to which the given Vaniſhing Point or Line appertains.

Now it being evident, that whatever Angle any two Lines make together, any two other Lines parallel to the former, whether in the ſame or in any other Plane, make together the ſame Angle[a]; and whatever Inclination any Line hath to one Plane, it hath the ſame Inclination to all other Planes which are parallel to that Plane; and alſo, that if any Number of parallel Planes be cut by other parallel Planes, the Inclination of all thoſe Planes to each other is the ſame; hence it is, that the Inclination of Lines or Planes to each other, is determined by their Vaniſhing Points and Vaniſhing Lines; and the Relation between theſe being found, determines the Relation between all Lines and Planes whatſoever, to which thoſe Vaniſhing Points and Lines belong.

Therefore the Propoſitions of this Section, which concern the Properties of Vaniſhing Points and Lines, are general; and relate alike to all Lines and Planes whatſoever, to which the Vaniſhing Points and Lines in queſtion are applicable.

SCHOL.

When the Diſtance of the Picture is very large, the Lines neceſſary to be drawn for determining the Vaniſhing Points and Lines required, will frequently fall at an inconvenient Diſtance from the Center of the Picture, unleſs the Plane on which the Work is performed, be of a great Extent: but this may be remedied, by working on a ſeparate Plane as a Picture, with a Diſtance leſs than the true Diſtance in any certain Proportion, as a half, a third part, or a quarter of the true Diſtance; and finding the Places of the required Vaniſhing Lines and Points in this ſeparate Picture, the Poſitions of all which will be Similar to thoſe of the correſponding Vaniſhing Points and Lines in the true Picture; and all Lines thus found in the ſeparate Picture, will bear the ſame Proportion to the correſponding Lines in the true Picture, as the aſſumed Diſtance doth to the true Diſtance; by which Rule the required Vaniſhing Points and Lines, or ſo much of them as can come within the Bounds of the real Picture, may be tranſcribed into it.

For the aſſumed Picture may be conſidered as a Plane parallel to the real Picture, placed ſo much nearer to the Eye, as the aſſumed Diſtance is leſs than the real Diſtance; in which Caſe it is evident, that the Images of all Lines and Points in the one, will be Similar to, and alike ſituated with thoſe in the other, and in the ſame Proportion to each other as the Diſtances are[b].

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[a] 10 El. 11.

[b] Theor. 23. B. I. and 17 El. 11.

SECTION II.

Of the Images of Points, Lines, and plain Figures, whoſe Relations to the Picture, or to any known Original Plane, are given.

DEF. 10.

THE Seat of any Point of an Original Line on any Plane (the Length of the Support of that Point being known) and the Interſection of an Original Line with any Plane, are called *Points of Relation* of the Original Line to that Plane.

DEF. 11.

The Vaniſhing and Interſecting Points of any Line, are *General Points of Relation* of that Line to all Planes whatſoever.

DEF. 12.

Any Point in one Plane with its Seat on another Plane, or any Point in the common Interſection of thoſe Planes, are *Points of Relation* of the one Plane to the other.

DEF. 13.

The common Interſection of any two Planes is a *Line of Relation* of thoſe two Planes to each other.

DEF. 14.

The Vaniſhing and Interſecting Lines of any Plane, are *General Lines of Relation* of that Plane to all other Planes.

PROP. XXX. PROB. XVIII.

The Center and Diſtance of the Picture, and the Perpendicular Seat of an Original Point on the Picture, with the Length of its Support, being given; thence to find the Image of that Point.

Fig. 103. Let O be the Center of the Picture, and A the given Seat of the Original Point.

Having drawn O A, from O and A draw any two parallel Lines O I, A B; make O I equal to the Diſtance of the Picture, and A B equal to the given Support, and draw I B, which will cut O A in *a*, the Image of the Point required.

Dem. For the Original of A O being perpendicular to the Picture [a], and A being the Interſecting Point of the Support of the Original Point [b], A O is the Indefinite Image of that Support; and A*a* repreſenting a Line equal to A B [c], which was taken equal to that Support, *a* is therefore the Image of the Point required. *Q. E. I.*

[a] Cor. 2. Theor. 5. B. I.
[b] Cor. Def. 5.
[c] Cor. 3. Prob. 5. B. II.

COR.

The Lines O I and A B are the Vaniſhing and Interſecting Lines of a Plane paſſing through the Original of A O; and as any two parallel Lines drawn through O and A, may be taken as the Vaniſhing and Interſecting Lines of a Plane in which the Original of A O lies, this Plane may be choſen at pleaſure, as may be moſt convenient; and the Line A O being thereby reduced into a Plane, whoſe Vaniſhing and Interſecting Lines are given, it becomes manageable accordingly by the Rules already taught [d].

[d] Sect. 2. B. II.

For A being the Interſecting Point of A O, and conſequently a Point in the Plane of the Picture, any Line A B drawn through A, may be the Interſecting Line of a Plane paſſing through A O; and O being the Vaniſhing Point of that Line, I O drawn through O parallel to A B, muſt be the Vaniſhing Line of that Plane.

The ſame is to be underſtood of any other Line whatſoever, whoſe Vaniſhing and Interſecting Points are given.

PROP. XXXI. PROB. XIX.

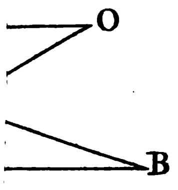
The Center and Diſtance of the Picture, and any two Points of Relation of an Original Line to the Picture being given; thence to find the Indefinite Image of that Line, its Seat on the Picture, the Angle it makes with its Seat, and the Vaniſhing and Interſecting Lines of the Plane of its Seat.

Fig. 104. 1. Let O be the Center of the Picture, and firſt, let A and B be the perpendicular Seats of any two Points of an Original Line on the Picture, the Length of the Supports of thoſe Points being known.

Through A and B draw G H, and through O draw E F parallel to it; take O *J* in E F equal to the Diſtance of the Picture, and having drawn O A and O B, by the help of the given Supports and of the Point *J*, find *a* and *b* the Images of the Points whoſe Seats are A and B [e]: through *a* and *b* draw *d z*, cutting E F and G H in *z* and *d*, and from O erect O I perpendicular to E F, and equal to the Diſtance of

[e] Prop. 30.

the



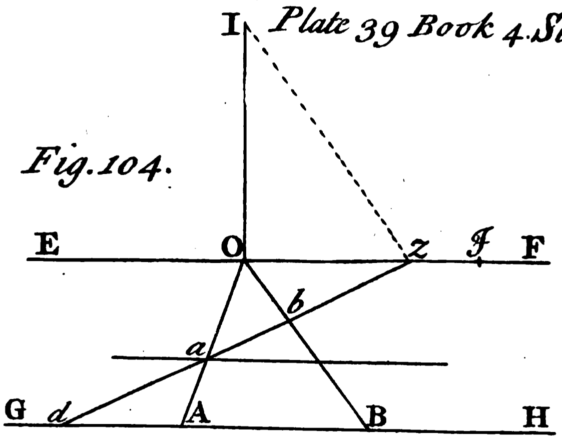


Fig. 104.

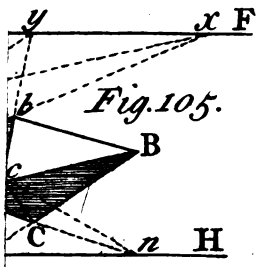
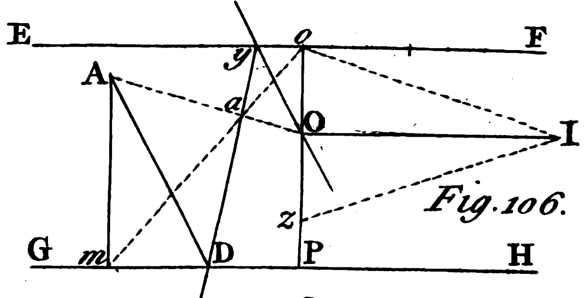


Fig. 105.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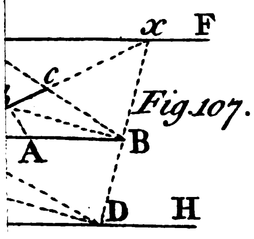


Fig. 107.

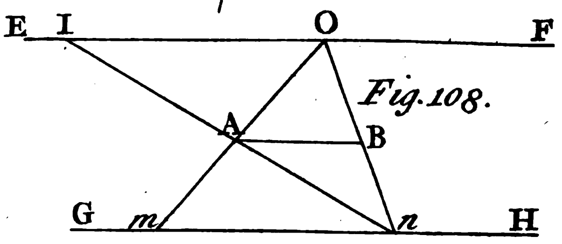
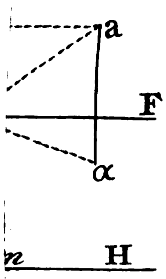


Fig. 108.



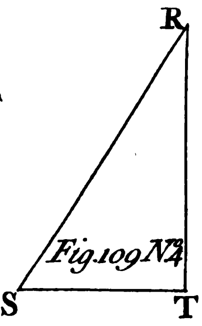


Fig. 109 Nº 4.

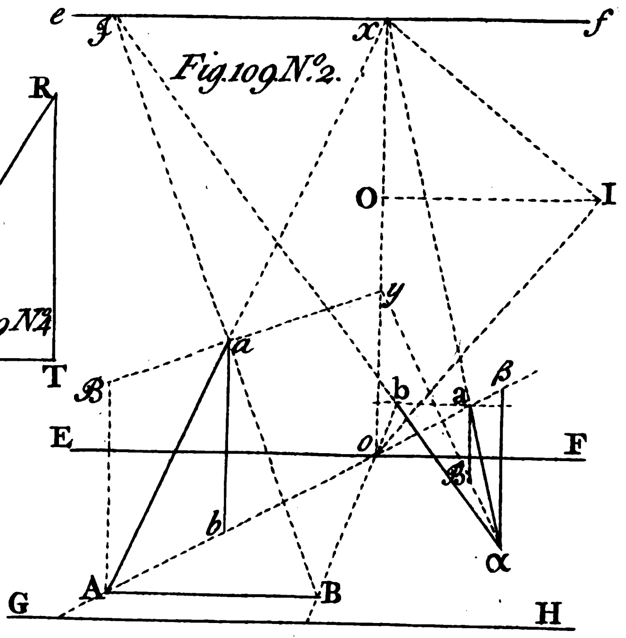


Fig. 109 Nº 2.

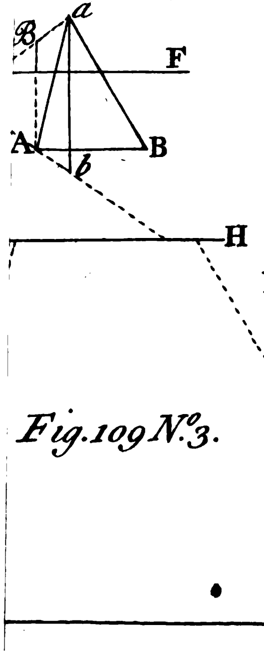


Fig. 109 Nº 3.

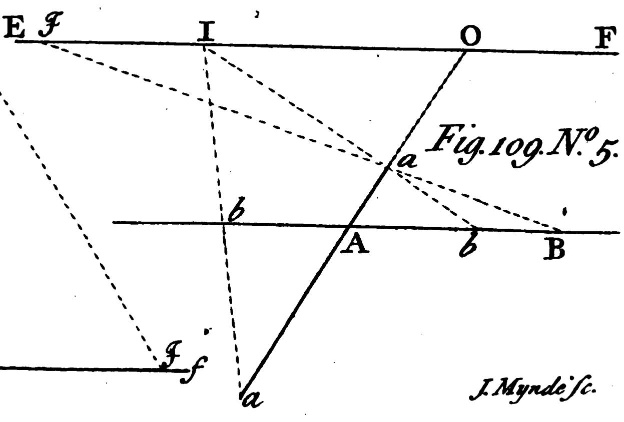


Fig. 109 Nº 5.

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the Picture, and draw Iz; then dz will be the Indefinite Image of the Original Line, z and d its Vaniſhing and Interſecting Points, GH its intire Seat on the Picture, and alſo the Interſecting Line of the Plane of its Seat, EF the Vaniſhing Line of that Plane, Iz the Radial of the Original Line, and IzO the Angle it makes with its Seat.

Dem. For GH drawn through the given Seats A and B, is the intire Seat of the Original Line on the Picture[a], and alſo the Interſecting Line of the Plane of its Seat[b]; and this Plane paſſing through the Perpendicular Supports of the Points of the Original Line, is therefore perpendicular to the Picture[c], conſequently EF drawn through O parallel to GH is the Vaniſhing Line of that Plane[d]; and a and b being the Images of two Points of the Original Line, dz drawn through a and b is the Indefinite Image of that Line, which being a Line in the Plane EFGH, z and d are its Vaniſhing and Interſecting Points, Iz its Radial, and IzO the Angle it makes with its Seat GH[e]. *Q.E.I.*

2. If the Support Bb of any Point b of the Original Line, with either of the Points d or z be given, the reſt of the things required may be found.

For d being given, dB gives GH, and its Parallel EF drawn through O; and db produced gives z.

Or z being given, zO gives EF, and its Parallel GH drawn through B, and zb produced gives d.

3. If the Support bB and the Line EF be given, together with the Angle of Inclination of the propoſed Line to its Seat, it being known which way that Inclination tends, all the reſt may be found.

For EF gives its Parallel GH drawn through B, and the Angle IzO made equal to the given Angle of Inclination, determines z, whence d is found as before.

4. If GH and the Image b of any Point of the propoſed Line be given, together with the Angle of Inclination of that Line to its Seat, the reſt may be found.

For GH gives its Parallel EF drawn through O, the Angle IzO determines z; and zb cuts GH in d.

5. Laſtly, if z and d be given, theſe alone determine all the reſt, z and O giving EF, and its Parallel GH is determined by d.

COR. 1.

If the Original Line be parallel to the Picture, its Image will be parallel to its Seat[f], and the Supports of all its Points will be equal: if therefore the Seats A and B of any two Points of that Line be given, the Image a of either of thoſe Points being found, will determine the intire Image of the Line propoſed; it being a Line drawn through a parallel to AB.

COR. 2.

If from O through any Point a of the Indefinite Image dz, a Line Oa be drawn cutting GH in A, A will be the Seat of the Original of a on the Picture[g].

PROP. XXXII. PROB. XX.

The Center and Diſtance of the Picture, and the Perpendicular Seats of the three angular Points of an Original Triangle on the Picture, with the Length of their Supports, being given; thence to find the Image of that Triangle, and the Vaniſhing and Interſecting Lines of its Plane.

Let O be the Center of the Picture, and A, B, and C the three given Seats. Fig. 105.

By the help of the given Seats and Supports, find the Indefinite Images lx and my of any two Sides ab and cb of the propoſed Triangle as lie moſt convenient[h], and through their Vaniſhing Points y and x draw EF, and through either of their Interſecting Points l or m draw GH parallel to EF; then through a and c already found, draw ac, and abc will be the Image of the propoſed Triangle, ABC will be its Seat, and EF and GH will be the Vaniſhing and Interſecting Lines of its Plane.

Dem. Becauſe all the Sides of a Triangle are in the ſame Plane[i], the Vaniſhing Points of any two Sides of that Triangle determine the Vaniſhing Line of its Plane[k], to which a Parallel being drawn through either of the Interſecting Points of thoſe Sides, it will be the Interſecting Line[l], conſequently EF drawn through the Vaniſhing Points y and x, is the Vaniſhing Line, and GH drawn parallel to EF through the Interſecting Point l or m, is the Interſecting Line of the Plane of the Triangle; and ab and cb being the determinate Images of the Sides of the Triangle whoſe Seats are

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[a] Def. 2.

[b] Cor. Def. 6.

[c] 18 El. 11.

[d] Cor. 1. Theor. 9. B. I.

[e] Theor. 11. B. I.

[f] Cor. 1. Theor. 15. B. I.

[g] Cor. Def. 5.

[h] Prop. 31.

[i] 2 El. 11.

[k] Cor. 2. Theor. 10. B. I.

[l] Cor. 2. Def. 10. B. I.

AB and CB, ac muſt neceſſarily be the Image of the third, which being produced both ways, will cut EF and GH in z and n, its Vaniſhing and Interſecting Points. *Q. E. I.*

COR.

If the Indefinite Image lx of any Side ab of the Triangle, and C the Seat of the oppoſite angular Point c, with the Length of its Support be given; the Vaniſhing and Interſecting Lines EF and GH may be thence found.

For lB drawn parallel to xO is the Seat of lx on the Picture[a]; Ob gives B, the Seat of any Point b taken at pleaſure in lx[b]; by the help of CB, and Oy drawn parallel to it, the Image c of the Point whoſe Seat is C, is found[c]; and bc produced cuts Oy and CB in y and m, its Vaniſhing and Interſecting Points[d], and yx and lm determine EF and GH.

And thus the Vaniſhing and Interſecting Lines of a Plane, paſſing through any given Line, and any Point out of that Line whoſe Seat and Support on the Picture are given, may be determined; ſeeing from any Point without a Line, a Triangle may be conſtituted on that Line.

PROP. XXXIII. PROB. XXI.

The Center of the Picture, and the Vaniſhing and Interſecting Lines of an Original Plane, with the Image of a Triangle in that Plane being given; thence to find the Perpendicular Seat of that Triangle on the Picture.

Fig. 105. Let O be the Center of the Picture, EFGH the given Plane, and abc the Image of a Triangle in that Plane.

Produce any two of the given Sides ab and cb to their Vaniſhing and Interſecting Points x, l, and y, m; through x and l draw xO, and lB, parallel to it, and from O through a and b draw OA, OB, then AB will be the Seat of ab, and A and B the Seats of a and b on the Picture[e]; then through y and m draw yO, and mC parallel to it, and through O and c draw OC, and C will be the Seat of c, and mC the Seat of cb on the Picture, which muſt neceſſarily alſo paſs through B the Seat of b, this laſt being a Point in cb. *Q. E. I.*

COR.

If the Original Plane be perpendicular to the Picture, the Perpendicular Seats of all Points or Lines in that Plane on the Picture, muſt fall in the Interſecting Line of that Plane[f].

PROP. XXXIV. PROB. XXII.

The Center and Diſtance of the Picture, and either of the Lines of Relation of an Original Plane to the Picture[g], and the Image of a Point in that Plane, with its Seat on the Picture being given; thence to find the other Line of Relation of that Plane to the Picture.

Fig. 106. 1. Let O be the Center of the Picture; and firſt, let EF be the Vaniſhing Line of a Plane, a the Image of a Point in that Plane, and A its Perpendicular Seat on the Picture.

Produce the Support Aa to its Vaniſhing Point O[h], and through O draw any Line Oy cutting EF in y, and having drawn AD parallel to Oy, draw ya cutting AD in D, and through D draw GH parallel to EF, and GH will be the Interſecting Line of the Original Plane propoſed.

Dem. For Oy and AD being the Vaniſhing and Interſecting Lines of a Plane paſſing through AO[i], and yD which paſſes through a, being the Image of a Line in that Plane, D is therefore the Interſecting Point of yD; but y being a Vaniſhing Point in EF, and a the Image of a Point in the Original Plane, ya is therefore alſo the Image of a Line in that Plane, and conſequently GH drawn through D the Interſecting Point of yD, parallel to EF, is the Interſecting Line of that Plane. *Q. E. I.*

Any two other parallel Lines Oo and Am drawn through O and A will equally ſerve the Purpoſe, as in the Figure.

2. If GH and the Support Aa be given, EF may thence be found. For the Parallels Oy and AD being drawn as before, y the Interſection of Da with Oy is the Point through which EF muſt be drawn parallel to GH.

[a] Art. 5. Prop. 31.

[b] Cor. 2. Prop. 31.

[c] Prop. 30.

[d] Prop. 31.

[e] Prop. 31. and Cor. 2.

[f] 38 El. 11.

[g] Def. 13, 14.

[h] Cor. 2. Theor. 5 B. I.

[i] Cor. Prop. 30.

3. If GH and the Angle of Inclination of the Original Plane to the Picture be given, and it be known which way that Inclination tends, EF may be thence found.

Through O draw OI parallel to GH, and equal to the Diſtance of the Picture; and having drawn the Vertical Line PO, draw I*o* cutting it in *o*, making the Angle I*o*O equal to the given Angle of Inclination; and EF drawn through *o* parallel to GH, will be the Vaniſhing Line ſought.

For I*o*O is the Angle of Inclination of the Plane EFGH to the Picture[a].

Here the Original Plane is ſuppoſed to incline upwards, but if the Inclination be the contrary way, EF muſt be drawn through *z*, where I*z* cuts PO, making I*z*O equal to the Angle of Inclination propoſed.

But if EF alone be given, the Problem is not determined; for although the Angle of Inclination I*o*O of the propoſed Plane to the Picture is thereby known, yet any Line GH parallel to EF may be taken as the Interſecting Line of a Plane inclining to the Picture in that Angle.

Note, *By the Plane* EFGH, *is meant the Plane whoſe Vaniſhing and Interſecting Lines are* EF and GH: *the ſame is to be underſtood of the Planes y*OAD, O*o*A*m*, *and of all other Expreſſions of the like ſort.*

PROP. XXXV. PROB. XXIII.

The Center and Diſtance of the Picture, and the Vaniſhing Line EF of an Original Plane, and the Image *ab* of a Line in that Plane of a known Length, being given; thence to find the Interſecting Line of that Plane. Fig. 107.

Produce *ab* to its Vaniſhing Point *x*, and ſet off the Diſtance of that Point at *y* in the Line EF[b], and having through *a* drawn CB parallel to EF, draw *yb* and *ya*; then take BC in the Line CB equal to the Original of *ab*, and draw CD parallel to *y*B, cutting *ya* in D, through which, GH being drawn parallel to EF, it will be the Interſecting Line deſired.

Dem. For *y* being the Point of Diſtance of the Vaniſhing Point *x*, the Originals of *ab* and *a*B are equal[c], and *a*B will be to CB the Length of its Original, as the Diſtance between EF and CB is to the Diſtance between EF and its Interſecting Line[d], or the Depth of the Original Plane[e].

Now in the Similar Triangles *ya*B, *a*CD, $Ba : aC :: ya : aD$

And by Compoſition $Ba : Ba + aC = BC :: ya : ya + aD = yD$.

But it is evident, that *ya* is to *y*D, as the Diſtance between EF and CB is to that between EF and GH, and conſequently the Interſecting Line GH is rightly determined. *Q. E. I.*

PROP. XXXVI. PROB. XXIV.

The Interſecting Line GH of a Plane, and the Image *ac* of a Line in that Plane, divided into two Parts in *b*, and the Proportion of the Originals of thoſe Parts being given; thence to find the Vaniſhing Line of that Plane. Fig. 107.

Through *a* draw *a*B parallel to GH, and from *a* ſet off two Parts *a*A and AB in the ſame Proportion to each other as are the Originals of *ab* and *bc*; draw A*b* and B*c* meeting in *z*, through which draw EF parallel to GH, and EF will be the Vaniſhing Line required.

Dem. For the Original of *a*B being parallel to the Picture, its Parts are in the ſame Proportion to each other as their Originals[f]; wherefore the Originals of *ac* and *a*B are divided in *b* and A in the ſame Proportion; the Originals of the Triangles *a*A*b*, *a*B*c* are therefore Similar, and conſequently the Originals of A*b* and B*c* are parallel[g]; their Images therefore meet in a Point of the Vaniſhing Line of the Plane in which they lie[h]; which Point being *z*, EF drawn through *z* parallel to GH, is the Vaniſhing Line ſought. *Q. E. I.*

COR.

If the given Image be *a*B parallel to GH, it will be ſufficient to know the Length of its Original, which being ſet off any where on GH, as at DL, and L*a* and DB being drawn, they will meet in *x* a Point in the Vaniſhing Line deſired[i].

PROP.

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[a] Cor. Prop. 13.

[b] Prop. 12.

[c] Cor. 1. Prob. 9. B. II.

[d] Cor. 1. Prob. 6. B. II.

[e] Cor. 3. Def. 18. B. I.

[f] Cor. 1. Theor. 23. B. I.

[g] 2 El. 6.

[h] Theor 5, and 10. B. I.

[i] Cor. 1. Prob. 6. B. II.

PROP. XXXVII. PROB. XXV.

Fig. 107. The Center and Diſtance of the Picture, and the determinate Image ac, of a Line divided into two Parts in b, being given, and the true Meaſures of thoſe Parts being known; thence to find the Vaniſhing and Interſecting Points of that Line.

Through a draw any Line aB, and in it take aA and AB in the ſame Proportion to each other, as are the Originals of ab and bc, and having drawn Ab and Bc meeting in z, draw zx parallel to aB, which will cut ac in x its Vaniſhing Point: and if the Diſtance of the Vaniſhing Point x be ſet off at y in the Line zx [a], and the Proportional Meaſure aB of the Part ab be found, and BC be taken equal to the Original of that Part; CD drawn parallel to yB will cut xB in D, through which LD drawn parallel to aB will give L the Interſecting Point of the Line propoſed.

Dem. For aB may be taken as the Image of a Line parallel to the Picture, and in the ſame Plane with ac, and z being a Vaniſhing Point in that Plane [b], zx parallel to aB is its Vaniſhing Line, and conſequently x is the Vaniſhing Point of ac. The reſt of the Conſtruction is the ſame with that of Prop. XXXV. whereby the Interſecting Line LD, and conſequently L the Interſecting Point of the Line ac, is found. *Q. E. I.*

PROP. XXXVIII. PROB. XXVI.

The Center and Diſtance of the Picture being given, and an Original Plane parallel to the Picture being propoſed, and the Diſtance between that Plane and the Picture being known; thence to find the Proportion of the Images of any Lines in that Plane to their Originals.

Fig. 108. Through O the Center of the Picture draw any Line EF, and any other Line GH parallel to it; and having from O drawn any Line Om cutting GH in m, take OI in EF equal to the Diſtance of the Picture, and mn in GH equal to the given Diſtance between the Picture and the parallel Plane, and draw In cutting Om in A; through A draw AB parallel to EF, and terminated in B by a Line On; then AB will be the Image of a Line in the parallel Plane, whoſe Original is equal to mn; and the Images of all Lines in the parallel Plane will be to their Originals as AB is to mn.

Dem. For EF and GH are the Vaniſhing and Interſecting Lines of a Plane perpendicular to the Picture, which muſt cut the parallel Plane in ſome Line parallel to EF [c], and Om being a Line in the Plane EFGH perpendicular to the Picture, and mA repreſenting a part of that Line equal to mn [d] the Diſtance between the Picture and the parallel Plane, A muſt therefore be the Image of a Point in the parallel Plane, and conſequently AB parallel to EF, is the Interſection of that Plane with the Plane EFGH; and becauſe of the Vaniſhing Point O, mn is the true Meaſure of AB [e], therefore AB is the Proportional Meaſure of all Original Lines in the parallel Plane which are equal to mn; and the Images of all Lines in the parallel Plane being proportional to their Originals [f], the Proportion of the Image of any Line in that Plane to its Original, will be as AB to mn. *Q. E. I.*

PROP. XXXIX. PROB. XXVII.

The Vaniſhing and Interſecting Lines of an Original Plane, and the Image of the Seat of a Point on that Plane, with the Length of its Support, being given; thence to find the Image of that Point.

This admits of three Caſes:

1. When the propoſed Support of the Original Point is parallel to the Vertical Line of the Original Plane, and conſequently to the Picture.

This happens either when the Original Plane inclines to the Picture, and the Oblique Seat of the Original Point on that Plane is given; or when the Original Plane is perpendicular to the Picture, in which Caſe the Perpendicular and Oblique Seats of the propoſed Point are the ſame [g].

2. When the propoſed Support inclines to the Picture.

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[a] Prop. 12.

[b] Prop. 36.

[c] Cor. Theor. 3. B. I.

[d] Prob. 6. B. II.

[e] Cor. 4. Prob. 6. B. II.

[f] Cor. 1. Theor. 23. B. I.

[g] Def. 3

This happens when the Original Plane is neither parallel nor perpendicular to the Picture, and the Perpendicular Seat of the propoſed Point on that Plane is given.

3. When the propoſed Support is perpendicular to the Picture.

This happens when the Original Plane is parallel to the Picture, and the Perpendicular Seat of the propoſed Point on that Plane is given.

CASE 1.

When the propoſed Support of the Original Point is parallel to the Vertical Line of the Original Plane.

Let E F G H be the given Plane, and A the Seat of an Original Point on that Plane. Fig. 109. N°. 1.

METHOD 1.

Through A draw any Line *g x*, cutting E F and G H in *x* and *g*; from A and *g* draw A *a* and *g b* perpendicular to E F, and having made *g b* equal to the Support of the propoſed Point, a Line *bx* will cut A *a* in *a* the Image of the Point required.

Dem. For A *a* perpendicular to E F, is the Indefinite Image of the Support of the propoſed Point, and *g b* being the Interſecting Line of a Plane paſſing through A *a* and *g x*, whoſe Vaniſhing Point is *x* [a], the Original of A *a* is equal to *g b* [b], which laſt being taken equal to the Support of the Original Point, A *a* is the determinate Image of that Support; wherefore *a* is the Image of the Point deſired. *Q. E. I.*

METHOD 2.

Having through A drawn *g x*, cutting E F and G H in *x* and *g* as before; from A draw A B parallel to E F, and having taken *g m* in G H equal to the Support of the propoſed Point, draw *m x* cutting A B in B; from A erect A *a* perpendicular to E F and equal to A B, and *a* will be the Image of the Point propoſed.

Dem. For the Originals of A B and A *a* being both parallel to the Picture, and meeting in the Original of A, they are in a Plane parallel to the Picture; wherefore A B and A *a*, which were taken equal, repreſent equal Lines [c], and A B repreſenting a Line equal to *g m*, which was made equal to the Support of the Original Point, A *a* is therefore the Image of that Support, and *a* the Image of the Point required. *Q. E. I.*

COR.

If the propoſed Point be behind the Directing Plane (in which Caſe the Image of its Seat will be in the Tranſprojective Part of the Plane E F G H [d]) the Tranſprojective Image of that Point may be found by either of theſe Methods.

Thus if a, be the Image of the Seat of the propoſed Point; having through a, drawn a *x* till it cut G H in *g*, and taken *g b* perpendicular to E F, and equal to the given Support as before; *b x* will cut a Line a *α* drawn from a, perpendicular to E F, in *α* the Tranſprojective Image of the propoſed Point: or *g m* being made equal to the given Support, produce *m x* till it cut a b drawn parallel to E F in b, and make a *α* equal to a b, and thereby the ſame Point *α* will be found.

For in the Plane *e f g b*, the Original of a *α* is equal to *g b*; and in the Plane E F G H, the Original of a b is equal to *g m* [e]: the reſt is evident.

And here the Original Point being ſuppoſed to be above the Plane E F G H, its Tranſprojected Image falls below its Seat a, the Tranſprojected Image a *α* of its Support being inverted [f].

GENERAL COROLLARY.

When the Oblique Seat of the propoſed Point is given, the Center of the Picture is not concerned; and therefore the Methods above propoſed ſerve alike, whether the Original Plane be perpendicular or inclining to the Picture.

CASE 2.

When the propoſed Support of the Original Point inclines to the Picture.

Let O be the Center of the Picture, E F G H the given Plane, and A the Image of the Seat of the propoſed Point on that Plane. Fig. 109. N°. 2, 3.

METHOD 1.

From A erect A *B* perpendicular to E F, and conſidering it as a Line parallel to the Picture, make it repreſent a Line equal to the Support of the propoſed Point [g]; and having found *x*, the Vaniſhing Point of Perpendiculars to the Plane E F G H [h], draw

x A;

[a] Cor. 1. Theor. 15. B. I.

[b] Cor. 4. Prob. 6. B. II.

[c] Cor. 2. Theor. 23. B. I.

[d] Cor. 4. Theor. 4. and Def. 23. B. I.

[e] Cor. 1. Prob. 6. B. II.

[f] Art. 21. Sect. 3. B. I.

[g] Method 1, or 2. Caſe 1.

[h] Prop. 20.

xA; then ſet off the Diſtance of the Vaniſhing Point x at y in the Line xo, and draw yB cutting xA in a, and a will be the Image of the propoſed Point, and Aa the Image of its Support.

Dem. Becauſe of the Vaniſhing Point x, the Original of Aa is perpendicular to the Plane EFGH, Aa is therefore the Indefinite Image of the Support of the propoſed Point; and becauſe the Original of AB is parallel to the Picture and to the Line xo, xo is the Vaniſhing Line of a Plane paſſing through AB[a], in which Plane the Line xA alſo lies; and y being the Point of Diſtance of the Vaniſhing Point x, AB and Aa repreſent equal Lines[b], and conſequently Aa is the determinate Image of the Support, and a the Image of the Point propoſed. *Q. E. I.*

METHOD 2.

Through x and A draw ef and AB, both parallel to EF; make AB in the Plane EFGH repreſent a Line equal to the Support of the propoſed Point, and having ſet off the Diſtance of the Vaniſhing Point x, at y in the Line ef, draw yB, which will cut Aa in the ſame Point a as before.

Dem. For it is evident, that xA and AB are in a Plane whoſe Vaniſhing Line is ef parallel to AB, and that therefore AB and Aa repreſent equal Lines. *Q. E. I.*

COR.

By either of theſe Methods, the Line xA is rendered manageable according to the Rules already ſhewn[c]; for by the firſt Method, xA is reduced into a Plane whoſe Vaniſhing Line is xo, and the proportional Meaſures on AB, a Line in that Plane parallel to the Picture, are known[d]; in the ſecond Method, the Line xA is reduced into a Plane whoſe Vaniſhing Line is ef, and the proportional Meaſures on AB, a Line in that Plane parallel to the Picture, are known; which laſt Line is alſo the Interſection of that Plane with the Plane EFGH.

The ſame Methods equally ſerve to render any other Line manageable, which inclines anywiſe to the Plane EFGH, its Vaniſhing Point and Interſection with that Plane being given; the Demonſtration being the ſame, whether x be the Vaniſhing Point of Perpendiculars to the Plane EFGH, or any other Vaniſhing Point out of EF.

METHOD 3.

If the Vaniſhing Point x be at an inconvenient Diſtance, it will be beſt to make uſe of the Oblique Seat of the propoſed Original Point, which may be found by its Perpendicular Seat in this manner.

Fig. 109. N°. 4. Any where a-part make a Rectangular Triangle RTS, having its Side RT equal to the perpendicular Support of the propoſed Point, and the Angle RST equal to the Inclination of the Picture to the Original Plane; then if R be conſidered as the Original Point, and T as its Perpendicular Seat on the Original Plane, the Hypotenuſe RS will be its oblique Support, and ST will be the Diſtance between its Perpendicular and Oblique Seats on that Plane; the Triangle RTS repreſenting the Triangle aAB in Fig. 88. Prop. III.

Fig. 109. N°. 2, 3. This being done, from o the Center of the Vaniſhing Line EF, through A the given Image of the Perpendicular Seat, draw vA, and having found a Part Ab in that Line, repreſenting a Line equal to ST[e], either beyond or on the hither Side of A, according as the Inclination of the Picture is towards or from the Eye, the Point b will then be the Image of the Oblique Seat of the Original Point on the Plane EFGH; by the help of which, and of the Oblique Support RS, the Image of the propoſed Point may be found, as in the firſt Caſe of this Problem.

Dem. For the Line which joins the Seats of the Original Point in the Plane EFGH, having o for its Vaniſhing Point[f], and A being the Image of the Perpendicular Seat; oA is the Indefinite Image of that Line; and Ab being made to repreſent a Line equal to ST, the Diſtance between the Perpendicular and Oblique Seats of the Original Point, b is therefore the Image of its Oblique Seat. *Q. E. I.*

COR.

Fig. 109. N°. 2. The Corollary of the preceeding Caſe is likewiſe applicable to all the Methods of this, as may be ſeen by the Figure, where a, is the Perpendicular Seat of α in the Tranſprojective Part of the Plane EFGH; only obſerving that β the Oblique Seat of α, in Tranſprojection, falls beyond a, the Point β repreſenting a Point nearer to the Eye

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[a] Cor. 1. Theor. 15. B. I.

[b] Cor. 1. Prob. 8. B. II.

[c] Sect. 2. B. II.

[d] Gen. Cor. Prob 6. B. II.

[e] Cor. 1. Prob. 8. B. II.

[f] Cor. Prop. 3.

than a, it being farther from the Vaniſhing Point o of the Line $o\beta$ in which they both lie[a].

CASE 3.

When the propoſed Support of the Original Point is perpendicular to the Picture.

Here the Original Plane being ſuppoſed parallel to the Picture, the Diſtance between the Picture and that Plane, as well as the Length of the Support of the Original Point on that Plane, muſt be known.

Let then O be the Center of the Picture, and A the Image of the Perpendicular Seat of the Original Point on the Original Plane. Fig. 109. N°. 5.

METHOD 1.

Draw A O, and through O and A draw any two parallel Lines E F and A B; take O I equal to the Diſtance of the Picture, and on A B take Ab ſo as to repreſent a Line in the Original Plane equal to the Support of the Original Point[b], either on the ſame or on the contrary Side of A from the Point I, according as the Original Point lies nearer or farther than the Original Plane; then Ib being drawn, it will cut A O in a, the Image of the Point deſired, and Aa will be the Image of its Support.

Dem. For the Support of the Original Point being perpendicular to the Picture, its Vaniſhing Point is O the Center of the Picture[c]; wherefore A O is the Indefinite Image of that Support, and I, being the Point of Diſtance of O, and Ab repreſenting a Line parallel to the Picture, the Originals of Ab and Aa are equal[d]; but the Original of Ab being by Conſtruction equal to the Support of the propoſed Point, Aa is the determinate Image of that Support, and therefore a is the Image of the Point propoſed. *Q. E. I.*

METHOD 2.

If inſtead of making I O equal to the Diſtance of the Picture, a Diſtance JO be taken equal to that between the Eye and the Original Plane; then A B being made equal to the propoſed Support, a Line JB will give the ſame Point a as before.

Dem. For an Original Line in a Plane parallel to the Picture, being to its Image, as the Diſtance of the Eye from the Original Plane, is to its Diſtance from the Picture[e]; if JO be taken equal to the Diſtance of the Eye from the Original Plane, it will have the ſame Proportion to I O, as the true Meaſure of the Original Line has to its proportional Meaſure on the Line A B; and therefore the true Meaſure being ſet off on A B, the Line JB will determine the Point a[f]. *Q. E. I.*

COR.

If the Original Point be behind the Eye, its Image is found after the ſame manner as at Prob. V. Book II. only uſing the proportional Meaſure of the Support on the Line A B, inſtead of the true Meaſure of that Support; which laſt ought to be uſed in caſe A B were the Interſecting Line of the Plane E F A B.

PROP. XL. PROB. XXVIII.

Any two Points of Relation of a Line to an Original Plane being given; thence to find the Indefinite Image of that Line, its Seat on, and Interſection with the Original Plane, and the Vaniſhing and Interſecting Lines of the Plane of its Seat.

CASE 1.

When the Supports of the Points whoſe Seats are given or required, are parallel to the Picture.

1. Let E F G H be the given Plane; and firſt, let B and C be the Images of the Oblique Seats of two Points of an Original Line on that Plane, the Length of their Supports being known. Fig. 110. N°. 1, 2.

Through B and C draw Dy, cutting E F and G H in y and D, then find the Images b and c of the Points whoſe Seats are given[g], and through b and c draw dz cutting Dy in A, and from y and D draw yz and Dd, both perpendicular to E F, cutting dz in z and d; then dz will be the Indefinite Image of the Original Line, z and d its Vaniſhing and Interſecting Points, A the Image of its Interſection with the Original Plane, Dy the Image of its Oblique Seat on that Plane, and zy and Dd the Vaniſhing and Interſecting Lines of the Plane of its Seat[h].

Dem.

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[a] Cor. 4. Theor. 4. B. I.

[b] Prop. 38.

[c] Cor. 2. Theor. 5. B. I.

[d] Cor. 1. Prob. 8. B. II.

[e] Theor. 23. B. I.

[f] Cor. 5 Prob. 8. B. II.

[g] Caſe 1. Prop. 39.

[h] Def. 6.

Dem. The Line Dy which paſſes through B and C, is the Image of the Seat of the Original Line on the Plane EFGH[a], and y and D are its Vaniſhing and Interſecting Points; and dz paſſing through b and c, is the Image of the Original Line, which being in the ſame Plane with its Seat, the Interſection A of dz and Dy is therefore the Image of the Interſection of the Original Line with the Plane EFGH; and y being a Vaniſhing Point, and D an Interſecting Point in the Plane of the Seat of the Original Line, the Lines yz and Dd drawn perpendicular to EF, are the Vaniſhing and Interſecting Lines of that Plane[b], which by their Interſections with dz determine z and d, the Vaniſhing and Interſecting Points of the Original Line. *Q. E. I.*

2. If the Support bB of any Point b of the Original Line, and either of the Points d, A, or z be given; the reſt of the things required may be found.

For d being given, dD perpendicular to GH gives D, DB gives y, and yz parallel to dD cuts db produced in z, and the Interſection of dz and Dy gives A.

If A be given, AB gives Dy, y and D give yz and Dd, and Ab produced determines z and d.

Or if z be given, zy is found, yB produced gives D, and conſequently Dd, and zb produced gives A and d.

3. If any two of the three Points d, A, and z, be given, every thing elſe may be found.

For d and A being given, dD, Dy and yz are thereby found, and dA produced gives z.

If A and z be given, zy, yD, and Dd are thereby found, and zA produced gives d.

Or if d and z be given, zy and Dd are thence found, and the Interſection of dz and Dy gives A.

4. If any two Points B and C in the Seat Dy, and any one Point b in the Image of the propoſed Line be given, together with the Angle of Inclination of that Line to its Seat, it being known which way that Inclination tends, all the reſt may be thence found.

For by BC, the Lines Dy, yz, and Dd are determined, and if a Point z be found in yz, ſubtending with y an Angle equal to the given Angle of Inclination[c], a Line drawn through z and b will be the Indefinite Image required, and will cut Dy and Dd in A and d.

But in order to find the Point z by the Angle it ſubtends with y, the Center and Diſtance of the Picture muſt be known, neither of which are concerned in the other *Data*.

COR. 1.

Fig. 110. N°. 3.

When the Original Line is parallel to the Plane EFGH, but not to the Picture, the Vaniſhing Point z of that Line coincides with y the Vaniſhing Point of its Seat[d], and the Original Line being parallel to its Seat, they neither interſect nor make any Angle with each other; ſo that in this Caſe, the Point A, and the Angle of Inclination of the Original Line to its Seat, have no Place: however the Images of the Original Line and its Seat may be found by the remaining *Data* as before; ef and dD drawn through z and d perpendicular to EF, being the Vaniſhing and Interſecting Lines of the Plane of the Seat of the propoſed Line on the Plane EFGH.

COR. 2.

Fig. 110. N°. 4.

When the Original Line is parallel to the Picture, but not to the Plane EFGH, its Oblique Seat on the Original Plane is alſo parallel to the Picture, and conſequently to EF[e]; and the Image Ac of the Original Line makes the ſame Angle with AC the Image of its Seat, as their Originals do[f].

Hence the Image Ac may be found by the Images B and C of the Seats of any two Points of that Line, the Length of their Supports being given[g]; or by the Interſection A of the propoſed Line with its Seat, and the Angle they make together; or laſtly, by that Angle, and the Seat and Support of any other Point of the Original Line.

For the Oblique Supports Bb, Cc, of all the Points of the Original Line, being parallel to the Vertical Line of the Plane EFGH, and at an equal Diſtance from the Picture, they muſt all lie in a Plane $bBCc$ parallel to the Picture; the Interſection BC of which Plane, with the Plane EFGH, muſt be parallel to GH[h], and conſequently to EF.

COR. 3.

If the Original Line be parallel to the Plane EFGH, as well as to the Picture; it

[a] Def. 4.

[b] Prop. 2.

[c] Prop. 24.

[d] Cor. 3. Theor. 10. B. I.

[e] Cor. 1. Theor. 15. B. I.

[f] Cor. 3. Theor. 2. B. I.

[g] Caſe 1. Prop. 39.

[h] 16 El. 11.

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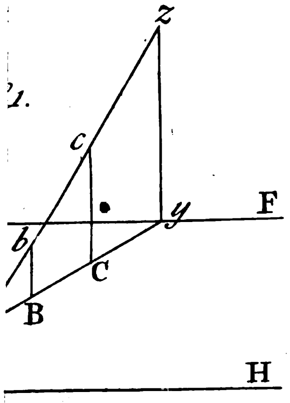
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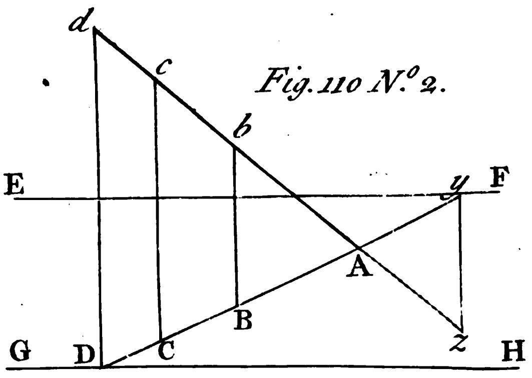


Fig. 110 N°. 2.

ig. 110 N°3.

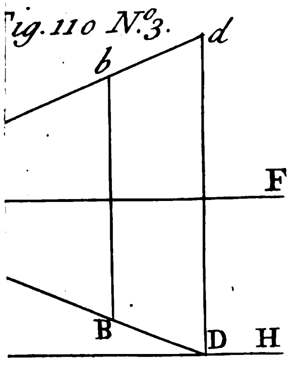
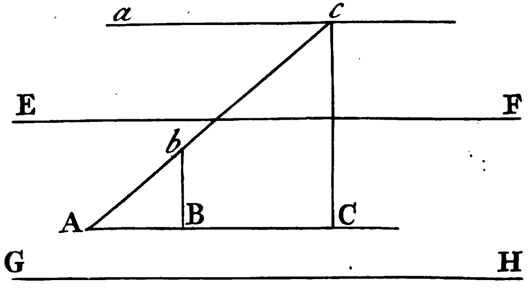


Fig. 110 N°4.



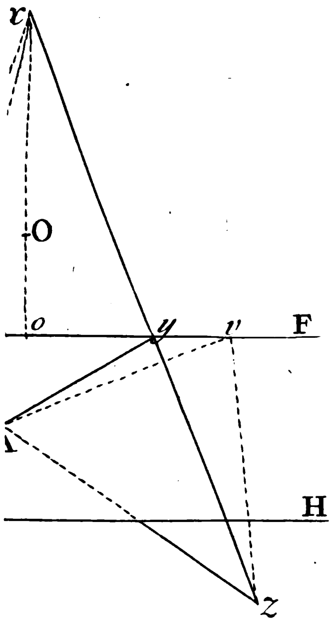
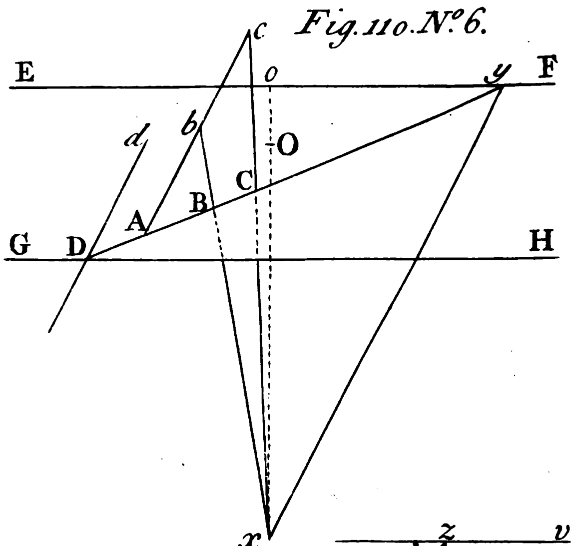


Fig. 110. N°6.



110 N°7.

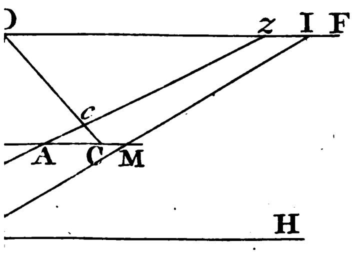
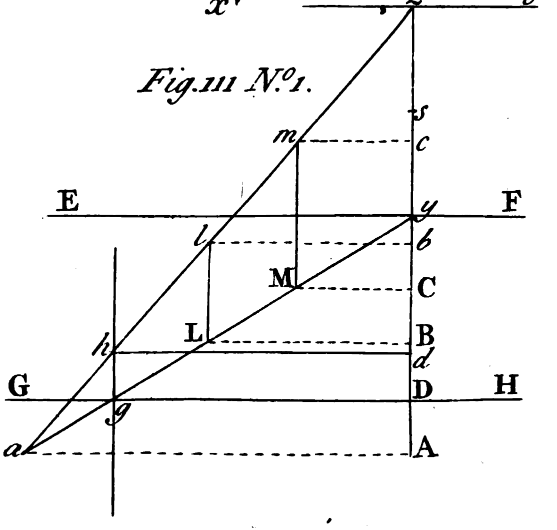


Fig. 111 N°1.



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being then parallel to its Seat, their Images may be found by the Seat and Support of any one Point of the Original Line.

For if the Support Cc of any Point c of the Original Line be found, ac and AC drawn through c and C parallel to EF, will be the Images of the Original Line and its Oblique Seat on the Plane EFGH.

CASE 2.

When the Supports of the Points whoſe Seats are given or required, incline to the Picture.

Fig. 110. N°. 5.

Let O be the Center of the Picture, and EFGH the given Plane, inclining to the Picture; and firſt, let B and C be the Images of the Perpendicular Seats of two Points of the propoſed Line on that Plane, the Length of their Supports being known.

Through B and C draw Dy, cutting EF and GH in y and D, as before; and having found x the Vaniſhing Point of Perpendiculars to the Plane EFGH[a], draw xB, xC, the Indefinite Images of the propoſed Supports, and find b and c the Images of the Points whoſe Seats are given[b]; then draw xy, to which through D draw Dd parallel, and through b and c draw dz, cutting xy and Dd in z and d; then dz will be the Indefinite Image of the propoſed Line, z and d its Vaniſhing and Interſecting Points, Dy the Image of its Perpendicular Seat, and A the Image of its Interſection with the Plane EFGH, and xy and Dd will be the Vaniſhing and Interſecting Lines of the Plane of its Seat.

Dem. The Plane of the Seat of the propoſed Line being in this Caſe perpendicular to the Plane EFGH, its Vaniſhing Line muſt paſs through x[c]; and Dy which paſſes through B and C, being the Perpendicular Seat of the propoſed Line on the given Plane, and y its Vaniſhing Point, the Vaniſhing Line of the Plane of its Seat muſt alſo paſs through y; and therefore xy is the Vaniſhing Line, and conſequently Dd parallel to xy, is the Interſecting Line of the Plane of the Perpendicular Seat of the propoſed Line on the Plane EFGH: all the reſt is evident. *Q. E. I.*

The ſame correſponding Points being given as in the laſt Caſe, they will ſerve to determine all the reſt; ſave only that here, the Interſecting Point d of the Original Line is not alone ſufficient to determine the Point D, ſeeing Dd muſt be parallel to yz, and therefore cannot be found unleſs D, y, z, or A be known; if either of the three firſt of theſe with d be given, the Practice is evident; but if d and A be only given, recourſe muſt be had to the following Method.

From d draw any Line $d\Delta$, cutting GH in Δ, and draw ΔA, cutting EF in v, from whence drawing vz parallel to $d\Delta$, it will cut dA in z its Vaniſhing Point, by which every thing elſe may then be determined.

For the Originals of dA and Δv which meet in A, are in the ſame Plane[d], and d and Δ being the Interſecting Points of theſe two Lines, $d\Delta$ is the Interſecting Line of their Plane, and v being a Vaniſhing Point in the ſame Plane, vz parallel to $d\Delta$ is the Vaniſhing Line of that Plane, and conſequently z is the Vaniſhing Point of dA.

SCHOL.

This Caſe differs very little from the preceeding, ſave that the Lines Bb, Cc, yz, and all others which were there directed to be drawn perpendicular to EF, muſt here be all drawn to the Point x, excepting only the Interſecting Line Dd, which muſt in both Caſes be drawn parallel to its correſponding Vaniſhing Line yz.

COR. 1.

When the Original Line is parallel to the Plane EFGH but not to the Picture, the Method is the ſame as before; ſave that the Vaniſhing Line of the Plane of the Seat, inſtead of being drawn perpendicular to EF, muſt be drawn through x, as juſt mentioned.

COR. 2.

When the Original Line is parallel to the Picture, but not to the Original Plane, its Perpendicular Seat on that Plane is not parallel to the Picture, nor is the Angle made by the Original Line with its Seat equal to that made by their Images; if therefore the Angle of Inclination of the Original Line to its Seat, together with their Interſection A, or its equivalent bB, the Support of a Point b of the Original Line, be alone given, the following Method muſt be uſed.

Fig. 110. N°. 6.

Find the Vaniſhing Line xy of Planes perpendicular to the Plane EFGH, whoſe

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[a] Prop. 20.

[b] Caſe 2. Prop. 39.

[c] Cor. 3. Prop. 20.

[d] 2 El. 11.

Interſections with that Plane make with their own Interſecting Lines, an Angle equal to the Angle of Inclination propoſed[a], and from y through A or B draw Dy; then Ab drawn through A or b parallel to xy, will be the Image of the Line propoſed, and Dy the Image of its Perpendicular Seat on the Plane EFGH.

For Ab being the Image of a Line parallel to the Picture in the Plane xyDd[b], and this Plane being perpendicular to the Plane EFGH, the Interſection Dy of theſe two Planes is therefore the Perpendicular Seat of Ab on the Plane EFGH; but the Originals of Dd and Ab being parallel, they make the ſame Angle with the Original of Dy, wherefore Ab is the Image of a Line parallel to the Picture, which makes with its Perpendicular Seat Dy on the Plane EFGH, an Angle equal to the Angle of Inclination given, and A or b being by Suppoſition the Image of a Point of the propoſed Original Line, Ab is therefore the Indefinite Image of that Line.

COR. 3.

If the Original Line be parallel to the Picture and alſo to the Plane EFGH, it will then likewiſe be parallel to its Seat; and the Images of that Line and its Seat may be found by the Seat and Support of any one Point of the Original Line, as before[c]; but the Plane of the Seat of that Line will not be parallel to the Picture, but will have its Vaniſhing Line paſſing through x parallel to EF.

CASE 3.

When the Supports of the Points whoſe Seats are given or required, are perpendicular to the Picture.

Fig. 110. N°. 7. 1 Let O be the Center of the Picture; and firſt, let B and C be the Images of the Perpendicular Seats of two Points of an Original Line on a Plane parallel to the Picture, the Length of their Supports, and the Diſtance between the Picture and the parallel Plane being known.

Through B and C draw BC, and through O draw EF parallel to it; then find b and c the Images of the Points whoſe Seats are given[d], and draw bc cutting BC in A, and EF in z; then from any Point B in BC, ſet off BM repreſenting a Line equal to the Diſtance between the Picture and the parallel Plane[e], and having drawn OB, from I the Point of Diſtance of the Vaniſhing Point O, draw IM cutting OB in m, through which draw GH parallel to EF, cutting bc in d; then dz will be the Indefinite Image of the Original Line, z and d its Vaniſhing and Interſecting Points, BM the Image of its Seat on the Original Plane, and A its Interſection with that Plane, and EF and GH will be the Vaniſhing and Interſecting Lines of the Plane of its Seat, which laſt is alſo the Seat of dz on the Picture.

Dem. For BC being the Seat of the propoſed Line on the parallel Plane, the Original of that Seat is parallel to the Picture, and conſequently to the Vaniſhing Line of the Plane of the Seat of the propoſed Line[f], which Plane being perpendicular to the Picture, its Vaniſhing Line muſt paſs through O, wherefore EF drawn through O parallel to BC is the Vaniſhing Line of that Plane; and the Original of Om being a Line in this Plane perpendicular to the Picture, and Bm repreſenting a part of that Line equal to the Diſtance between the Picture and the parallel Plane[g], m is therefore the Interſecting Point of Om, and conſequently GH drawn through m parallel to EF, is the Interſecting Line of the Plane of the Seat of the propoſed Line on the parallel Plane, and likewiſe the Seat of that Line on the Picture[h]: the reſt needs no farther Demonſtration. *Q. E. I.*

2. If any two of the three Points z, A, and d, or any one of them with the Support bB of any Point of the propoſed Line, be given; the intire Image of that Line and its Seat may be thence found.

For if z be given, by the help of O, EF is found, and the Direction of AB and GH which are parallel to it; if then A or its equivalent Bb be alſo given, the Image bz and its Seat AB are found; and Bm in the Line OB being made to repreſent a Line equal to the Diſtance between the Picture and the parallel Plane, a Point m in the Interſecting Line GH is thereby had, whence that Line and the Interſecting Point d of dz are determined.

Again, the Point A by the help of Bb, gives AB and Ab, and the Center O determines EF parallel to AB, whence z is found, and GH and the Point d are determined as before.

Laſtly, the Point d with O gives dO, in which dN being made to repreſent a Line equal to the Diſtance between the Picture and the parallel Plane, a Point N in the Seat

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[a] Prop. 29.

[b] Cor. 2. Theor. 15. B.I.

[c] Cor 3. Caſe 1.

[d] Caſe 3. Prop. 39.

[e] Prop. 38.

[f] Cor. 1. Theor. 15. B.I.

[g] Cor. 1. Prob. 8. B.II.

[h] Cor. Def. 6.

is thereby found, which with B determines AB, and thence its parallel EF, and db gives A and z.

3. The Angle of Inclination of the propoſed Line to its Seat, ſupplies the place of the Image of one Point of that Line.

For AB being parallel to the Picture, the Vaniſhing Point z in the Line EF is found, by making it ſubtend with O an Angle equal to the Complement of the Angle of Inclination given[a].

COR.

If the Original Line be parallel to the Picture, it is then alſo parallel to the Original Plane, and conſequently to its Seat on that Plane; whence the Image of the Support of any Point of the Original Line, with the Image of any one other Point, either of the Original Line or its Seat, being given, the intire Images of both may be found.

GENERAL COROLLARY.

It is evident, that if through any Point c of the Image Ac, a Line cC be drawn, perpendicular to EF in the firſt Caſe[1], from the Point x in the ſecond Caſe[2], or from the Point O in the third Caſe[3], cutting the Seat BC in C, the Point C will be the Seat of c on the propoſed Plane.

PROP. XLI. PROB. XXIX.

Any two Points of Relation of a Line to an Original Plane being given; thence to find the Indefinite Image of that Line, and its Seat on, and Interſection with the Original Plane, when the Plane of that Seat paſſes through the Eye.

This happens in three Caſes:

1. When the Supports given or required, are parallel to the Picture, and the Directing Point of the propoſed Line lies in the Eye's Director of the Original Plane.

Here, the Directing Line of the Plane of the Oblique Seat of the propoſed Line, is the Eye's Director of the given Plane[b], which Plane therefore paſſes through the Eye; and the propoſed Line and its Seat, and the Supports of all its Points, being in that Plane, their Images muſt all be in the Image of that Plane, which is only a ſtraight Line[c].

2. When the Supports incline to the Picture, and the propoſed Line lies in the Vertical Plane.

In this Caſe, the Plane of the Seat of the propoſed Line coinciding with the Vertical Plane, that Line and its Seat, and alſo the Perpendicular as well as the Oblique Supports of all its Points, being in that Plane[d], their Images muſt all coincide with the Vertical Line.

3. When the Supports are perpendicular to the Picture, and the propoſed Line lies in a Plane paſſing through the Eye perpendicular to the Picture.

In this Caſe, the Plane which paſſes through the Eye and the propoſed Line, is itſelf the Plane of the Perpendicular Seat of that Line, either on the Picture, or on any other Original Plane parallel to the Picture; the whole Image of which Plane is therefore a ſtraight Line paſſing through the Center of the Picture.

In either of theſe Caſes, the Methods ſhewn in the laſt Propoſition will not ſerve, and therefore the following may be uſed.

CASE 1.

When the Supports given or required, are parallel to the Picture, and the Directing Point of the propoſed Line falls in the Eye's Director of the Original Plane.

Let EFGH be the given Plane; and let B and C be the given Oblique Seats of two Points of the Propoſed Line on that Plane, the length of their Supports being known. Fig. 111. N°. 1.

Through B and C draw Dy, cutting EF and GH in y and D; then Dy will be the Indefinite Image of the Seat of the Propoſed Line, and y and D the Vaniſhing and Interſecting Points of that Seat; and the ſame Line Dy produced both ways, will alſo be the Indeterminate Image of the Original Line, and of the Supports of all its Points, and likewiſe the Vaniſhing Line of the Plane of its Seat[e], which Line muſt be perpendicular to EF, the Eye's Director being by Suppoſition the Directing Line of that Plane[f]. This being premiſed, from y draw any Line yg in the Plane EFGH,

[a] Caſe 2. Prob. 3. B. II.

[1] Fig. 110. N°. 1, 2, 3, 4.

[2] Fig. 110. N°. 5, 6.

[3] Fig. 110. N°. 7.

[b] Cor. 1. Prop. 2.

[c] Cor. 1. Theor. 17. B. I.

[d] Cor. Prop. 4.

[e] Theor. 19. B. I.

[f] Cor. 1. Def. 10. B. I.

cutting GH in g, and through g draw gh parallel to Dy; and having from the given Seats B and C drawn BL, CM, parallel to EF, cutting gy in L and M, find the Indefinite Image of a Line whoſe Seat on the Plane EFGH is gy, and which hath the Supports of two of its Points whoſe Seats are L and M, equal reſpectively to the Supports of the Points of the Original Line whoſe Seats are B and C[a]; and let the Line thus found be hz, whoſe Vaniſhing and Interſecting Points are z and h, and its Interſection with the Plane EFGH is a, and l and m the Images of the Points in that Line whoſe Seats are L and M; then from l, m, h, and a, draw Parallels to EF, cutting Dy in b, c, d, and A, and dz will be the Indefinite Image of the propoſed Original Line, z and d its Vaniſhing and Interſecting Points, c and b the Images of the Points whoſe Seats are C and B, and A will be the Image of the Interſection of the Original Line with the Plane EFGH.

Dem. Becauſe the Original of LB is parallel to the Picture, the Original of the Parallelogram lLBb is in a Plane parallel to the Picture, and therefore Bb and Ll repreſent equal Lines[b]; but the Original of Ll is by Conſtruction equal to the Support of the Point of the Original Line whoſe Seat is B, therefore b is the Image of that Point; and for the like Reaſon, c is the Image of that Point of the required Line whoſe Seat is C: and becauſe of the Vaniſhing Point y, the Originals of LB and MC are equal and parallel[c], to which the Originals of lb and mc being alſo equal and parallel, the Figure $mlbc$ repreſents a Parallelogram, whoſe Sides lm and bc therefore repreſent Parallels; and conſequently z, the Vaniſhing Point of lm, is alſo the Vaniſhing Point of bc[d]; and becauſe the Originals of lz and bz are parallel, they are in the ſame Plane, and lb repreſenting a Line in that Plane parallel to the Picture, it is alſo parallel to the Vaniſhing and Interſecting Lines of that Plane[e]; wherefore vz and hd drawn through z and h parallel to EF, are the Vaniſhing and Interſecting Lines of that Plane, and therefore d is the Interſecting Point of bc: Laſtly, becauſe the Interſection of the Plane $vzhd$ with the Plane EFGH is parallel to EF[f], the Line aA drawn through a parallel to EF, is the Image of the Interſection of thoſe two Planes, and conſequently A is the Interſection of bc with the Plane EFGH. *Q. E. I.*

CASE 2.

When the Supports given or required, incline to the Picture, and the propoſed Original Line lies in the Vertical Plane of the Original Plane.

Fig. 111. N°. 2. Let O be the Center of the Picture, and EFGH the given Plane, and let B and C be the perpendicular Seats of two Points of the propoſed Line on that Plane, the length of their Supports being known.

Through B and C draw Dy, the Seat of the propoſed Line on the Plane EFGH, which in this Caſe, coincides with Po the Vertical Line of that Plane; and having found x the Vaniſhing Point of Perpendiculars to the Plane EFGH, proceed as in the laſt Caſe; ſave that the Supports lL and mM, inſtead of being drawn perpendicular to EF, muſt be drawn from the Vaniſhing Point x[g], the Conſtruction in both Caſes being in all other reſpects the ſame; and thereby the ſubſtituted Line hz, and thence the Indefinite Image dz of the propoſed Line, and the other required Points in that Line will be found.

Dem. For by reaſon of the Vaniſhing Point x, the Originals of xl and xb are parallel, and therefore in the ſame Plane; and LB repreſenting a Line in that Plane parallel to the Picture, to which the Original of lb is alſo parallel, lLBb repreſents a Parallelogram in that Plane, whoſe Sides lL and bB therefore repreſent equal Lines; and conſequently Bb is the Image of the Support, and b the Image of the Point whoſe Seat is B: and for the like Reaſon, c is the Image of the Point of the required Line whoſe Seat is C. The reſt is demonſtrated as in the preceeding Caſe. *Q. E. I.*

CASE 3.

When the Supports given or required, are perpendicular to the Picture, and the propoſed Line lies in a Plane paſſing through the Eye perpendicular to the Picture.

Fig. 111. N°. 3. Let O be the Center of the Picture, and B and C the given Seats of two Points of the propoſed Line on an Original Plane parallel to the Picture, the length of the Supports of thoſe Points, and the Diſtance of the Original Plane from the Picture being known.

Through B and C draw BC, which muſt paſs through O, and repreſents the Original Line and its Seat, and is alſo the Vaniſhing Line of the Plane of its Seat; then draw any Line LM parallel to BC, repreſenting a Line in the Original Plane parallel

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[a] Caſe 1. Prop. 40.

[b] Cor. 2. Theor. 23. B. I.

[c] Cor. 1. Prob. 6. B. II.

[d] Theor. 5. B. I.

[e] Cor. 1. Theor. 15. B. I.

[f] Theor. 15. B I.

[g] Caſe 2. Prop. 40.

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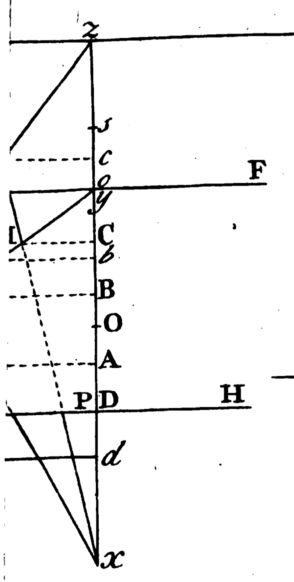


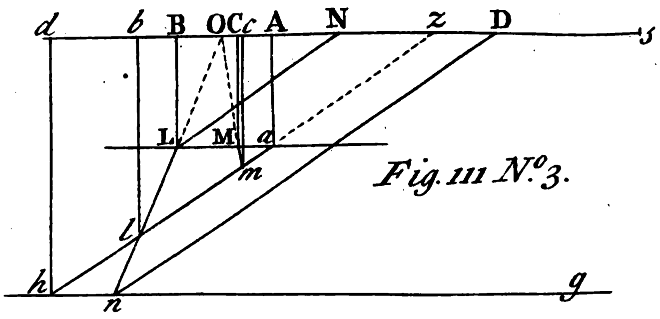


Fig. 111 N.º 3.

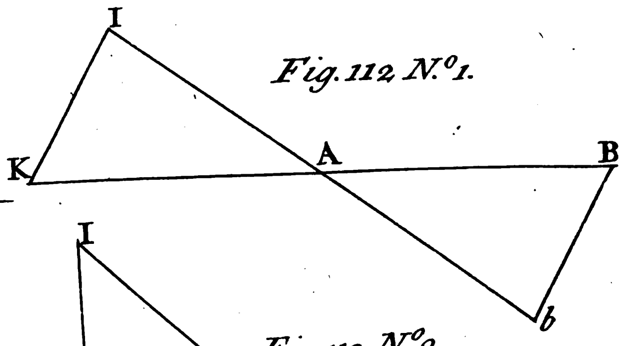
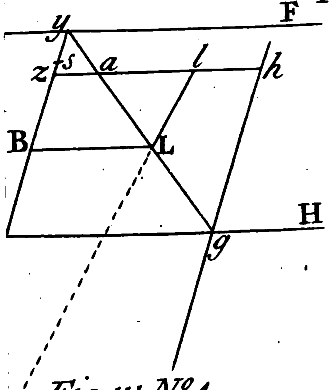


Fig. 112 N.º 1.



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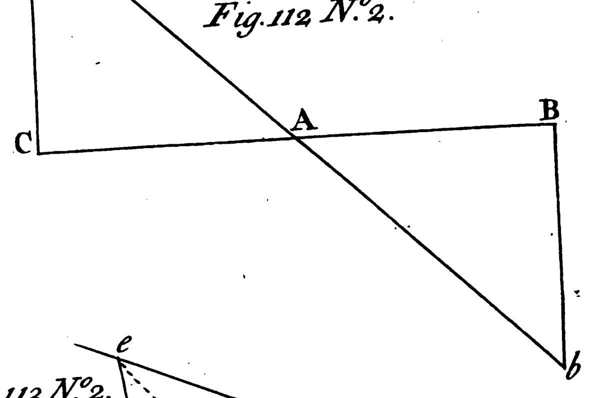
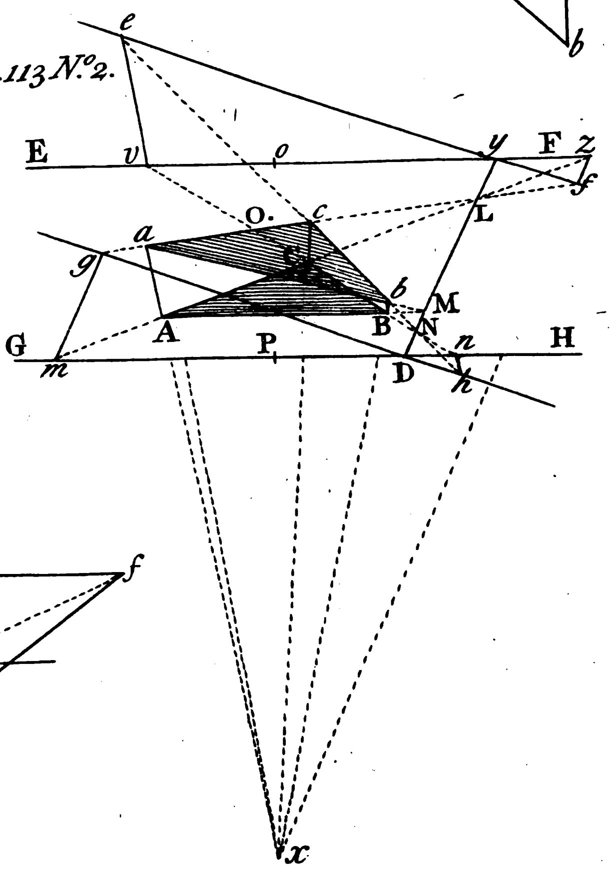
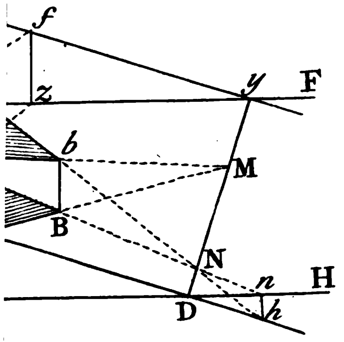


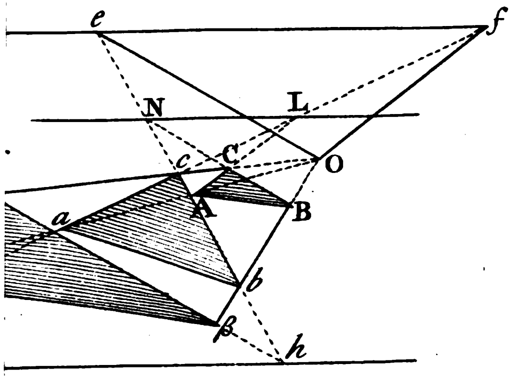
Fig. 112 N.º 2.

Fig. 113 N.º 2.



113 N.º 1.





J. Mynde sc.

to the Seat of the propoſed Line, and draw BL, CM, perpendicular to BC, cutting LM in L and M: then find the Indefinite Image *bz* of a Line whoſe Seat on the Original Plane is LM, and which hath the Perpendicular Supports of two of its Points whoſe Seats are L and M, equal reſpectively to the Supports of the Points of the required Line whoſe Seats are B and C[a], and from each of the Points *l*, *m*, *a*, and *b*, in the Line *bz* thus found, draw Perpendiculars to BC, cutting it in *b*, *c*, A, and *d*; and *dz* will be the Indefinite Image of the required Line, *z* and *d* its Vaniſhing and Interſecting Points, *b* and *c* the Images of the Points whoſe Seats are B and C, and A will be the Interſection of the required Line with the Original Plane.

Dem. Becauſe LM repreſents a Line in the Original Plane, parallel to BC the Seat of the propoſed Line on that Plane, BLMC repreſents a Parallelogram in the Original Plane; and becauſe of the Vaniſhing Point O, the Originals of LB and *lb*, which are parallel to the Picture, are equal, wherefore the Original of *l*LB*b* is a Parallelogram, whoſe Sides L*l* and B*b* repreſent equal Lines; conſequently, B*b* is the Image of the Support of the Point whoſe Seat is B; and for the like Reaſon, C*c* is the Image of the Support of the Point whoſe Seat is C: and becauſe the Originals of *bl* and *cm* are parallel and equal to the Originals of BL and CM, the Original of *blmc* is a Parallelogram, whoſe Sides *bc* and *lm* repreſenting Parallels, they are in the ſame Plane, and have the ſame Vaniſhing Point *z*; and becauſe *lb* is a Line in this Plane parallel to the Picture, *bd* parallel to *lb* is the Interſecting Line of that Plane, wherefore *d* is the Interſecting Point of *dz* the Line required. The reſt is evident. *Q. E. I.*

COR. 1.

In all theſe Caſes, *zygb*[1], or *z*O*gb*[2], the Plane of the Seat of the ſubſtituted Line *bz*, is parallel to the Plane of the Seat of the propoſed Original Line, they having the ſame Vaniſhing Line *zy* or *z*O; and the Points L, *l*, M, *m*, are the Oblique Seats of B, *b*, C, *c*, on the ſubſtituted Plane, BL, CM, *bl*, *cm*, repreſenting Lines parallel to the Picture and to the common Vertical Line of thoſe Planes; and conſequently B, *b*, C, *c*, are the Oblique Seats of the Points L, *l*, M, *m*, on the Plane of the Seat of the Original Line[b]; and the Original and ſubſtituted Lines and their Seats on the Original Plane, are mutually the Oblique Seats of each other on the Planes *zygb* and *zy*, or *z*O*gb* and *z*O.

COR. 2.

Hence, the ſame correſponding Points of the Original Line and its Seat on the Original Plane being given, as in the laſt Propoſition, they will be ſufficient for finding the Indefinite Images of them both; by finding the Oblique Seats of the given Points on a ſubſtituted Plane, and thence the Indefinite Images of a ſubſtituted Line and its Seat on the Original Plane; for then any Point of the ſubſtituted Line or its Seat, being transferred to the Plane of the Seat of the Original Line, by its Oblique Support on that Plane, will mark a correſponding Point of the Image of the Original Line or its Seat on the Original Plane.

SCHOL.

This Method of uſing a ſubſtituted Line inſtead of the Original Line, when the Image of this laſt coincides with its Seat, may likewiſe be of Service, when the Images of the Original Line and its Seat fall ſo cloſe together, as to be inconvenient for determining the Points required in either of them, from the given correſponding Points of the other.

PROP. XLII. PROB. XXX.

The Image of an Original Line which paſſes through the Eye, being given; thence to find the Indefinite Image of its Seat on a given Original Plane, and alſo the Image of the Seat of any Point of that Line, the Diſtance of which from its Interſecting Point is known.

When an Original Line paſſes through the Eye, its Image is only a Point, which Point repreſents every poſſible Point of the Original Line, and conſequently its Vaniſhing and Interſecting Points, and its Interſection with all Planes that it can cut[c].

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[a] Caſe 3. Prop. 40.

[1] Fig. 111. N°. 1, 2.

[2] Fig. 111. N°. 3.

[b] Prop. 5.

[c] Theor. 8. and Cor. and Theor. 18. B. I.

CASE. 1.

When the Supports of the Points whoſe Seats are required, are parallel to the Picture.

Fig. 111. N°. 1. Let EFGH be the given Plane, and z the Indefinite Image of a Line which paſſes through the Eye.

From z draw zD perpendicular to EF, cutting EF and GH in y and D, and from y draw any Line yg cutting GH in g, through which draw gb parallel to zy; then Dy will be the Indefinite Image of the Oblique Seat of the Line z on the Plane EFGH, and $zygb$ will be a ſubſtituted Plane parallel to the Plane of the Seat of that Line; ſeeing this laſt Plane paſſes through the Eye's Director [a], and muſt therefore cut the Plane EFGH in Dy a Line paſſing through z perpendicular to EF, which Line is the intire Image of that Plane [b]; then from z draw any Line zb in the Plane $zygb$, cutting gb in b its Interſecting Point, and make bl in this Line repreſent a Line equal to the given Diſtance between any propoſed Point of the Original Line and its Interſecting Point; and having found L, the Oblique Seat of l on the Plane EFGH, transfer that Seat to B in the Line Dy, by LB drawn parallel to EF, and B will be the Image of the Seat of the propoſed Point of the Original Line on the Plane EFGH, and Bz the Image of its Oblique Support on that Plane.

Dem. Becauſe of the Vaniſhing Point z, the Originals of zb and of the Line z being parallel, the Originals of l and of the propoſed Point in the Line z, are equally diſtant from the Picture, and their Oblique Supports on the Plane EFGH are therefore in a Plane parallel to the Picture, in which Plane the Line LB alſo lies; wherefore B, where LB cuts the Seat Dy of the Original Line, is the Seat of the propoſed Point of that Line on the Plane EFGH, and z being the Image of that Point, zB is therefore the Image of its Support. *Q. E. I.*

CASE 2.

When the Supports of the Points whoſe Seats are required, incline to the Picture.

Fig. 111. N°. 4. Let O be the Center of the Picture, EFGH the given Plane, and z the Indefinite Image of the propoſed Line.

From x, the Vaniſhing Point of Perpendiculars to the Plane EFGH, through z draw xy, cutting EF and GH in y and D, and from y draw any Line yg, cutting GH in g, from whence draw gb parallel to xy; then yD will be the Indefinite Image of the Perpendicular Seat of the Line z on the Plane EFGH, and $zygb$ a ſubſtituted Plane parallel to the Plane of the Seat of that Line; xy being the whole Image of a Plane paſſing through the given Line z perpendicular to the Plane EFGH [c]; then from z draw zb in the Plane $zygb$ parallel to EF, and uſing zb as a ſubſtituted Line, find bl repreſenting a part of that Line equal to the Diſtance between any propoſed Point of the Original Line and its Interſecting Point, and having found L, the Perpendicular Seat of l on the Original Plane, transfer that Seat to B in the Line Dy by LB parallel to EF, and thereby B the Seat of the propoſed Point of the Original Line will be found.

Dem. Becauſe of the Vaniſhing Point z, the Originals of zb and of the Line z being parallel, the Originals of l and of the propoſed Point in the Line z are equally diſtant from the Picture, and therefore the Original of zl which connects thoſe Points, is parallel to the Picture, and being parallel to EF, is alſo parallel to the Plane EFGH, and conſequently to its Perpendicular as well as Oblique Seats on that Plane; and L being the Perpendicular Seat of l, a Point in the Line zl, on that Plane, BL drawn through L parallel to EF is the intire Seat of that Line [d], and therefore B is the Perpendicular Seat of z, the Image of the propoſed Point of the Line z on that Plane. *Q. E. I.*

SCHOL.

It is here neceſſary, that the ſubſtituted Line zb be not only parallel to the given Line z, but that its Image be alſo parallel to EF, to the end that any Line zl, which connects the Images z and l, of any two Points in the Original and ſubſtituted Lines, equally diſtant from the Picture, may alſo be parallel to the Plane EFGH, and conſequently to its Perpendicular Seat BL on that Plane; which Precaution is not neceſſary when the Oblique Seats are only wanted, as in the preceeding Caſe, ſeeing all Points equally diſtant from the Picture, at whatever different Heights they be above the Plane EFGH, have their Oblique Seats on that Plane in a Line parallel to EF, the Oblique Supports of all thoſe Points lying in a Plane parallel to the Picture.

[a] Cor. Prop. 2.

[b] Cor. 1. Theor. 17. and Theor. 19. B. I.

[c] Cor. 3. Prop. 20.

[d] Cor. 3. Caſe 2. Prop. 40.

CASE 3.

When the Supports of the Points whoſe Seats are required, are perpendicular to the Picture.

Let O be the Center of the Picture, and z the Indefinite Image of the propoſed Line, and let the Original Plane be parallel to the Picture, and the Diſtance between them known. Fig. 111. N°. 3.

Through z and O draw zO, which will be the intire Image of the Plane of the Perpendicular Seat of the Line z, either on the Picture, or on any Plane parallel to it; then having drawn any Line gb parallel to Oz, as the Interſecting Line of a ſubſtituted Plane parallel to the Plane Oz, find LM the Interſection of this Plane with the propoſed Original Plane[a]; then from z draw any Line zb in the Plane Ozgb, for a ſubſtituted Line, cutting the parallel Plane in a, and having the Diſtance of the propoſed Point of the Line z, either from its Interſecting Point, or from its Interſection with the parallel Plane, given, make either bl or al in the Line zb repreſent that Diſtance, and draw lO cutting LM in L, and from L draw LN parallel to zb cutting zO in N, and N will be the Seat of the propoſed Point of the Line z on the parallel Plane; and if the Line lO be produced till it cut gb in n, a Line nD parallel to bz will cut Oz in D, the Perpendicular Seat of the propoſed Point in the Line z on the Picture.

Dem. For the Originals of zb and of the Line z being parallel, the Original of l is at an equal Diſtance from the Picture with the propoſed Point of the Line z; wherefore the Original of the Line lz which connects thoſe Points, is parallel to the Picture, and conſequently to its Seats both on the Picture and the parallel Plane; but L is the Seat of l on the parallel Plane, and n the Seat of the ſame Point l on the Picture[b], conſequently LN and nD parallel to lz are the Seats of lz on the parallel Plane and the Picture, wherefore N and D are the Seats of z, the Image of the propoſed Point in the Line z, on thoſe two Planes. *Q. E. I.*

[a] Prop. 38.

[b] Gen. Cor. Prop. 40.

LEM. 2.

Let Ib repreſent an Original Line paſſing through the Eye at I, and KB the Oblique Seat of that Line on any Plane not parallel to the Picture, which Seat muſt neceſſarily paſs through K the Point of Station, the Eye's Director IK being the Oblique Support of the Point I of the propoſed Line on the Original Plane. Fig. 112. N°. 1.

1. Now if a Point b be taken in the Original Line Ib, as far beyond A its Interſection with its Seat, as A is from I, the Oblique Seat B of that Point will be as far beyond A as A is from K.

For the Oblique Support Bb of the Point b, being parallel to IK, the Triangles IKA, ABb are Similar; wherefore IA and Ab being by Suppoſition equal, AK and AB are alſo equal. *Q. E. D.*

Or let Ib be the Original Line paſſing through the Eye, and CB the Perpendicular Seat of that Line on any Plane, which Seat muſt neceſſarily paſs through C, where that Plane is cut by a Perpendicular IC, drawn to it from the Eye, C being the Perpendicular Seat of the Point I of the Original Line on that Plane. Fig. 112. N°. 2.

2. Then if a Point b be taken in the Original Line, as far beyond A its Interſection with its Seat, as A is from I, the Perpendicular Seat B of that Point will be as far beyond A, as A is from C.

For the Perpendicular Support bB of the Point b being parallel to IC, the Triangles ICA, ABb are Similar; wherefore IA and Ab being equal, AC and AB are alſo equal. *Q. E. D.*

PROP. XLIII. PROB. XXXI.

The Image of an Original Line which paſſes through the Eye being given; thence to find the Image of the Seat of a Point of that Line on a given Plane, as far diſtant beyond the Interſection of the propoſed Line with that Plane, as that Interſection is diſtant from the Eye.

CASE 1.

When the Support of the Point whoſe Seat is required, is parallel to the Picture.

Let EFGH be the given Plane, and z the Indefinite Image of the propoſed Line. Fig. 111.
Through z draw Dy perpendicular to EF, which will be the Indefinite Image of N°. 1.
the

the Oblique Seat of the Line z on the Plane E F G H[a], then biſect zy in s, and s will be the Image of the Seat of the Point required.

Dem. For z being the Image of the Interſection of the Original Line with its Seat[b], and y being the Vaniſhing Point of that Seat, the Point s which biſects zy, is the Image of a Point in that Seat as far diſtant beyond the Original of z, as this laſt is from its Directing Point[c]; but the Directing Point of the Seat Dy is the Point of Station of the Original Plane, conſequently s is the Image of the Oblique Seat of a Point in the propoſed Line z, as far beyond its Interſection with the Plane E F G H, as that Interſection is from the Eye[d]. *Q. E. I.*

CASE 2.

When the Support of the Point whoſe Seat is required, inclines to the Picture.

Fig. 111. N°. 2, and 4.

Let O be the Center of the Picture, E F G H the given Plane, and x the Vaniſhing Point of Perpendiculars to that Plane, and let z be the Indefinite Image of the propoſed Line.

Through x and z draw Dy the Indefinite Image of the Perpendicular Seat of the propoſed Line on the given Plane[e]; then find a Point s between z and y, ſo that xy may be Harmonically divided in x, z, s, and y[f], and s will be the Image of the Seat of the Point required.

Dem. Becauſe the Indefinite Image xy of the Seat of the propoſed Line, is Harmonically divided in x, z, s, and y, of which y is its Vaniſhing Point, the Original of xy is biſected by the Originals of x, z, and s[g], of which z repreſents the middle Point; but z is the Image of the Interſection of the Original Line with its Seat, and x is the Image of a Point in that Seat where a Perpendicular from the Eye cuts it, x being the Indefinite Image of that Perpendicular; wherefore s is the Image of a Point in the Seat of the propoſed Line, as far beyond its Interſection with that Line, as that Interſection is from the Interſection of the Seat with a Perpendicular from the Eye, and conſequently s is the Image of the Seat of the Point deſired[h]. *Q. E. I.*

CASE 3.

When the Support of the Point whoſe Seat is required, is perpendicular to the Picture.

Fig. 111. N°. 3.

Let O be the Center of the Picture, and z the Indefinite Image of the propoſed Line.

Draw zO the Image of the Perpendicular Seat of the propoſed Line, either on the Picture, or on any Original Plane parallel to it[i], and take zs in that Line equal to zO, and s will be the Image of the Point deſired.

Dem. For z is the Image of the Interſection of the propoſed Line with its Perpendicular Seats, both on the Picture and on the parallel Plane, and O is the Image of the Interſections of thoſe Seats with a Perpendicular from the Eye; and the Seats themſelves being, the one a Line in the Plane of the Picture, and the other a Line parallel to it, zs and zO which are equal, repreſent equal Lines[k]; wherefore s is the Image of a Point in either Seat, as far beyond its Interſection with the propoſed Line, as that Interſection is from the Interſection of the ſame Seat with a Perpendicular from the Eye, and conſequently s is the Image of the Seat of the Point required[l]. *Q. E. I.*

COR.

If the propoſed Line z be perpendicular to the Original Plane, the Point z will coincide with O the Vaniſhing Point of Perpendiculars to that Plane; and that Point is then, not only the Indefinite Image of the propoſed Line, but alſo of its Perpendicular Seat on the Original Plane, and conſequently of the Perpendicular Seat of every Point of the propoſed Line on that Plane.

PROP. XLIV. PROB. XXXII.

The Center and Diſtance of the Picture, and the Images of the Seats of the three angular Points of a Triangle on an Original Plane, with the Length of their Supports, being given; thence to find the Image of that Triangle, and the Vaniſhing and Interſecting Lines of its Plane.

CASE

[a] Caſe 1. Prop. 42.

[b] Theor. 18. B. I.

[c] Theor. 26. B. I.

[d] Lem. 2.

[e] Caſe 2. Prop. 42.

[f] Lem. 1. B. III.

[g] Cor. 5. Lem. 8. B. III.

[h] Lem. 2.

[i] Caſe 3. Prop. 42.

[k] Cor. 1. Theor. 23. B. I.

[l] Lem. 2.

CASE 1. *and* 2.

When the Supports of the Points whoſe Seats are given, are either parallel or inclining to the Picture.

Let EFGH be the given Plane, and A, B, and C, the given Seats of the angular Points of the Triangle on that Plane. Fig. 113. N°. 1, 2.

Compleat the Seat ABC of the propoſed Triangle, then by the help of the Seats AC and BC of any two of its Sides which lie moſt convenient, and the given Supports of the angular Points, find the Indefinite Images gf and he of thoſe two Sides, and in them the Points a, c, and b, whoſe Seats and Supports are given[a]; through the Vaniſhing Points e and f draw ef, and through either of the Interſecting Points g or h, draw gh parallel to it, cutting EF and GH in y and D, and draw Dy; then abc will be the Image of the propoſed Triangle, ef and gh the Vaniſhing and Interſecting Lines of its Plane, and Dy the Interſection of that Plane with the Plane EFGH.

Dem. For f and e being the Vaniſhing Points of the Sides ac and bc of the Triangle abc, and g and h being the Interſecting Points of the ſame two Sides, ef and gh are the Vaniſhing and Interſecting Lines of the Plane of the Triangle[b], and Dy drawn through y and D, the Interſections of theſe with the correſponding Lines of the Plane EFGH, is the Image of the Interſection of thoſe two Planes[c]. *Q. E. I.*

SCHOL.

The firſt of theſe Figures repreſents the Caſe, when the Supports are parallel to the Picture, and the other, when the Supports incline to it: in the firſt, the Center of the Picture is not concerned; in the laſt, the Supports, as well as the Vaniſhing Lines of the Planes of the Seats, are drawn from x the Vaniſhing Point of Perpendiculars to the Plane EFGH; but in both, the Interſecting Lines of thoſe Planes are parallel to their reſpective Vaniſhing Lines[d].

CASE 3.

When the Supports are perpendicular to the Picture.

Let O be the Center of the Picture, and A, B, and C, the Perpendicular Seats of the angular Points of the Triangle on a Plane parallel to the Picture, the Diſtance of which from the Picture is known. Fig. 113. N°. 3.

By the help of O, and the Seats AC and BC of any two Sides of the Triangle as lie moſt convenient, and the given Length of the Supports, find the Indefinite Images gf and he of thoſe two Sides[e], whence the Image abc, and the Vaniſhing and Interſecting Lines ef and gh of the Plane of the Triangle will be found as before: and the Original Plane being here parallel to the Picture, and the Interſection L of any Side ac with its Seat AC, being a Point in the Interſection of that Plane with the Plane of the Triangle, LN drawn parallel to ef, will be the Image of the Interſection of thoſe two Planes[f]; and $eO\gamma h$ and $fO\gamma g$ being the Planes of the Seats of bc and ac, $\gamma\beta$ and $\gamma\alpha$ are the Perpendicular Seats of bc and ac, and conſequently $\alpha\beta\gamma$ the perpendicular Seat of the Triangle abc on the Picture[g]. *Q. E. I.*

COR.

In all theſe Caſes, wherever the Image of any Side of the Triangle cuts its Seat on the Original Plane, that Interſection muſt be a Point in the Interſection of that Plane with the Plane of the Triangle.

SCHOL.

By comparing this Problem with what was ſhewn at each of the Caſes of Prop. XL. it will appear, that if any two Points of Relation of either Side of the Triangle to the Original Plane, and any one Point of Relation of either of the other Sides to that Plane be given, all the other things may be thence found.

Hence, this Problem ſerves for finding the Vaniſhing and Interſecting Lines of a Plane paſſing through a given Line, and any Point out of that Line whoſe Seat and Support on an Original Plane are given[h]: but as this admits of great Variety, according to the Situation of the given Point, which may lie either before or behind the Eye, or in the Directing Plane, or may be a Point at an infinite Diſtance, all theſe different Caſes will be more proper to be taken into Conſideration in the next Book, where we ſhall treat of Projections.

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[a] Caſe 1 and 2. Prop. 40.

[b] Cor. 2. Theor. 10. B. I.

[c] Theor. 16. B. I.

[d] Schol. Caſe 2. Prop. 40.

[e] Caſe 3. Prop. 40.

[f] Cor. Theor. 3. B. I.

[g] Cor. Def. 6.

[h] Cor. Prop. 32.

PROP. XLV. PROB. XXXIII.

The Indefinite Image of a Line being given; thence to find the Image of the Interſection of that Line with any given Plane.

CASE 1.

When the given Plane is either perpendicular or inclining to the Picture.

Fig. 114. N°. 1. Let EFGH be the given Plane, and dz the Indefinite Image of a Line out of that Plane.

Through z and d, the Vaniſhing and Interſecting Points of the given Line, draw any two parallel Lines zy and dD, cutting EF and GH in y and D, and draw yD, which will cut dz in A, the Image of its Interſection with the Plane EFGH.

Dem. For zy and dD are the Vaniſhing and Interſecting Lines of a Plane paſſing through the Original of dz[a], and yD being the Image of a Line in that Plane, the Originals of yD and dz therefore cut each other in the Original of A; but yD is the Image of the Interſection of the Plane zydD with the Plane EFGH[b]; therefore the Point A, where dz cuts Dy, is the Image of the Interſection of the Original Line with the Plane EFGH. *Q. E. I.*

COR.

Any two of the three Points d, A, and z, being given, the third may be thence found.

If [illegible] and A be given, draw any Line zy, cutting EF in y, then yA gives a Point D, and Dd drawn parallel to zy, cuts zA in d.

Or if d and A be given, any Line dD gives a Point D, and DA a Point y, through which a Parallel to dD being drawn, the Point z is thereby found.

CASE 2.

When the propoſed Original Plane is parallel to the Picture, and the Diſtance between them is known, the Center and Diſtance of the Picture being alſo given.

Fig. 114. N°. 2. Through z the Vaniſhing Point of the given Line dz, and O the Center of the Picture, draw ef, and through the Interſecting Point d draw gb parallel to it; and having from O drawn any Line OD cutting gb in D, make Da in that Line repreſent a Line equal to the Diſtance between the Picture and the Original Plane[c], and aA drawn parallel to ef, will cut dz in A its Interſection with the propoſed Plane.

Dem. For $efgb$ being a Plane paſſing through dz perpendicular to the Picture, and a being a Point in the Interſection of that Plane with the parallel Plane, Aa parallel to ef repreſents the Interſection of thoſe two Planes[d]; wherefore A is the Interſection of dz with the Original Plane. *Q. E. I.*

COR.

Any two of the three Points d, A, and z, being given, the third may be thence found.

For z and A being given, zO and Aa are determined; then having found the Proportion of the Images of Lines in the parallel Plane to their Originals[e], draw any Line Oa, and having on Aa ſet off the proportional Meaſure of the Diſtance between the Picture and the parallel Plane, make aD repreſent that Diſtance[f], whereby Dd, and conſequently d, will be found.

Or if d and A be given, draw OA, and by the ſame Method as before, make Ag repreſent the Diſtance already mentioned, and thereby gb, and conſequently Aa and Oz will be found.

SCHOL.

By this general Method, the Interſection A of a Line dz with any given Plane, is more conveniently determined, than by its Seat on that Plane, eſpecially when the Images of the Original Line and its Seat fall cloſe together.

CASE 3.

When the Original of the given Line is parallel to the Picture, and the Diſtance between them is known.

Fig. 114. N°. 3. Let O be the Center of the Picture, EFGH the given Plane, and Ab the given Image of a Line parallel to the Picture.

Through

[a] Cor. Prop. 30.

[b] Theor. 16, B. I.

[c] Prob. 8. B. II.

[d] Prop. 38.

[e] Prop. 38.

[f] Prob. 8. B. II.

Fig. 114. N°1.

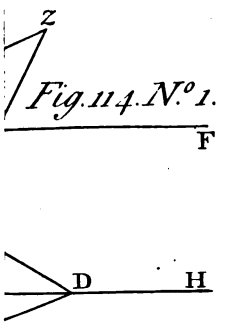
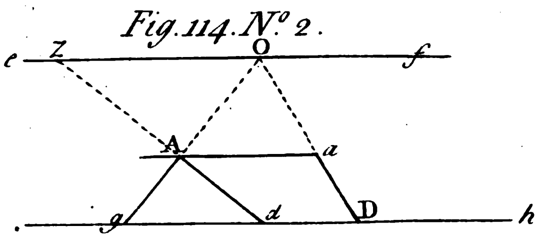


Fig. 114. N°2.



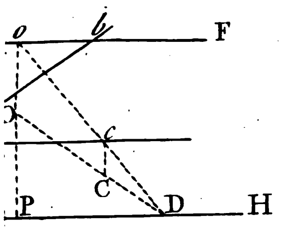
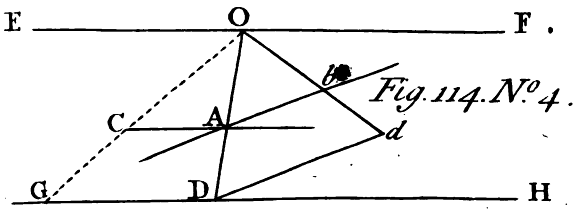


Fig. 114. N°4.



5 N°1.

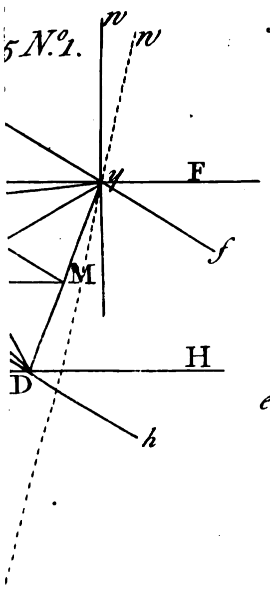


Fig. 115. N°2.

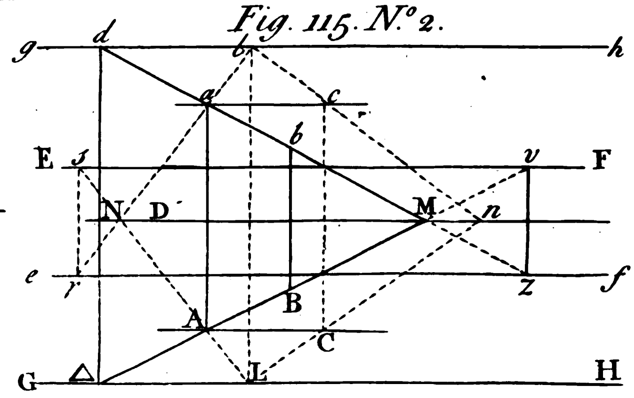
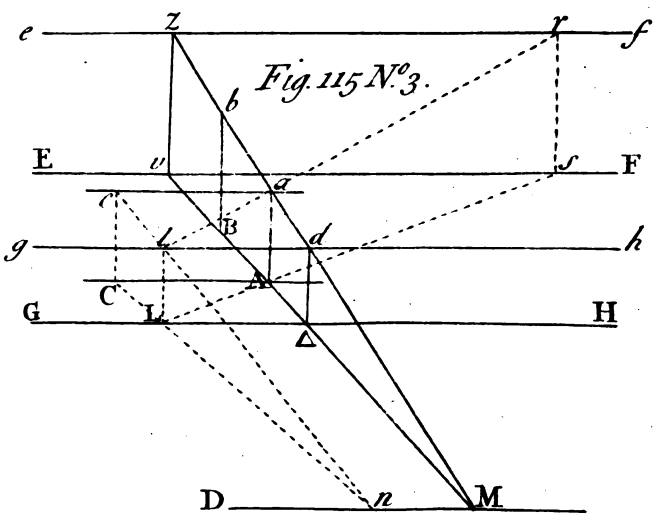
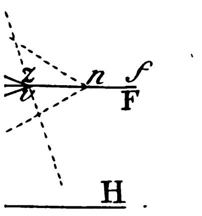


Fig. 115 N°3.





7. 115. N°5.

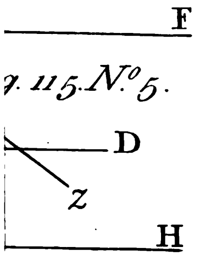
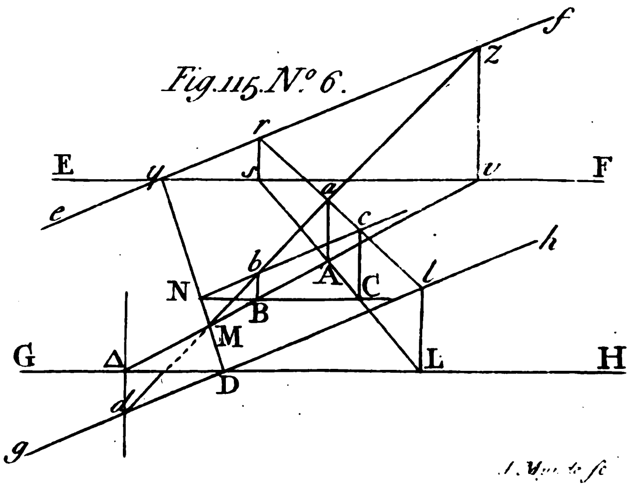


Fig. 115. N°6.



Through O and *o* draw any two Lines OD, *o*D, meeting in any Point D of the Line GH, and make DC in the Line OD repreſent the Diſtance between the Original Line and the Picture; from C draw C*c* perpendicular to EF, cutting *o*D in *c*, through which draw *c*A parallel to EF, which will cut A*b* in A its Interſection with the Plane EFGH.

Dem. For the Original of OD being perpendicular to the Picture, C is the Image of the Interſection of that Line with a Plane parallel to the Picture, paſſing through the Original of A*b*; and *c* being the Oblique Seat of C on the Plane EFGH, and the Support C*c* being therefore parallel to the Picture, it lies wholly in the parallel Plane, and *c* is therefore a Point in the Interſection of that Plane with the Plane EFGH; wherefore *c*A parallel to EF is the Image of the Interſection of thoſe two Planes, conſequently A is the Interſection of A*b* a Line in this parallel Plane, with the Plane EFGH. *Q. E. I.*

And here, *c*A is alſo the Oblique Seat of A*b* on the Original Plane.

COR.

If the Vaniſhing Line EF paſs through the Center of the Picture, the Practice is ſhortened; for then, any Line OG being drawn in the Plane EFGH, and a part GC made to repreſent the Diſtance between the Picture and the propoſed Original Line, CA parallel to EF determines A the Interſection of A*b* with the given Plane. Fig. 114. Nº. 4.

PROP. XLVI. PROB. XXXIV.

The Center and Diſtance of the Picture, and an Original Plane being given; and any one Line of Relation of another Plane to that Plane, with one Point of Relation of thoſe two Planes, being alſo given [a]; thence to find the Vaniſhing and Interſecting Lines of this laſt Plane, and the Image of its Interſection with the other.

CASE 1.

When the Vaniſhing Lines of the propoſed Planes interſect.

1. Let O be the Center of the Picture, and EFGH the given Plane; and firſt, let *ef* be the Vaniſhing Line of another Plane, *a* the Image of a Point in that Plane, and A the Image of its Seat on the Plane EFGH. Fig. 115. Nº. 1.

METHOD 1.

Through *y* the Interſection of the given Vaniſhing Lines, draw *yw*, either perpendicular to EF, or from *x* the Vaniſhing Point of Perpendiculars to the Plane EFGH, according as *a*A is the Oblique or Perpendicular Support of *a* on that Plane; then from *y* through A draw *y*A cutting GH in *m*, from whence draw *mg* parallel to *yw*, cutting a Line *ya* in *g*, and through *g* draw *gb* parallel to *ef*, cutting GH in D, and draw D*y*; then *gb* will be the Interſecting Line of the Plane *efa*, and D*y* the Image of its Interſection with the Plane EFGH.

Dem. Becauſe *y* is a Point in the Vaniſhing Line *ef*, *ya* is the Image of a Line in the Plane *efa*, the Seat of which Line on the Plane EFGH is *ym* which paſſes through A [b], and *mg* being the Interſecting Line of the Plane of the Seat of *ya*, *g* is its Interſecting Point; conſequently *gb* drawn through *g* parallel to *ef*, is the Interſecting Line of the Plane *efa*; and *y* and D being the Interſections of the Vaniſhing and Interſecting Lines of theſe two Planes, D*y* is therefore the Image of their common Interſection [c]. *Q. E. I.*

METHOD 2.

When *a*A is the Oblique Support of *a*, the ſame things may be found more conveniently in this manner.

From A draw AM parallel to EF, and from *a* draw *a*M parallel to *ef*, cutting AM in M; through *y* and M draw *y*D cutting GH in D, through which draw *gb* parallel to *ef*, and D*y* and *gb* will be the Lines ſought.

Dem. For *a*A, AM, and *a*M being each of them Lines parallel to the Picture [d], and meeting in A and *a*, the Triangle *a*AM is in a Plane parallel to the Picture, the Interſections of which Plane with the Planes EFGH and *efa* are AM and *a*M; wherefore M is a Point in the common Interſection of theſe two Planes: the reſt is evident. *Q. E. I.*

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[a] Def. 12, 13, 14.

[b] Cor. 1. Caſe 1 and 2. Prop. 40.

[c] Theor. 16. B. I.

[d] Cor. 2. Theor. 15. B. I.

2. If the Support aA of any Point a in the required Plane, and either of the Lines gb or Dy be given, the reſt may be thence found.

If gb be given, cutting GH in D; draw DA cutting EF in v, from whence vz being drawn, either perpendicular to EF, or from the Vaniſhing Point x, according as aA is the Oblique or Perpendicular Support of a, draw Da cutting vz in z, through which ef being drawn parallel to gb, it will be the Vaniſhing Line ſought, whence Dy is alſo found.

For Da is the Image of a Line in the required Plane, and vz being the Vaniſhing Line of the Plane of its Seat on the Plane EFGH, z is its Vaniſhing Point, and conſequently a Point in the Vaniſhing Line required.

Or, when aA is the Oblique Support of a; draw AM parallel to EF, and aM parallel to gb, which will give M a Point in Dy, whereby that Line, and conſequently ef are found.

If Dy be given, DA gives v, and conſequently vz the Vaniſhing Line of the Plane of the Seat of Da, whence zy and gb are found; or yA gives mg, the Interſecting Line of the Plane of the Seat of ya, whence gb and ef are determined; or laſtly, when Aa is the Oblique Support of a, AM being drawn parallel to EF till it cut Dy in M, the Line aM will be parallel to ef and gb, which muſt paſs through y and D.

3, Any one of the three Lines ef, gb, and yD, being given, with one Point in either of the other two, not in the given Line, the others may be thence found.

For by the given Point, the Line in which it lies, is found, and thence the third.

4. If either Dy or gb, or the Indefinite Image Dz of any Line in the required Plane, be given, together with the Angle of Inclination of that Plane to the Plane EFGH, it being known which way that Inclination tends, the other things required may be thence found.

If Dy be given, through y draw a Vaniſhing Line ef of Planes which incline to the Plane EFGH in the propoſed Angle[a], and gb drawn through D parallel to ef, will give $efgb$ the Plane required.

If gb be given, find a Vaniſhing Line ef of Planes inclining to the Plane EFGH, in the given Angle, and making with EF an Angle zyE equal to the Angle gDG[b]; and ef will be the Vaniſhing Line ſought, whence y, and conſequently Dy are had.

For ef the Vaniſhing Line ſought, muſt be parallel to gb the Interſecting Line given.

Or if Dz be given, through z draw a Vaniſhing Line of Planes which incline to the Plane EFGH in the given Angle[c], and gb drawn through D the Interſecting Point of Dz, parallel to ef, will give $efgb$ the Plane deſired.

But if ef alone be given, the Problem is not determined; for although the Angle of Inclination of the required Plane to the Plane EFGH may be thence found[d], yet any Line gb parallel to ef may be taken as the Interſecting Line of a Plane inclining to the Plane EFGH in that Angle.

CASE 2.

When the Vaniſhing Lines of the propoſed Planes are parallel.

Fig. 115. N°. 2, 3. 1. Let EFGH be the given Plane; and firſt, let ef parallel to EF be the Vaniſhing Line of another Plane, a the Image of a Point in that Plane, and A its Seat on the Plane EFGH.

Through A draw any Line $v\Delta$, cutting EF and GH in v and Δ, and through v draw vz, either perpendicular to EF, or from the Vaniſhing Point of Perpendiculars to the Plane EFGH, according to the kind of the given Support, cutting ef in z, and from Δ draw Δd parallel to vz; then through z and a draw za, till it cut Δd in d, and $v\Delta$ in M, and through d and M draw gb and DM parallel to EF, and gb will be the Interſecting Line of the Plane efa, and DM will be the Interſection of that Plane with the Plane EFGH.

Dem. Becauſe z is a Vaniſhing Point in ef, and a is the Image of a Point in the Plane efa, za is the Image of a Line in that Plane, and vA is its Seat on the Plane EFGH[e]; and $zv\Delta d$ being the Plane of the Seat of that Line, d is therefore the Interſecting Point, and M the Image of the Interſection of za with the Plane EFGH; and conſequently gb drawn through d parallel to ef, is th Interſecting Line of the Plane efa, and DM drawn through M parallel to EF, is the Image of the Interſection of the Planes EFGH and $efgb$[f]. *Q. E. I.*

2. If the Support aA of any Point a in the required Plane, and either of the Lines gb or DM be given, the reſt may be thence found.

If

1

[a] Prop. 25.

[b] Prop. 27.

[c] Prop. 26.

[d] Prop. 23.

[e] Caſe 1 and 2. Prop. 40.

[f] Theor. 15. B. I.

If gh be given; through A draw any Line Δv, cutting EF and GH in v and Δ, from v draw vz as before directed, and through Δ draw Δd parallel to it, cutting gh in d; and the Interſections of da with vz and Δv, will give z and M, through which ef and DM being drawn parallel to EF, they will be the other Lines required.

If DM be given; through A draw any Line Δv, cutting EF, GH, and DM, in v, Δ, and M, and having drawn vz and Δd as before, through M and a draw dz, whereby z and d, and conſequently ef and gh will be found.

3. Any two of the three Lines ef, gh, and DM, being given, the third may thence be found.

If ef and gh be given; draw any two parallel Lines zv and Δd, cutting the Vaniſhing and Interſecting Lines of both Planes reſpectively in v and z, Δ and d; through the Interſections v and Δ of theſe Lines with EF and GH, draw $v\Delta$, and through their Interſections z and d with ef and gh, draw zd, and through M the Interſection of $v\Delta$ with zd, draw DM parallel to EF, and that will be the Interſection of the Planes propoſed.

For vz and Δd are the Vaniſhing and Interſecting Lines of a Plane paſſing through $v\Delta$ and zd, which two Lines are the Interſections of that Plane with the Planes EFGH and $efgh$.

If ef and DM be given; zv and Δd being drawn as before, draw $v\Delta$ cutting DM in M, and zM will cut Δd in d, through which gh muſt paſs.

And laſtly, if gh and DM be given; the Point d in Δd being then known, dM will give z, through which ef paſſes.

It is evident alſo, that if any one Point in either of the Lines ef, gh, or DM, be given, the whole of that Line is given, ſeeing it muſt be parallel to EF.

4. If either DM or gh be given, together with the Angle of Inclination of the required Plane to the Plane EFGH, it being known which way that Inclination tends; the other Lines of Relation of the required Plane may be thence found.

For having found a Vaniſhing Line ef parallel to EF, whoſe Planes incline to the Plane EFGH in the given Angle[a], ef will be the Vaniſhing Line ſought; wherefore DM or gh being alſo given, the other is thence found.

But if ef alone be given, the Problem is not determined, for the reaſon already mentioned[b].

[a] Cor. 2. Caſe 1 and 2. Prop. 25.

[b] Caſe 1.

COR.

By the Place of M, it is determined whether the Interſection of the propoſed Planes be in the Perſpective, Projective, or Tranſprojective Parts of thoſe Planes, according as that Point falls with reſpect to their Vaniſhing and Interſecting Lines[c]; and if the Lines which by their mutual Interſection ſhould produce the Point M, be parallel, then the Interſection of the propoſed Planes is their common Directing Line[d]; and in this Caſe, if any Plane $zv\Delta d$ cut both the propoſed Planes, their Interſections zd and $v\Delta$ with that Plane will be parallel, and for that reaſon, the one may be found by the other, as well as if they met in a known Point M.

[c] Def. 21, 22, 23, and 26. B. I.

[d] Cor. 5. Theor. 12. B. I.

CASE 3.

When the Vaniſhing Lines of the propoſed Planes coincide.

Here, the propoſed Planes being parallel, they have no Interſection, ſo that ef and DM both coincide with EF, and nothing remains to be found, but gh the Interſecting Line of the Plane required.

Let then EFGH be the given Plane, and EF the common Vaniſhing Line of the two Planes, and let a be the Image of a Point in the required Plane, and A its Seat on the Plane EFGH. Fig. 115. N°. 4.

Through A draw any Line Δv, cutting EF and GH in v and Δ, from Δ draw Δd perpendicular to GH, or parallel to xv, according to the kind of the given Seat, and through v and a draw va cutting Δd in d; then gh drawn through d parallel to EF will be the Interſecting Line deſired.

Dem. For Δv and dv are the Interſections of the propoſed Planes with a Plane $v\Delta d$, which paſſes through aA the Support of the given Point. *Q. E. I.*

CASE 4.

When the Plane required is parallel to the Picture.

Here, the Plane required having no Vaniſhing or Interſecting Line, nothing remains to be found but its Interſection with the given Plane, and to this purpoſe, the Diſtance between the Picture and the parallel Plane muſt be known.

Fig. 115. N°. 5. Let then O be the Center of the Picture, and EFGH the given Plane.

Through O draw any Vaniſhing Line zv, cutting EF in v, and from v draw any Line $v\Delta$, cutting GH in Δ, and draw ΔO, and having taken Δc in that Line, repreſenting a Line equal to the Diſtance between the Picture and the parallel Plane, from c draw cM parallel to zv, cutting $v\Delta$ in M, from whence draw MD parallel to EF, and that will be the Interſection deſired.

Dem. For cM is the Interſection of the parallel Plane with the Plane $zv\Delta$[a], and $v\Delta$ being the Interſection of the Plane $zv\Delta$ with the Plane EFGH, M is therefore a Point in the Interſection of the parallel Plane with the Plane EFGH, and conſequently MD drawn through M parallel to EF, is the Image of the Interſection of thoſe two Planes[b]. *Q. E. I.*

PROP. XLVII. PROB. XXXV.

Any two Planes, with the Image of a Line in one of them, being given; thence to find the Seat of that Line on the other Plane.

CASE 1.

When the Vaniſhing Lines of the given Planes interſect.

Fig. 115. N°. 6. Let $efgh$ and EFGH be the given Planes, Dy their common Interſection, and ab the Image of a Line in the Plane $efgh$, whoſe Seat on the Plane EFGH is required.

1. Produce ab to its Vaniſhing and Interſecting Points z and d, from z draw zv either perpendicular to EF, or through the Vaniſhing Point of Perpendiculars to the Plane EFGH, according as the Oblique or Perpendicular Seat of ab on that Plane is required, and draw $d\Delta$ parallel to zv; then Δv will be the Seat of ab on the Plane EFGH, whence A and B the Seats of any Points a and b of the propoſed Line on that Plane, may be found[c]. *Q. E. I.*

2. The Interſection M of ab with the Plane EFGH, ſupplies the Place of d and Δ, or of z and v; ſeeing M is a Point in Δv, which therefore may be found by the help of any other Point in that Line.

3. If z be out of reach; through any Point a of the given Line, draw a Line lr in the Plane $efgh$, whoſe Vaniſhing and Interſecting Points r and l can be conveniently had, and find its Seat Ls, and thence A the Seat of a, as before; and MA will give the Seat of ab.

4. If d be out of reach; through any Point N in the common Interſection Dy of the given Planes, draw NC and Nc parallel reſpectively to EF and ef, and theſe may be uſed inſtead of the true Interſecting Lines GH and gh[d].

For the Originals of NC and Nc being Lines in the Planes EFGH, and $efgh$, parallel to the Picture[e], and meeting in N, they are in a Plane parallel to the Picture, and conſequently at an equal Diſtance from it; and therefore b, where Nc cuts dz, may be uſed as the Interſecting Point of dz, and B, where bB drawn parallel to zv cuts NC, may be uſed as the Interſecting Point of Δv the Seat of dz, bB being a Line parallel to the Picture, in $zv\Delta d$ the Plane of the Seat of dz, and the Point B where it cuts NC, being therefore a Point of Δv, the Interſection of that Plane with the Plane EFGH.

COR.

If the Image c of any Point in the Plane $efgh$ be given, and its Seat on the Plane EFGH be required; through c draw any Line lr in the Plane $efgh$, and find its Seat Ls, whence C the Seat of c will be found[f].

But if only the Oblique Seat of c on the Plane EFGH be required; through c draw cN parallel to ef, cutting Dy in N, from whence draw NC parallel to EF, and a Perpendicular to EF drawn from c, will cut NC in C the Seat deſired.

CASE 2.

When the Vaniſhing Lines of the given Planes are either parallel or coincide.

Fig. 115. N°. 2, 3, 4. Let $efgh$ and EFGH be the given Planes, and ab the given Line in the Plane $efgh$. The Practice in this Caſe differs in nothing from the preceeding, except in the fourth Article, which is performed in this manner:

Draw any Line Ll perpendicular to EF cutting the Interſecting Lines gh and GH in l and L, from whence to any Point n in the common Interſection of the given Planes draw ln, Ln; then at any convenient Diſtance from lL, draw Cc parallel to

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[a] Prop. 38.

[b] Cor. Theor. 3. B. I.

[c] Caſe 1 and 2, Prop. 40. and Gen. Cor.

[d] Gen. Cor. Prob. 6. B. II.

[e] Cor. 2. Theor. 15. B. I.

[f] Gen. Cor. Prop. 40.

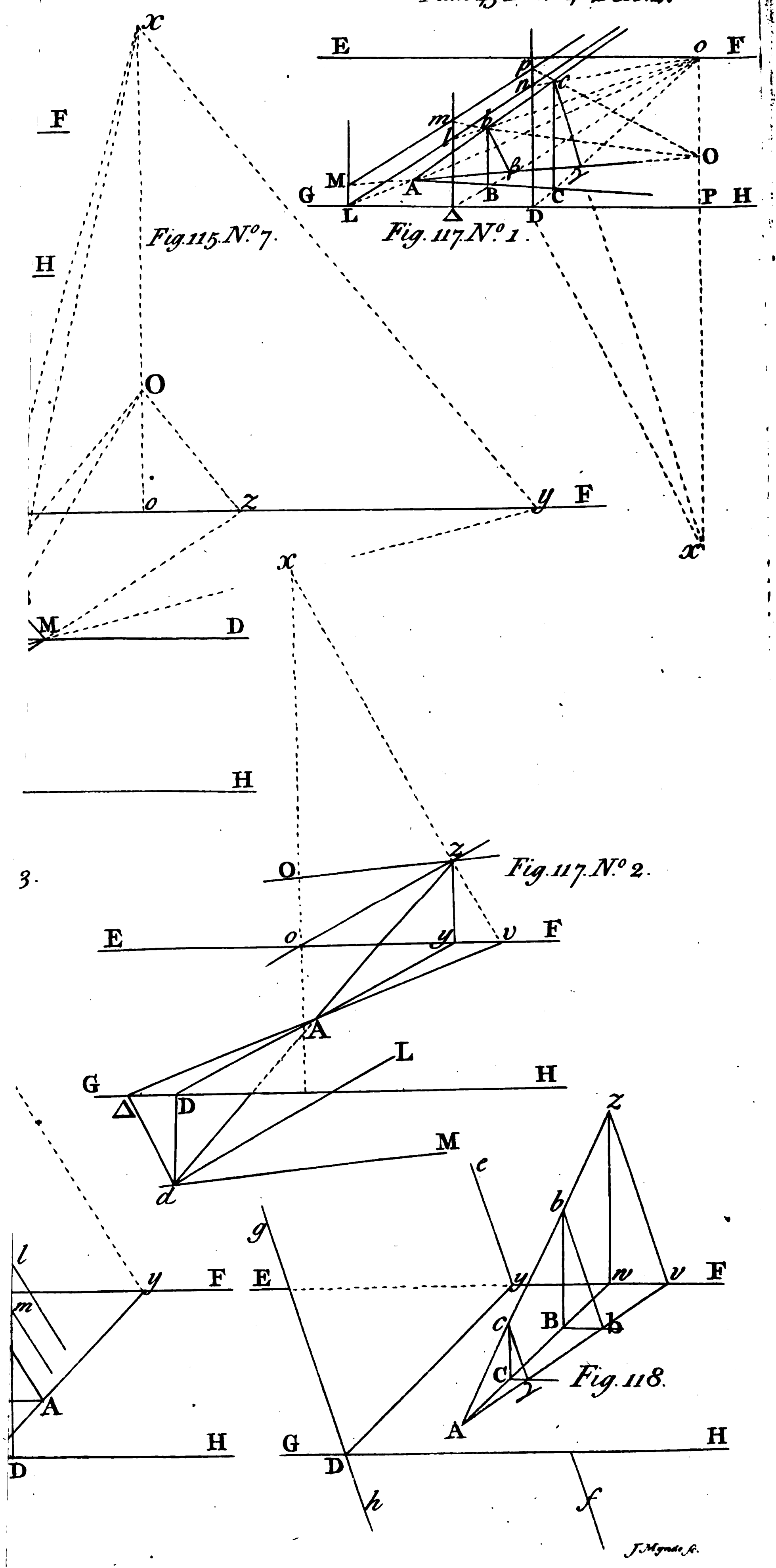

Fig. 115. N° 7.

Fig. 117. N° 1.

Fig. 117. N° 2.

Fig. 118.

3.

J. Mynde sc.

to it, cutting *ln* and L*n* in *c* and C, through which draw *ac*, AC, parallel to EF, and theſe may be uſed inſtead of the true Interſecting Lines *gb* and GH.

For the Originals of *c*C, *ac*, and AC, being parallel to the Picture, and meeting in *c* and C, they are in a Plane parallel to the Picture, the Interſections of which Plane with the given Planes are *ac*, AC, which Lines are therefore at an equal Diſtance from the Picture.

When the Vaniſhing Lines *ef* and EF coincide, the Points *z*, *v*, and M, being then all the ſame, it will be neceſſary, when *d* is out of reach, to find the Seat A of ſome Point *a* of the propoſed Line, in the manner before mentioned[a]. *Q. E. I.* Fig. 115. N°. 4.

CASE 3.

When either of the propoſed Planes is parallel to the Picture.

Let O be the Center of the Picture, EFGH one of the propoſed Planes, and MD its Interſection with the other Plane parallel to the Picture. Fig. 115. N°. 7.

1. And firſt, let *ab* be the Image of a Line in the Plane EFGH, whoſe Seat on the parallel Plane is required.

Produce *ab* to its Vaniſhing and Interſecting Points *z* and *d*, and draw *z*O, and through M the Interſection of *dz* with MD, draw MB parallel to *z*O; and MB will be the Perpendicular Seat of *ab* on the parallel Plane, whence A and B the Seats of *a* and *b* on that Plane may be found, by the Interſections of O*a* and O*b* with MB[b].

But the Oblique Seat of *ab* on the parallel Plane is the ſame with MD, the Interſection of that Plane with the Plane EFGH,

2. If AB be the given Line in the parallel Plane, and its Oblique Seat on the Plane EFGH be required; it is evident, it muſt fall in MD.

But if the Perpendicular Seat of AB on the Plane EFGH be deſired; from *x* the Vaniſhing Point of Perpendiculars to the Plane EFGH, draw *xy* parallel to AB, cutting EF in *y*, and a Line *y*M drawn from *y* through M, the Interſection of AB with DM, will be the Seat required; whence α and β the Seats of A and B on the Plane EFGH are found, by the Interſections of *x*A and *x*B with *y*M.

Dem. For *xy* is the Vaniſhing Line of a Plane perpendicular to the Plane EFGH, paſſing through the given Line AB, and is therefore the Vaniſhing Line of the Plane of the Perpendicular Seat of AB on the Plane EFGH; and M being a Point in the Interſection of theſe two Planes, and *y* being the Vaniſhing Point of that Interſection, *y*M is therefore the Image of that Interſection, and conſequently the Perpendicular Seat of AB on the Plane EFGH. *Q. E. I.*

PROP. XLVIII. PROB. XXXVI.

Any two Planes, with the Image of a Triangle in one of them, being given; thence to find the Seat of that Triangle on the other Plane.

This is done by finding the Seats of any two Sides of the given Triangle on the propoſed Plane[c], whereby the Seats of the three angular Points, and conſequently the intire Seat of the Triangle will be determined: and as ſufficient Methods have already been propoſed for doing this, in all poſſible Situations of the given Planes, either with reſpect to each other or to the Picture, it is unneceſſary to draw any Figures, or to enlarge farther on this Problem. *Q. E. I.*

PROP. XLIX. PROB. XXXVII.

The Center and Diſtance of the Picture, and an Original Plane being given, together with the Image of a Point out of that Plane, with either its Perpendicular or Oblique Seat on that Plane, or on the Picture; thence to find the other Seats of that Point, both on the Picture and Original Plane.

Let O be the Center of the Picture, EFGH the given Plane, and *x* the Vaniſhing Point of Perpendiculars to that Plane, and let *a* be the given Point, and A its Perpendicular Seat on the Plane EFGH. Fig. 116.

From *o*, the Center of the Vaniſhing Line EF, through the given Seat A, draw *o*A cutting GH in D, and draw Dα parallel to the Vertical Line *o*P; from *a* draw *a*B parallel to *o*P, cutting *o*D in B, and from *o* and O through *a* draw *o*β, Oα, cutting Dα in β and α; then B will be the Oblique Seat of *a* on the Plane EFGH, and β will be the Oblique, and α the Perpendicular Seat of *a* on the Picture.

Dem.

[a] Cor. Caſe 1.

[b] Caſe 3. Prop. 40. and Gen. Cor.

[c] Prop. 47.

Dem. For the Plane $oPD\alpha$ which paſſes through the Support aA of the Point a, being parallel to the Vertical Plane, it is the Plane in which all the Seats of the Point a lie, either on the Original Plane or the Picture[a]; wherefore aB parallel to oP is the Oblique Support, and B the Oblique Seat of a on the Original Plane[b], and $D\alpha$ being the Interſecting Line of the Plane $oPD\alpha$, α and β are the Interſecting Points of Oa and oa; α is therefore the Perpendicular, and β the Oblique Seat of a on the Picture, the Original of $O\alpha$ being perpendicular to the Picture, and the Original of $o\beta$ being parallel to the Line of Station of the Plane EFGH[c]. *Q. E. I.*

[a] Prop. 3.

[b] Prop. 1.

[c] Prop. 4. and Def. 3.

COR. 1.

It is evident, that if any two of the five Points, a, A, B, α, or β, be given, all the reſt may thence be found; for by any one of the Seats, the Plane $oPD\alpha$ is determined, and x, O, and o, are the Vaniſhing Points of three of the Supports aA, $a\alpha$, and $a\beta$, and the fourth aB is parallel to oP, and the Interſection of any two Supports gives the Point a.

COR. 2.

If the given Point were in the Vertical Plane, as at a, and its given Seat at A in the Line oP, a ſubſtituted Plane $oP\alpha D$ muſt be uſed, and the Points a and A being transferred to a and A in that Plane, by Parallels to EF, and the Seats of a in the Plane $oP\alpha D$ being thence found, they are to be transferred back to the Plane oP, by Parallels to EF, where they will mark the correſponding Seats of the Point a on oP[d].

[d] Prop. 41.

The ſame Method may be uſed when the given Point is ſo near the Vertical Plane, that the Plane in which its Seats lie, hath ſo little Depth as to be inconvenient for uſe, as the Plane $oPdg$; only obſerving, that the Seats A and B of the ſubſtituted Point a on the Plane EFGH, are to be transfered to the Interſection od of the Plane $oPdg$ with the Original Plane, and the Seats α and β of the Point a on the Picture, are to be transferred to dg the Interſecting Line of that Plane.

PROP. L. PROB. XXXVIII.

The Center and Diſtance of the Picture, and an Original Plane being given, together with the Image of a Line out of that Plane, with either its Perpendicular or Oblique Seat on that Plane, or on the Picture; thence to find the other Seats of that Line, both on the Picture and Original Plane.

1. Either of the Seats of the propoſed Line on the Original Plane being given; thence to find its other Seat on that Plane.

Fig. 117. N°. 1. Let O be the Center of the Picture, bc the given Line, and BC its Oblique Seat on the Plane EFGH.

Find B and C the Oblique Seats of any two Points b and c of the given Line on the Plane EFGH[e], and having drawn Bo and Co, from x the Vaniſhing Point of Perpendiculars to the given Plane, draw xb, xc, which will cut Bo and Co in β and γ, the Perpendicular Seats of b and c on the Plane EFGH[f], through which a Line $\beta\gamma$ being drawn, it will be the Perpendicular Seat of bc on that Plane[g].

[e] Gen. Cor. Prop. 40.

[f] Prop. 49.

[g] Def. 2.

Or if $\beta\gamma$ be given; by β and γ the Lines βo and γo, and thereby the Points B and C are determined, and by them BC is found.

The Interſection A of the given Line with the Plane EFGH, and the Seat of any one Point c of the propoſed Line, will likewiſe anſwer the purpoſe; ſeeing the Seat of bc, of either Sort, on the given Plane, muſt neceſſarily paſs through A. *Q. E. I.*

2. Either of the Seats of the propoſed Line on the Original Plane being given; thence to find either of the Seats of that Line on the Picture.

Having through the Seats B and C, or β and γ, of any two Points b and c of the given Line on the Original Plane, drawn oB and oC, as before, produce them till they cut GH in Δ and D, from whence draw Δm, Dp, perpendicular to EF; then from O and o through c, draw Op, on, cutting Dp in p and n, and from the ſame Points O and o through b, draw Om, ol, cutting Δm in m and l; and mp will be the Perpendicular Seat, and ln the Oblique Seat of bc on the Picture.

For $oPDp$ being the Plane in which the Seats of the Point c lie, p is the Perpendicular, and n the Oblique Seat of c on the Picture; and $oP\Delta m$ being the Plane in which the Seats of b lie, m is the Perpendicular, and l the Oblique Seat of b on the Picture[h]; and therefore mp drawn through the Perpendicular Seats of b and c, is the Perpendicular Seat of bc, and ln drawn through the Oblique Seats of b and c, is the Oblique Seat of bc on the Picture. *Q. E. I.*

[h] Prop. 49.

The

The Point A ſupplies the place of one Point in the given Line and its Seat; for oA being drawn cutting GH in L, L is the Oblique Seat of A on the Picture, and LM drawn parallel to oP, is the Interſecting Line of the Plane in which the Seats of A lie; wherefore OA cuts LM in M the Perpendicular Seat of A on the Picture, and M with any Point p in the Perpendicular Seat of bc, and L with any Point n in the Oblique Seat of bc, determine Mp and Ln the Perpendicular and Oblique Seats of bc on the Picture.

3. Either of the Seats of the propoſed Line on the Picture being given; thence to find either of the Seats of that Line on the Original Plane.

If the Perpendicular Seat mp be given; from any two Points m and p in that Line, draw Perpendiculars to EF, cutting GH in Δ and D, from whence draw Δo, Do; then Lines from O to m and p give two Points b and c in the Line bc, and the Seats of b and c being found in the Lines Δo and Do, thereby BC or $\beta\gamma$ is determined.

Or if the Oblique Seat ln be given; the Perpendiculars to EF from l and n give Δ and D, and conſequently Δo and Do, ol and on give b and c, whoſe Seats on the Lines Δo, Do, are found as before, and thence BC and $\beta\gamma$. *Q. E. I.*

4. Either of the Seats of the propoſed Line on the Picture being given; thence to find the other Seat of that Line on the Picture.

If mp be given; through m and p draw ml, pn, perpendicular to EF, then Om and Op give b and c, and ob and oc cut ml and pn in l and n, whence ln the Oblique Seat of bc on the Picture is found: and by the like Method, mp may be found, if ln be given. *Q. E. I.*

C O R. 1.

This Problem is here ſolved by the help of the Seats of two Points of the given Line, or by the Seat of one Point of that Line, and its Interſection with the Original Plane, ſuppoſing the Vaniſhing and Interſecting Points of that Line to be out of reach; but if theſe can be conveniently had, they alone are ſufficient for finding the Perpendicular or Oblique Seat of that Line, either on the Picture, or on any other given Plane, whatſoever, abſtracted from any other given Relation of that Line to any particular Plane.

Fig. 117. N°. 2.

Thus if z and d be the Vaniſhing and Interſecting Points of a given Line dz, and x the Vaniſhing Point of Perpendiculars to a given Plane EFGH; zy and dD perpendicular to EF, give Dy the Oblique Seat, and zv drawn from x, and $d\Delta$ parallel to it, give Δv the Perpendicular Seat of dz on the Plane EFGH[a]; and the Line dM parallel to zO, is the Perpendicular Seat, and dL parallel to zo, is the Oblique Seat of dz on the Picture[b].

C O R. 2.

Fig. 117. N°. 3.

If the Original of the given Line Ab be parallel to the Picture, it will be parallel to its Oblique and Perpendicular Seats on the Picture[c], and its Oblique Seat on the Plane EFGH will be parallel to EF[d]; whence if any one Point B, m, or l, in either of thoſe Seats be found, the whole of that Seat is determined; and the Perpendicular Seat of Ab on the Plane EFGH is found, by drawing xy parallel to Ab, and from y through A or β drawing $y\beta$[e].

C O R. 3.

If the Original Plane be parallel to the Picture, the Indefinite Supports of any Points of the given Line on the Original Plane, are alſo the Supports of the ſame Points on the Picture, and the correſponding Seats of the given Line on the Picture and Original Plane are parallel[f].

PROP. LI. PROB. XXXIX.

The Center and Diſtance of the Picture, and the Image of a Triangle being given, with its Perpendicular or Oblique Seat, either on the Picture, or on a given Original Plane; thence to find the other Seats of that Triangle on the Picture and Original Plane.

This is done by finding the required Seat of any Side of the given Triangle by its given Seat[g], and then finding the required Seat of the remaining angular Point of the Triangle by its given Seat[h], whence the deſired Seat of the Triangle will be compleated. *Q. E. I.*

D E F. 15.

Fig. 118.

Let EFGH and $efgb$ be two given Planes, and Dy their common Interſection, and let b be the Image of a Point out of thoſe Planes, and B its Oblique Seat on the Plane EFGH.

Through B draw Bb parallel to EF, and from b draw bb parallel to ef, cutting Bb

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[a] Caſe 1 and 2. Prop. 40.

[b] Prop. 31.

[c] Cor. 1. Prop. 31.

[d] Cor. 2. Caſe 1. Prop. 40.

[e] Caſe 3. Prop. 47.

[f] 16 El. 11.

[g] Prop. 50.

[h] Prop. 49.

in b; then b is called the *Parallel Seat*, and $b\mathrm{b}$ the *Parallel Support* of b on the Plane EFGH with reſpect to the Plane $efgh$.

COR.

The Vaniſhing Lines of all Planes which paſs through $b\mathrm{b}$ are parallel to ef or coincide with it, and the Interſection of any ſuch Plane with the Plane $efgh$ is alſo parallel to ef.

For the Originals of $b\mathrm{B}$ and Bb being parallel to the Picture, the Original of $b\mathrm{b}$ is alſo parallel to the Picture, and conſequently to the Vaniſhing Lines of all Planes that can paſs through it[a]; theſe Vaniſhing Lines are therefore parallel to ef, wherefore the common Interſections of thoſe Planes with the Plane $efgh$, are alſo parallel to ef[b].

[a] Cor. 1. Theor. 15. B. I.

[b] Theor. 15. B. I.

DEF. 16.

The Parallel Seat of any Line bc on the Plane EFGH, with reſpect to the Plane $efgh$, is a Line $\gamma\mathrm{b}$ drawn through γ and b the Parallel Seats of any two Points c and b of the Line bc.

COR.

The Line bc and its Parallel Seat $\gamma\mathrm{b}$ are in a Plane whoſe Vaniſhing Line zv is parallel to ef.

For the Originals of $b\mathrm{b}$ and $c\gamma$ being parallel to the Picture, they are parallel to the Vaniſhing Line zv of the Plane which paſſes through them.

PROP. LII. PROB. XL.

The Center and Diſtance of the Picture, and the Image of a Point, with its Perpendicular or Oblique Seat on a known Plane, being given; thence to find the Perpendicular or Oblique Seat of that Point on any other given Plane.

CASE 1.

When the Vaniſhing Lines of the given Planes interſect.

Fig. 119. N°. 1. Let O be the Center of the Picture, EFGH and $efgh$ the given Planes, and $\mathrm{D}y$ their common Interſection; and let a be the propoſed Point, and A its Perpendicular Seat on the Plane EFGH.

METHOD 1.

By the given Seat A find the Oblique Seat B of the given Point on the Plane EFGH[c]; from B draw BL parallel to EF, cutting $\mathrm{D}y$ in L, from whence draw $\mathrm{L}\beta$ parallel to ef, and from a draw $a\beta$ perpendicular to ef, cutting $\mathrm{L}\beta$ in β; then β will be the Oblique Seat of a on the Plane $efgh$; and if from w, the Center of the Vaniſhing Line ef, through β, a Line $w\beta$ be drawn, and cut in α by az drawn from a to z the Vaniſhing Point of Perpendiculars to the Plane $efgh$, α will be the Perpendicular Seat of a on that Plane.

Dem. For the Originals of $a\mathrm{B}$, BL, and $\mathrm{L}\beta$, which meet in B and L, being all parallel to the Picture[d], they are in a Plane parallel to the Picture, in which Plane the Line $a\beta$ alſo lies; and $\mathrm{L}\beta$ being the Interſection of this Plane with the Plane $efgh$, and $a\beta$ being perpendicular to ef, the Point β, where it cuts $\mathrm{L}\beta$, is therefore the Oblique Seat of a on that Plane; and if β be the Oblique Seat, α is the Perpendicular Seat of that Point on that Plane[e], *Q. E. I.*

[c] Prop. 49.

[d] Cor. 2. Theor. 15. B. I.

[e] Prop. 49.

METHOD 2.

Through z the Vaniſhing Point of Perpendiculars to the Plane $efgh$, draw zv parallel to ef, cutting EF in v, and having found a, the Parallel Seat of a on the Plane EFGH[f], draw $v\mathrm{a}$ cutting $\mathrm{D}y$ in M, and having drawn $\mathrm{M}\alpha$ parallel to ef, draw za, which will cut $\mathrm{M}\alpha$ in α, the Perpendicular Seat of a on the Plane $efgh$, whence its Oblique Seat β is found, by the Interſection of $w\alpha$ with $a\beta$ drawn perpendicular to ef[g].

Dem. For zv being the Vaniſhing Line of a Plane perpendicular to the Plane $efgh$[h], paſſing through $a\mathrm{a}$ the Parallel Support of a on the Plane EFGH[i], and $v\mathrm{a}$ being the Interſection of the Plane $zva\mathrm{a}$ with the Plane EFGH, the Point M where $v\mathrm{a}$ cuts $\mathrm{D}y$, is a Point in the Interſection of the Plane $zva\mathrm{a}$ with the Plane $efgh$; wherefore $\mathrm{M}\alpha$ parallel to ef is the whole of that Interſection[k]; and the Original of za being a Line in the Plane $zva\mathrm{a}$ perpendicular to the Plane $efgh$, the Point α where it cuts $\mathrm{M}\alpha$, is therefore the Perpendicular Seat of a on the Plane $efgh$[l]. *Q. E. I.*

[f] Def. 15.

[g] Prop. 49.

[h] Cor. 3. Prop. 20.

[i] Cor. Def. 15.

[k] Cor. Def. 15.

[l] Prop. 49.

COR. 1.

If through v and D a Line $v\mathrm{D}$ be drawn, and produced beyond D at pleaſure, the Angle $\mathrm{F}v\mathrm{D}$ will contain the Space, within which the Parallel Seats of all Points on the

Plate 44 Book 4 Sect. 2.

º1

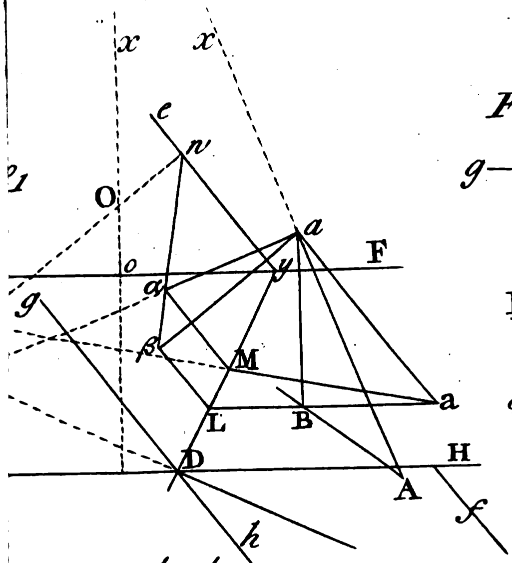
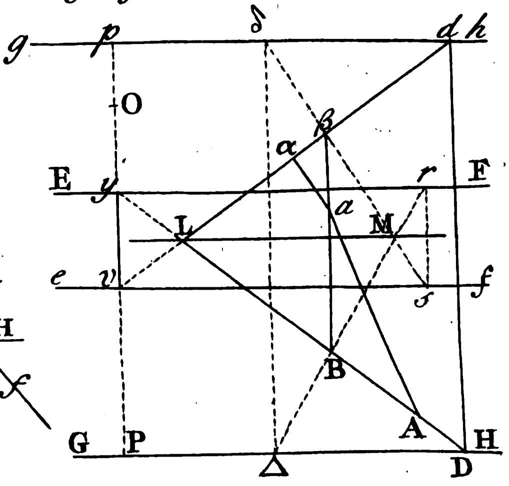
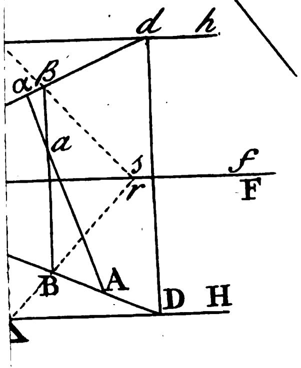


Fig. 119 Nº 2.





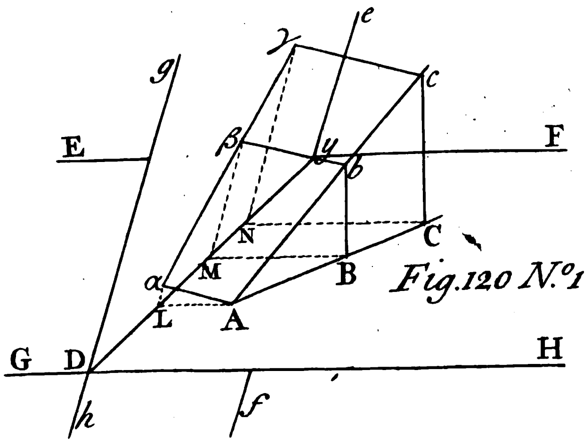


Fig. 120 Nº 1.

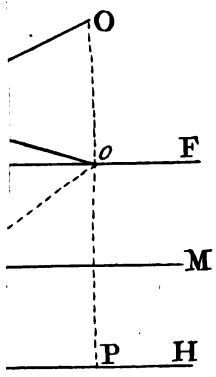
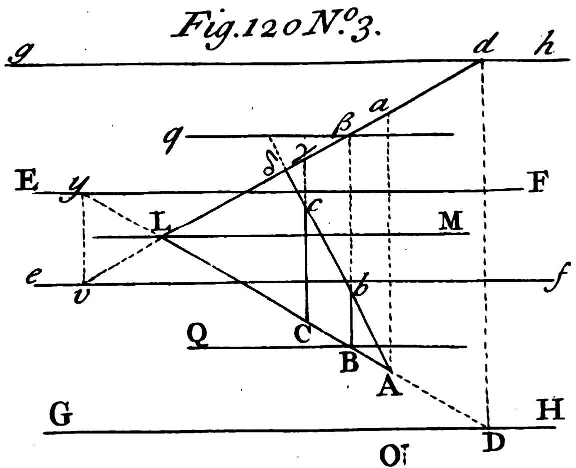


Fig. 120 Nº 3.



120 Nº 2.

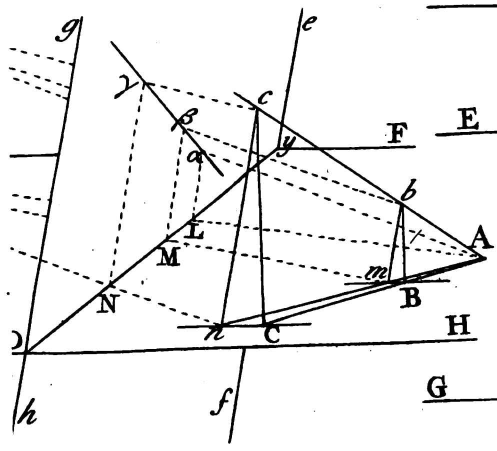
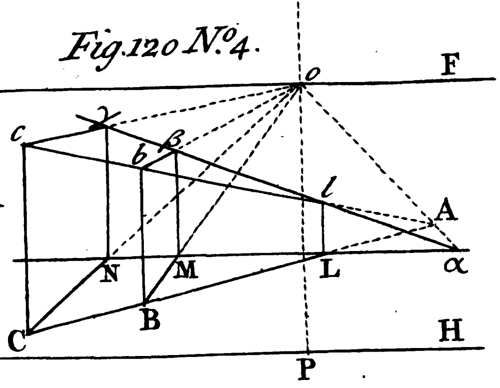


Fig. 120 Nº 4.



J. Mynde sc.

the Plane EFGH muſt lie, whoſe Perpendicular Seats on the Plane $efgb$ can fall any where within the Perſpective Part of that Plane, and the Line vD is the Boundary of that Space.

For if a Line drawn from v to a, the Parallel Seat of any Point a on the Plane EFGH, fall any where within the Angle FvD, it muſt cut Dy in ſome Point M, between D and y, and the Perpendicular Seat of a on the Plane $efgb$ muſt be in Mα drawn parallel to ef: but if va fall without that Angle, it muſt cut Dy either below D or beyond y; in the firſt Caſe, the Perpendicular Seat of a on the Plane $efgb$ will fall in the Projective Part, and in the other Caſe, it will fall in the Tranſprojective Part of that Plane.

C O R. 2.

If the Plane $efgb$ be perpendicular to the Picture, the Vaniſhing Point z, and conſequently v, will be infinitely diſtant, and aM will coincide with aL or BL, and the Perpendicular and Oblique Seats of a on the Plane $efgb$ will be the ſame.

C A S E 2.

When the Vaniſhing Lines of the given Planes are either parallel or coincide.

Let O be the Center of the Picture, and EFGH, $efgb$, the given Planes; and let a be the given Point, and A its Perpendicular Seat on the Plane EFGH. Fig. 119. N°. 2, 3.

Through O draw Pp the common Vertical Line of the given Planes[a], cutting EF and ef in their Centers y and v; through y and the given Seat A draw yD, cutting GH in D, and having drawn Dd parallel to Pp, cutting gb in d, draw dv; then PpDd will be the Plane in which the Seats of a on both the given Planes lie[b], which Seats are therefore to be found in yD and vd, the Interſections of the Plane PpDd with thoſe Planes; the Perpendicular Seats, by Lines drawn from a to the Vaniſhing Points of Perpendiculars to the reſpective Planes, and the Oblique Seats, by Bβ drawn through a parallel to Pp[c]. *Q. E. I.*

C O R. 1.

If the Oblique Seat B on the Plane EFGH were given, and the Oblique Seat β on the Plane $efgb$ only required; it is not neceſſary, that the Vaniſhing Line vy ſhould paſs through the Center of the Picture, but any other Vaniſhing Line rs parallel to Pp may be uſed, whence a Plane $rs\Delta\delta$ paſſing through B being drawn, the Oblique Seat β will be found in $s\delta$, the Interſection of that Plane with the Plane $efgb$[d].

C O R. 2.

When the Vaniſhing Lines EF and ef coincide, the Points v and y are the ſame; but this makes no material Difference in the Practice, as may be ſeen by the Figures; and when the Vaniſhing Lines are parallel, any Point M in the common Interſection of the given Planes may be uſed inſtead of the Vaniſhing Line rs, when the Oblique Seats only are wanted.

C A S E 3.

When one of the propoſed Planes is parallel to the Picture.

Let O be the Center of the Picture, EFGH one of the propoſed Planes, and LM its Interſection with the other Plane parallel to the Picture; and let a be the given Point, and A or B its Perpendicular or Oblique Seat on the Plane EFGH. Fig. 119. N°. 4.

Through A or B draw Ao, cutting LM in L, and having drawn Lα parallel to oP, through a draw aO, ao, which will cut Lα in α and β, the Perpendicular and Oblique Seats of a on the parallel Plane.

Dem. For the Seats of a on the parallel Plane, are in the ſame Plane oPDb in which its Seats on the Picture lie, and Lα being the Interſection of this Plane with the parallel Plane, the ſame Lines aO, ao, which mark the Seats a and b of the Point a on the Picture, alſo cut Lα in α and β, the correſponding Seats of a on the parallel Plane. *Q. E. I.*

C O R.

If either of the Seats α or β be given, the Line αL and the Point L are thereby found, whence oL is determined, in which the Seats A and B lie.

PROP. LIII. PROB. XLI.

The Center and Diſtance of the Picture, and the Image of a Line bc, with its Perpendicular or Oblique Seat on a known Plane EFGH, Fig. 120. N°. 1, 2, 3.

being

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[a] Cor. 2. Theor. 14. B. I.

[b] Prop. 3.

[c] Gen. Cor. Prop. 40.

[d] Prop. 5.

being given; thence to find the Perpendicular or Oblique Seat of that Line on any other given Plane $efgh$.

CASE 1.

When the Vaniſhing Lines of the given Planes interſect.

Fig. 120. N°. 1. Find the Oblique Seats B and C of any two Points b and c of the given Line on the Plane EFGH, and thence the Oblique Seats β and γ of the ſame two Points on the Plane $efgh$[a]; and $\beta\gamma$ will be the Oblique Seat of bc on that Plane, whence its Perpendicular Seat on that Plane may be found[b].

Fig. 120. N°. 2. Or if the Perpendicular Seat of bc on the Plane $efgh$ be firſt deſired; it is found by m and n the Parallel Seats of b and c, and the Vaniſhing Points v and z[c]. *Q. E. I.*

In either Caſe, A the Interſection of bc with the Plane EFGH, is equivalent to a Point in the propoſed Line, and its Seat on that Plane.

CASE 2.

When the Vaniſhing Lines of the given Planes are either parallel or coincide.

Fig. 120. N°. 3. If bc be the given Line, and BC its Oblique Seat on the Plane EFGH; produce BC to its Vaniſhing and Interſecting Points y and D, and compleat the Plane $yvDd$; then vd the Interſection of this Plane with the Plane $efgh$, is the Oblique Seat of bc on that Plane, in which Line the Oblique Seats β, γ, of the Points b and c, and alſo δ, the Interſection of bc with the Plane $efgh$, are found[d], whence the Perpendicular Seat of bc on that Plane may be had, if required[e]. *Q. E. I.*

COR.

If the Original of the given Line bc were parallel to the Picture, and BQ were its Oblique Seat on the Plane EFGH; through B the Seat of any Point b of that Line, draw any Line yD, and having compleated the Plane $yvDd$, and thence found β, the Oblique Seat of b on the Plane $efgh$[f], a Line βq drawn through β parallel to EF, will be the Oblique Seat of bc on that Plane.

For the Original of bc being here ſuppoſed parallel to the Picture, its Oblique Seat on the Plane $efgh$ is alſo parallel to the Picture; wherefore β being the Seat of one Point of that Line, βq is its intire Seat on that Plane.

The Figure here referred to, ſerves equally for the Caſe when the Vaniſhing Lines of the given Planes coincide, as when they are parallel, if LM be conſidered as their common Vaniſhing Line.

CASE 3.

When one of the propoſed Planes is parallel to the Picture.

Fig. 120. N°. 4. Let LM be the Interſection of the Plane EFGH with a Plane parallel to the Picture, bc the given Line, and BC its Oblique Seat on the Plane EFGH.

By the Oblique Seats C and B of the Points c and b on the Plane EFGH, find γ and β their Oblique Seats on the parallel Plane[g], and $\gamma\beta$ will be the Oblique Seat of bc on that Plane; and by the like way, the Perpendicular Seat of that Line might be found, if the Lines from b and c, inſtead of being drawn to o the Center of the Vaniſhing Line EF, were drawn to O the Center of the Picture. *Q. E. I.*

COR.

The Interſection A of the Line bc with its Seat BC on the Plane EFGH, ſerves to find a Point α in its Oblique Seat on the parallel Plane; and the Interſection L of the Seat BC with the parallel Plane, ſerves to find the Interſection l of the Line bc with that Plane.

Note, *What was ſaid at Cor.* 1. *Prop.* L. *is equally applicable here.*

It would be ſuperfluous to add a Problem for finding the Seat of a given Triangle on a propoſed Plane, from its given Seat on another Plane; that being only a Compoſition of the two laſt Propoſitions[h].

GENERAL COROLLARY.

The ſeveral Methods propoſed in this Section for finding the Image of a Triangle, and the Vaniſhing and Interſecting Lines of its Plane, and for finding the Seat of a Triangle on any propoſed Plane, ſerve alſo to find the Images or Seats of any other Plain Figures[i], of which it is therefore unneceſſary to give farther Examples.

STEREO-

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[a] Meth. 1. Prop. 52.

[b] Prop. 50.

[c] Meth. 2. Prop. 52.

[d] Cor. 1. Caſe 2. Prop. 52.

[e] Prop. 50.

[f] Cor. 1. Caſe 2. Prop. 52.

[g] Caſe 3. Prop. 52.

[h] Prop. 51.

[i] Schol. Prob. 30. B. II.

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By *J. HAMILTON*, Eſq; F. R. S.

VOL. II.

LONDON,
Printed for the AUTHOR,
By W. BOWYER, in *White Fryars*, near the *Temple*, and Sold by S. AUSTEN, at the *Angel*
and *Bible* in *St. Paul's Church-Yard*.
MDCCXXXVIII.

STEREOGRAPHY,

OR A

COMPLEAT BODY

OF

PERSPECTIVE,

In all its BRANCHES.

BOOK V.

SECTION I.

Of the Projections of Points, Lines, and plain Figures, on a given Plane from a given Point.

DEF. 1.

IN general, the Projection of an Object on a Plane, is the Interſection of that Plane with ſtraight Lines, either parallel between themſelves, or elſe proceeding from ſome one common Point, and paſſing through the ſeveral Points of the Object.

When theſe Lines are parallel between themſelves, they form the Geometrical Projection; and when they all proceed from ſome one Point, they form the Stereographical Projection of the Object on the propoſed Plane, as already deſcribed[a].

But the Projection here meant, is the Shadow of an Object on a Plane, produced by Rays of Light either parallel between themſelves, or proceeding from ſome one Luminous Point, and which paſſing by the Extremities of the Object, project and define its Shadow on the propoſed Plane: theſe Rays are called the *Projecting Lines*; and the Shadow thus produced, is called the *Projection of the Object*.

DEF. 2.

When the Rays are parallel, as the Rays of Light which proceed from the Sun, Moon, or any other immenſely diſtant Luminary may be taken to be, with reſpect to any Objects here on Earth which can be ſeen at the ſame View; the Images of thoſe Rays, if they be not parallel to the Picture, muſt all meet in one common Vaniſhing Point.

If theſe parallel Rays flow from before the Eye, ſo as to throw the Shadows of Objects towards the Eye; their Vaniſhing Point, which in that Caſe muſt lie above the Vaniſhing Line of the Plane of the Horizon, is called *a Projecting Point at an infinite Diſtance before the Directing Plane*; which Point repreſents the direct Image of the Luminary from whence the Rays flow, conſidered as a Point at an infinite Diſtance before the Eye, without regard to its apparent Diameter.

DEF. 3.

If the parallel Rays flow from behind the Eye, ſo as to throw the Shadows beyond the Objects; their Vaniſhing Point, which in that Caſe muſt fall below the Vaniſhing

[a] Sect. 3. B. I.

Line of the Plane of the Horizon, is called *a Projecting Point at an infinite Diſtance behind the Directing Plane*; which Point then repreſents the Tranſprojected Image of the Luminary from whence the Rays flow, conſidered as a Point at an infinite Diſtance behind the Eye[a].

D E F. 4.

If the parallel Rays be alſo parallel to the Picture, ſo as to throw the Shadows of Objects ſideways; they will be parallel to their Images, which can then have no Vaniſhing Point, ſo that the Projecting Point hath in this Caſe no Image; but as a Line drawn from the Eye parallel to thoſe Rays, will fall wholly in the Directing Plane, the *Projecting Point* is then ſaid to be *at an infinite Diſtance in the Directing Plane.*

D E F. 5.

When the Rays which define the Shadow, meet in ſome one Point, as the Rays of Light which flow from a Candle or Torch, or any other Luminous Point at a moderate Diſtance; if that Point be before the Eye, then its Image is called *a Projecting Point at a moderate Diſtance before the Directing Plane.*

D E F. 6.

If the Rays meet in a Point behind the Eye, then the Tranſprojected Image of that Point is called *a Projecting Point at a moderate Diſtance behind the Directing Plane.*

D E F. 7.

If the Rays meet in a Point in the Directing Plane, that Point has then no Image, and the *Projecting Point* is in that Caſe ſaid to be *at a moderate Diſtance in the Directing Plane.*

D E F. 8.

A Plane paſſing through any given Line and a Projecting Point, is called the *Projecting Plane* of that Line; the Interſection of that Plane with the propoſed Original Plane being the Projection of the given Line on the Original Plane.

D E F. 9.

The Plane on which the Shadow or Projection of any Object is required, when it is not deſcribed by its Letters, is called the *Plane of the Projection.*

S C H O L.

The Deſign of this Section being to ſhew how, from the given Image of any Object, to determine the Image of its Shadow or Projection on a propoſed Plane from a given Point, as it ought to appear in the Picture; there will be no occaſion to take any Meaſures from the Original Object itſelf, every thing being to be performed by the help of its Image; which will therefore be conſidered as the Object whoſe Shadow or Projection is ſought, and ſhall be generally ſo called, to avoid Circumlocution.

And as for this Purpoſe, the Relation of the Objects, and of the Projecting Point, to the Plane of the Projection, or to ſome other given Plane muſt be known, we ſhall here conſtantly ſuppoſe their Oblique Seats on ſome Plane or other to be given, if not otherwiſe mentioned; theſe being the moſt univerſally convenient for uſe, as ſerving alike, whether the Plane of the Projection be perpendicular or anywiſe inclining to the Picture, and being at any time eaſily found, if any other Relation of the Objects to the propoſed Plane be given, as has already been fully ſhewn[b].

P R O B. I.

An Original Plane not parallel to the Picture, and a Point with its Seat on that Plane, being given; thence to find the Projection of that Point on the given Plane, from a Projecting Point whoſe Seat on the ſame Plane is alſo given.

C A S E 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane[c].

Fig. 121. N°. 1, 2. Let EFGH be the Plane of the Projection, S the Projecting Point, and T its Seat on that Plane; and let *a* and *b* be two given Points, and A and B their Seats on the ſame Plane.

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[a] Cor. 4. Theor. 4. B. I.

[b] Sect. 2. B. IV.

[c] Def. 5, 6.

Plate 45 Book 5 Sect. 1.

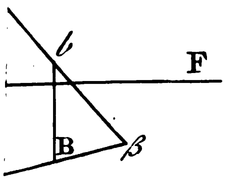
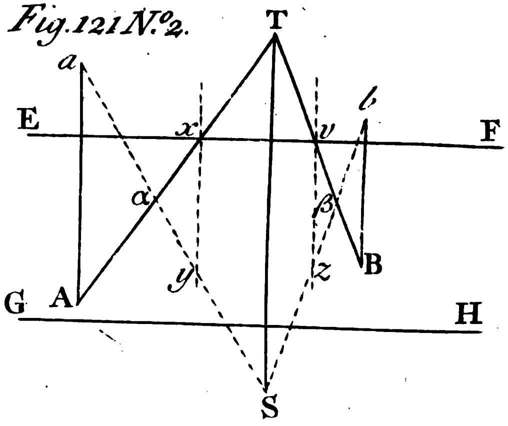
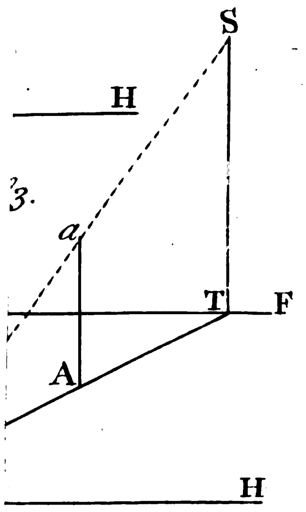


Fig. 121 N°. 2.





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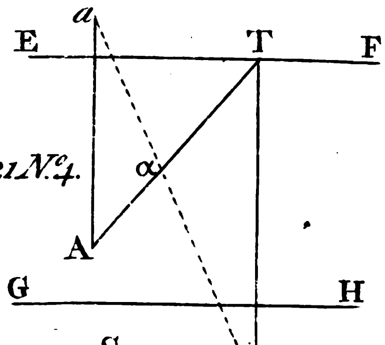
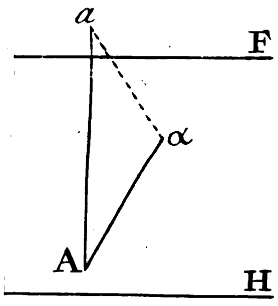


Fig. 121 N°. 4.

g. 121 N°. 5.



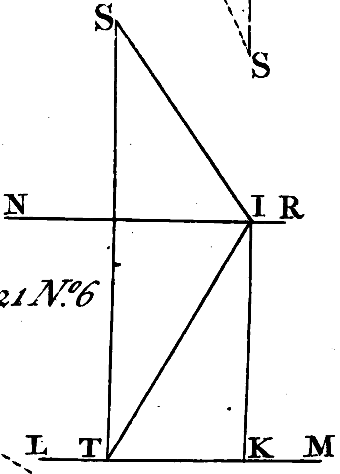
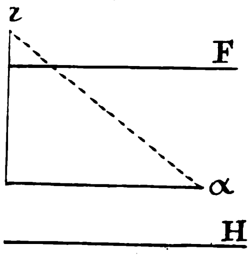


Fig. 121 N°. 6

121 N°. 7.



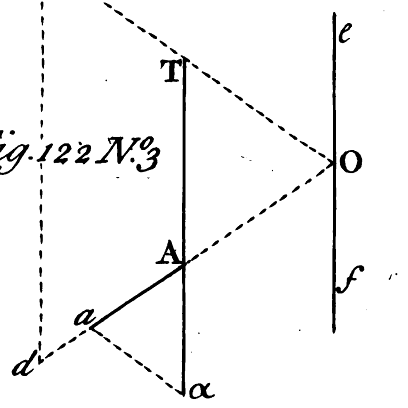
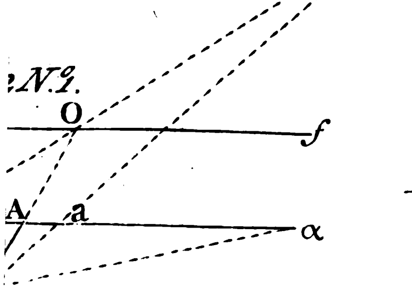


Fig. 122 N°. 3

N°. 2.



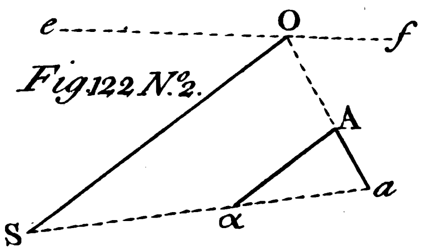


Fig. 122 N°. 2.

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Through T and the given Seats A and B, draw TA, TB, and from S draw Sa, Sb, cutting TA and TB in α and β; then α and β will be the Projections of a and b on the Plane EFGH from the Point S.

Dem. For TA and TB being the Seats of the Projecting Lines Sa and Sb on the Plane EFGH, α and β are the Interſections of thoſe Lines with that Plane[a], and are therefore the Projections of a and b on that Plane from the Point S. *Q. E. I.*

S C H O L.

In the firſt Figure, the Original of the Projecting Point S is ſuppoſed to lie before the Eye, and the Point a is ſuppoſed nearer, and the Point b farther from the Eye than S; whence the Projection of a falls towards the Eye at α, and the Projection of b forwards at β. Fig. 121. N°. 1.

In the ſecond Figure, the Original of the Projecting Point S is ſuppoſed to lie behind the Eye, whence its Seat T falls in the Tranſprojective Part of the Plane EFGH, and its Support ST is inverted, the Point S falling below T[b]; Sa and Sb are the Complements of the Images of the Projecting Lines, which proceed from S to a and b, and TA and TB are the Complements of their Seats on the Plane EFGH[c], the Vaniſhing Points of which Seats are x and v; wherefore xy and vz are the Vaniſhing Lines of the Planes of the Seats of Sa and Sb, and y and z their Vaniſhing Points[d]; and both the Points a and b being before the Eye, and the Original of the Projecting Point S behind, the Projections α and β are thrown forward. Fig. 121. N°. 2.

C A S E 2.

When the Projecting Point is at an Infinite Diſtance before or behind the Directing Plane[e].

Let EFGH be the Plane of the Projection, S the Projecting Point, conſidered as a Vaniſhing Point, and let a be the Point whoſe Projection is ſought, and A its Seat on the given Plane. Fig. 121. N°. 3, 4.

From S draw ST perpendicular to EF, cutting it in T; then the Interſection α of TA with Sa, will be the Projection deſired.

Dem. For TA being the Seat of Sa on the Plane EFGH, α is its Interſection with that Plane, and conſequently the Projection ſought. *Q. E. I.*

S C H O L.

Here S being the Vaniſhing Point of the Projecting Lines, which are ſuppoſed to be parallel, T is the Vaniſhing Point of their Oblique Seats on the Plane EFGH, T is therefore conſidered as the Seat of S on that Plane.

In the firſt Figure, the Rays are ſuppoſed to flow from a Point at an infinite Diſtance before the Eye, of which Point S is the direct Image, and therefore the Point a is projected towards the Eye[f]. Fig. 121. N°. 3.

In the ſecond Figure, the Luminous Point is ſuppoſed to be at an infinite Diſtance behind the Eye, and S is its Tranſprojected Image, and falls below its Seat T, and therefore the Point a is projected forward[g]. Fig. 121. N°. 4.

C A S E 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane[h].

Here, the Projecting Point being in the Directing Plane, it is the Directing Point of the Projecting Lines, and its Oblique Seat being alſo in the ſame Plane, that Seat is the Directing Point of the Seats of the Projecting Lines on the Plane of the Projection, and neither the Projecting Point, nor its Seat can have any Image: theſe are however ſupplied in the following manner.

Let EFGH be the Plane of the Projection, a the Point whoſe Projection is required, and A its Seat on that Plane. Fig. 121. N°. 5.

Any where a-part draw the Directing Plane NRLM, wherein let IK be the Eye's Director, NR the Parallel of the Eye, and LM the Directing Line of the Plane of the Projection[i]; and let S be the Place of the Projecting Point in the Directing Plane, and T its Seat on the Directing Line, and draw SI and TI: then from any Point i in EF the Vaniſhing Line of the Plane of the Projection, draw si, ti, inclining the ſame way and in the ſame Angles to EF, as SI and TI do to NR in the Directing Plane, and from A and a draw $A\alpha$, $a\alpha$, parallel reſpectively to ti and si, and their Interſection α will be the Projection deſired. Fig. 121. N°. 6. Fig. 121. N°. 5.

Dem. For S and T being the Directing Points of the Projecting Lines, and of their

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[a] Prop. 40. B. IV.

[b] Cor. Caſe 1. Prop. 39. B. IV.

[c] Theor. 4. and Def. 24. B. I.

[d] Prop. 40. B. IV.

[e] Def. 2, 3.

[f] Def. 2.

[g] Def. 3.

[h] Def. 7.

[i] Schol. Prob. 2. B. II.

Seats on the Plane of the Projection, and S I and T I being their Directors, the Images of all the Projecting Lines will be parallel to S I, and the Images of their Seats will be parallel to T I[a]; and si and ti in the Picture, being by Conſtruction parallel to S I and T I in the Directing Plane, $a\alpha$ parallel to si is the Image of the Projecting Line which paſſes through a, and $A\alpha$ parallel to ti is the Image of the Seat of that Line, and conſequently the Interſection α of theſe two, is the Projection of a on the Plane EFGH from the propoſed Projecting Point in the Directing Plane. *Q. E. I.*

D E F. 10.

The Lines si and ti transferred to E F as above directed, are called the *Directions of the Projecting Lines and their Seats.*

C O R. 1.

The Directions si and ti ſerve equally for finding the Projections of all Points whatſoever on the Plane E F G H from the Projecting Point propoſed, the Images of the Projecting Lines being conſtantly parallel to si, and the Images of their Seats to ti, in whatever part of the Picture the Point A falls; ſeeing S and T muſt ſtill continue to be the Directing Points of the Projecting Lines, and of their Seats on the Plane E F G H.

C O R. 2.

The given Plane E F G H may be made to ſerve the purpoſe of the ſeparate Directing Plane thus:

Having taken any Point i in E F, take another Point s, in the ſame Poſition with reſpect to i, as the Projecting Point in the Directing Plane hath with regard to the Eye; then st being drawn perpendicular to E F till it cut G H in t, si and ti will be the Directions ſought.

For the Diſtance between E F and G H being the ſame as that between N R and L M[b], the Triangle sit in the Picture is every way Similar and equal to the Triangle S I T in the Directing Plane.

C A S E 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane[c].

Fig. 121. N°. 7. Here, the Projecting Lines being ſuppoſed parallel to each other and to the Picture, they are parallel to their Images, and their Oblique Seats on the Plane of the Projection are parallel to the Picture, and conſequently to the Vaniſhing Line of that Plane[d]; wherefore if si be drawn, inclining the ſame way, and in the ſame Angle to E F, as the Projecting Lines are ſuppoſed to do to their Oblique Seats on the Plane E F G H, si will be the Direction of the Projecting Lines, and their Seats being parallel to E F, ti in this Caſe coincides with E F.

If then through the Seat A of the given Point a, $A\alpha$ be drawn parallel to E F, a Line $a\alpha$ drawn from a parallel to si, will cut $A\alpha$ in α the Projection deſired; $A\alpha$ being the Oblique Seat of the Projecting Line $a\alpha$ on the Plane E F G H. *Q. E. I.*

P R O B. II.

An Original Plane parallel to the Picture, and a Point with its Seat on that Plane, being given; thence to find the Projection of that Point on the given Plane, from a Projecting Point whoſe Seat on the ſame Plane is given.

C A S E 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 122. N°. 1. Let O be the Center of the Picture, S the Projecting Point, and T its Perpendicular Seat on a Plane parallel to the Picture; and let a be the given Point, and A its Perpendicular Seat on the ſame Plane.

Through the given Seats T and A draw T A till it be cut by Sa in α, and α will be the Projection deſired.

Or if s be the Projecting Point, and T its Perpendicular Seat on the parallel Plane; sa will cut T A in a, the Projection of a from the Point s.

Dem. For T A being the Perpendicular Seat of the Projecting Lines Sa and sa on the

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[a] Cor. 1. Def. 18. and Theor. 6. B. I.

[b] Cor. 3. Def. 18. B. I.

[c] Def. 4.

[d] Cor. 2. Caſe 1. Prop. 40. B. IV.

the parallel Plane, α and a are the Interſections of Sa and sa with that Plane, and conſequently the Projections ſought. *Q. E. I.*

SCHOL.

The Point S here repreſents a Projecting Point between the Eye and the Plane of the Projection, wherefore its Seat T falls between S and O the Vaniſhing Point of its Support ST: and the Point s is the Tranſprojected Image of a Projecting Point behind the Eye, and therefore falls on the contrary Side O from its Seat T, Os being the Tranſprojective Part of sT the Complement of the Image of its Support.

But the Projecting Point cannot lie any where between O and T, for then its Original would be behind the Plane of the Projection; and in Order to the forming any viſible Projection, it is neceſſary, that the Projecting Point and the Object ſhould be both on the ſame Side of the Plane of the Projection with the Eye.

Thus if the Projecting Point were ſuppoſed to be at q; the Point n where qa cuts TA, is not the Projection of a, but only marks the Point where the parallel Plane is cut by a Line drawn from q to a; or the perſpective Appearance of a on the parallel Plane as ſeen by an Eye at q.

COR.

If through O, a Line ef be drawn parallel to TA; ef will be the Vaniſhing Line of a Plane perpendicular to the Picture, paſſing through the Projecting Points S and s, and the Support Aa of the given Point.

For O being the Vaniſhing Point of aA, ST, and sT, and TA repreſenting a Line parallel to the Picture, ef drawn through O parallel to TA is the Vaniſhing Line of a Plane paſſing through aA and TA, and conſequently through S and s, perpendicular to the Picture[a].

CASE 2.

When the Projecting Point is at an Infinite Diſtance behind the Directing Plane.

Let O be the Center of the Picture, and S the Projecting Point, conſidered as the Tranſprojective Image of a Luminous Point at an infinite Diſtance behind the Eye; and let a be the given Point, and A its Seat on the Plane of the Projection. Fig. 122. No. 2.

Draw SO, and through A draw $A\alpha$ parallel to it, then Sa will cut $A\alpha$ in α, the Projection required.

Dem. For S being a Vaniſhing Point, SO is the Vaniſhing Line of a Plane perpendicular to the Picture, paſſing through S and the Support Aa of the given Point, the Interſection of which Plane with the Plane of the Projection, which is parallel to the Picture, is therefore parallel to SO; and conſequently $A\alpha$ parallel to SO, is the Seat of the Projecting Line Sa on that Plane, and α is therefore the Projection ſought. *Q. E. I.*

SCHOL.

If ef be the Vaniſhing Line of the Plane of the Horizon, the Point S muſt fall below it; it being neceſſary, that the Luminous Point, of which S is the Tranſprojective Image, ſhould be above the Horizon: if the Point S were above ef, it would then repreſent the direct Image of a Luminous Point at an infinite Diſtance before the Eye, in which Caſe the Projecting Point being behind the Plane of the Projection with reſpect to the Eye, no viſible Projection could be made; or if it could be ſuppoſed to be the Tranſprojected Image of any Luminous Point behind the Eye, the Original of that Point muſt then lie below the Horizon, from whence therefore no Projection could be formed.

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Let O be the Center of the Picture, a the propoſed Point, and A its Seat on the Plane of the Projection, the Diſtance of which from the Picture muſt be known, when the Seat of the Projecting Point on that Plane is not given, as it is here ſuppoſed not to be. Fig. 122. N°. 3.

Take the Point s in the ſame Situation with reſpect to O, as the Projecting Point is ſuppoſed to have with regard to the Eye in the Directing Plane, and draw sO, and having found d the Interſecting Point of the given Support Aa[b], draw sd; then through A draw $A\alpha$ parallel to sd, and from a draw $a\alpha$ parallel to sO, and their Interſection α will be the Projection deſired.

Dem. For here sO being by Conſtruction parallel to the Director of the Projecting Lines, sO is their Direction[c], and Od being the Indefinite Image of the Support Aa, it

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[a] Cor. 1. Theor. 15. B. I.

[b] Cor. Caſe 2. Prop. 45. B. IV.

[c] Def. 10. and Cor. 2. Caſe 3. Prob. 1.

it is parallel and equal to its Director [a]; wherefore if sOd be ſuppoſed to be the Directing Plane, O the Place of the Eye, and s the Projecting Point in that Plane; Od will repreſent the Director of the given Support Aa, and d its Directing Point; wherefore d will be a Point in the Directing Line of a Plane paſſing through Aa and the Projecting Point s, and s being alſo a Directing Point in the ſame Plane, sd will be the Directing Line of that Plane [b]; wherefore $A\alpha$ parallel to sd is the Interſection of that Plane with the Plane of the Projection, which is here ſuppoſed parallel to the Picture [c], and conſequently α, the Interſection of $A\alpha$ with $a\alpha$, drawn parallel to sO the Direction of the Projecting Lines, is the Projection required. *Q. E. I.*

COR. 1.

The Direction sO of the Projecting Lines continues the ſame in all Situations of the Point a; but ds, to which their Seats on the Plane of the Projection are to be made parallel, changes as oft as the Point A is varied; but while that continues, the Place of a in the Line Aa makes no Alteration in sd.

COR. 2.

If $A\alpha$ be produced till it cut sO in T, T will be the Seat of the Projecting Point on the Plane of the Projection; which Point therefore being known, a Line TO will give the Direction of the Projecting Lines, and their Seats on the Plane of the Projection muſt all paſs through T.

For ds, conſidered as a Line in the Picture, is the Interſecting Line of the Plane which paſſes through Aa and the Projecting Point, it paſſing through d the Interſecting Point of Aa, parallel to the Directing Line of that Plane; of which ef drawn through O parallel to sd, is the Vaniſhing Line; and AT being the Interſection of that Plane with the Plane of the Projection, and Os being the Image of a Line in that Plane perpendicular to the Picture, and tending to the Projecting Point, the Interſection T of Os with $A\alpha$ is therefore the Perpendicular Seat of the Projecting Point on the Original Plane.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

In this Caſe the Projecting Lines being parallel to the Picture, they are alſo parallel to the Plane of the Projection, ſo that the Projection of any Point on that Plane muſt be at an infinite Diſtance; however, the Indefinite Projection of the Support of any Point on that Plane is found in this manner.

Fig. 122. N°. 4. Let A be the Seat of the propoſed Point a on the Plane of the Projection.

Having found sO the Direction of the Projecting Lines, through A draw Am parallel to it, and Am will be the Indefinite Projection of the Support Aa of the given Point; but as the Projection of a ought to be determined by a Line drawn through a parallel to sO, that Line being then parallel to Am, can never meet it to determine the Projection.

Dem. For sO not only repreſents the Direction of the Projecting Lines, but is here alſo the Vaniſhing Line of a Plane paſſing through the Support Aa, and the Projecting Line of the Point a; the Interſection of which Plane with the Plane of the Projection muſt be Am parallel to sO. *Q. E. I.*

SCHOL.

The Original Plane being here ſuppoſed parallel to the Picture, and being conſidered without regard to any other Plane which cuts the Picture, the Perpendicular Seats of the Projecting Point and of the Point whoſe Projection is ſought on that Plane, are ſuppoſed to be given; except when the Projecting Point is infinitely diſtant, in which Caſe it can have no Seat on that Plane: but if any other Plane inclining to the Picture were concerned, then the Oblique Seats of thoſe Points on the Plane of the Projection might be uſed, and the Vaniſhing Point of their Oblique Supports being then in the Center of the Vaniſhing Line of the inclining Plane [d], that Center muſt in ſuch Caſe be uſed in all reſpects inſtead of O the Center of the Picture.

PROB. III.

An Original Plane not parallel to the Picture, and the Indefinite Image of a Line with its Seat on that Plane, being given; thence

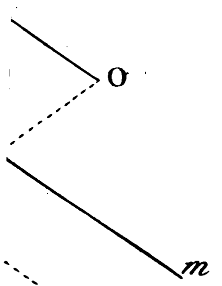
[a] Cor. 1. Def. 18. B. I.

[b] Cor. 2. Theor. 10. B. I.

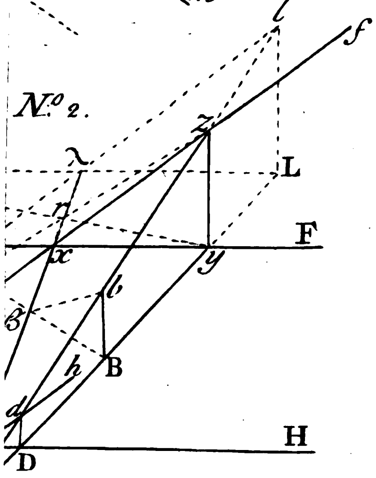
[c] 16 El. 11.

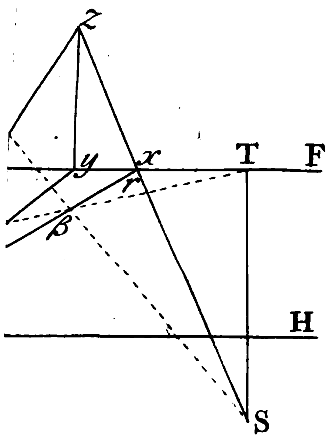
[d] Prop. 49. B. IV.

122. N°. 4.



N°. 2.





N°. 6.

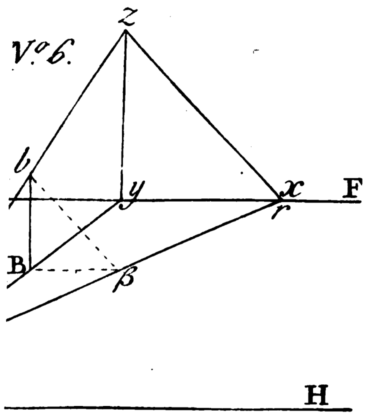


Fig. 123. N°. 1.

Fig. 123 N°. 3.

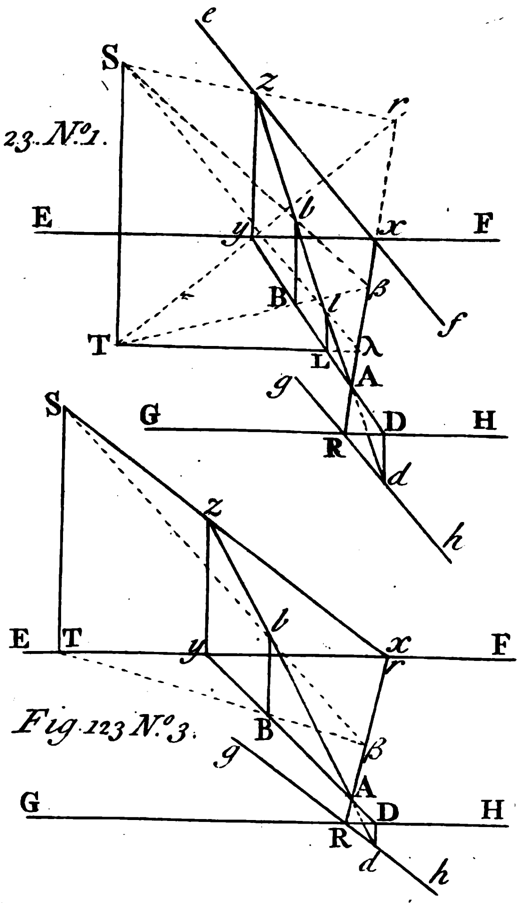


Fig. 123. N°. 5.

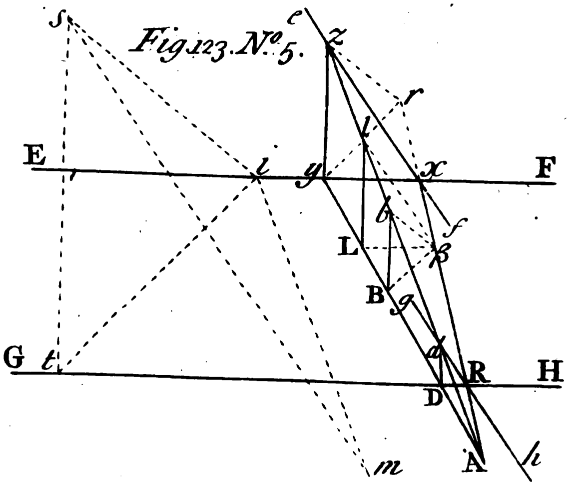
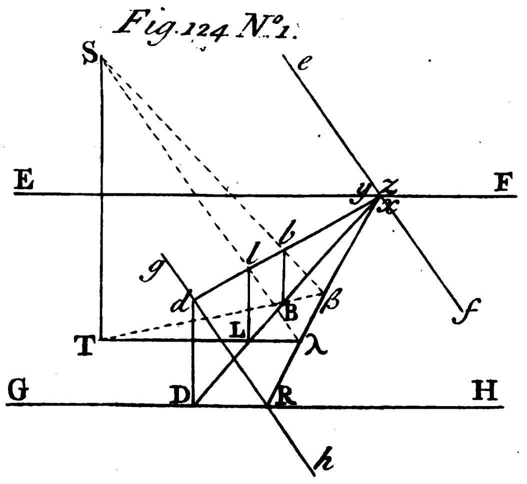


Fig. 124 N°. 1.



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to find the Projection of that Line on the given Plane from any given Projecting Point, and the Vaniſhing and Interſecting Lines of the Projecting Plane[a].

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Let EFGH be the Plane of the Projection, S the Projecting Point, and T its Seat on that Plane; and let dz be the given Line, and Dy its Seat on the ſame Plane. Fig. 123. N°. 1, 2.

Draw TL parallel to EF, cutting Dy in L, and Ll perpendicular to EF, cutting dz in l; and having drawn Sl, through z and d draw ef and gh parallel to Sl, cutting EF and GH in x and R; then Rx will be the Indefinite Projection of dz, and ef and gh the Vaniſhing and Interſecting Lines of the Projecting Plane.

Dem. For Ll being the Oblique Support of the Point l of the Line dz on the Plane EFGH, the Originals of ST, TL, and Ll, and conſequently of the Projecting Line Sl, are all in a Plane parallel to the Picture; wherefore Sl is parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane which paſſes through Sl and dz[b]; and therefore ef and gh, drawn through z and d the Vaniſhing and Interſecting Points of dz, parallel to Sl, are the Vaniſhing and Interſecting Lines of the Projecting Plane, and conſequently Rx the Interſection of this Plane with the Plane EFGH is the Projection of dz on that Plane. *Q. E. I.*

COR. 1.

If from T through y a Line Ty be drawn, till it be cut in r by Sz, drawn through S and z; r will be a Point in the Indefinite Projection of dz, and alſo in the Projections of all other Lines whatſoever which have z for their Vaniſhing Point.

Becauſe of the Vaniſhing Line yz, to which ST is parallel, Sz and Ty are in the ſame Plane, and therefore meet in r; but Ty is a Line in the Plane EFGH, and Sz is a Line in the Projecting Plane, therefore their Interſection r is a Point in Rx the common Interſection of thoſe two Planes; and y being the Vaniſhing Point of the Seats of all Lines whoſe Vaniſhing Point is z, theſe two Points, and alſo the Points S and T, remain the ſame, whichever of the Lines z be propoſed to be projected; wherefore the Point r alſo continues the ſame, and is therefore a Point in the Indefinite Projections of all Lines whoſe Vaniſhing Point is z, on the Plane EFGH from the Point S.

Here z may be conſidered as a Point of dz, having y for its Seat, and Sz may be taken as the Projecting Line of the Point z, and Ty as the Seat of that Projecting Line on the Plane EFGH, and conſequently r as the Projection of z on that Plane.

DEF. 11.

The Point r is called the *Focus* of the Projection Rx; and repreſents the Interſection of the Plane of the Projection, with a Line drawn from the Projecting Point, parallel to the Line whoſe Projection is ſought; the Originals of Sz and dz being parallel, they having the ſame Vaniſhing Point z.

COR. 2.

If Ty and Sz be parallel, the *Focus* r being then infinitely diſtant, its Original is at their common Directing Point; wherefore in this Caſe the Projection of dz, and of all other Lines which have z for their Vaniſhing Point, will be parallel to Ty or Sz[c].

CASE 2.

When the Projecting Point is at an infinite Diſtance before or behind the Directing Plane.

Here the Projecting Point S being a Vaniſhing Point, Sz is the Vaniſhing Line of the Projecting Plane, and gh drawn through d parallel to it, is the Interſecting Line, whence the Indefinite Projection Rx is found as before; and T the Seat of S being alſo a Vaniſhing Point in EF, Ty coincides with EF, wherefore r the *Focus* of the Projection coincides with x its Vaniſhing Point, as is ſufficiently evident by the Figures. *Q. E. I.* Fig. 123. N°. 3, 4.

COR.

If the Line Sz ſhould happen to be parallel to EF, then x the Vaniſhing Point of

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[a] Def. 8.

[b] Cor. 1. Theor. 15. B. I.

[c] Cor. 5. Theor. 12. B. I.

of the Projection being infinitely diſtant, the Projection muſt paſs through A parallel to EF, its Original being parallel to the Picture[a].

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Fig. 123. N°. 5. Having either on a ſeparate Directing Plane, or on the given Plane $EFGH$, drawn the Triangle sit, as already directed, and thereby found the Directions si and ti of the Projecting Lines and their Seats[b]; from i draw im parallel and equal to the given Line zd, making the Point i correſpond to the Point z of that Line, and draw sm; then through z and d draw ef and gb parallel to sm, and theſe will be the Vaniſhing and Interſecting Lines of the Projecting Plane, whence the Indefinite Projection Rx is found as before.

Dem. For im repreſenting the Director, and m the Directing Point of dz[c], sm repreſents the Directing Line of the Projecting Plane[d], wherefore ef and gb drawn through z and d parallel to sm, are the Vaniſhing and Interſecting Lines of that Plane. *Q. E. I.*

And here r the *Focus* of the Projection, is found by the Interſection of zr and yr, drawn from z and y parallel to the Directions si and ti.

COR.

The Vaniſhing and Interſecting Lines of the Projecting Plane may alſo be found without drawing im or sm; by finding the Projection β of any Point b of the propoſed Line dz[e], and from β drawing βL parallel to EF, cutting the Seat Dy in L, from whence Ll being drawn perpendicular to EF cutting dz in l, the Line $l\beta$ will be parallel to ef and gb, whence theſe Lines are determined.

For it is evident, that $l\beta$ is a Line in the Projecting Plane parallel to the Picture.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

Fig. 123. N°. 6. Here si the Direction of the Projecting Lines, is parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane; wherefore zx and gb drawn through z and d parallel to si, are the Vaniſhing and Interſecting Lines of that Plane.

Dem. For si is by Conſtruction parallel to the Projecting Lines, which are by Suppoſition parallel to the Picture[f]. *Q. E. I.*

And here, as in Caſe 2. the Vaniſhing Point x, and the *Focus* r of the Projection Rx, are the ſame.

GENERAL COROLLARY 1.

Fig. 124. N°. 1, 2, 3, 4, 5, 6. When the Original of the given Line dz is parallel to the Plane of the Projection, but not to the Picture, the Vaniſhing Points of its Seat and Projection, and alſo the *Focus* of that Projection all coincide with the Vaniſhing Point z of the given Line, and the Vaniſhing and Interſecting Lines of the Projecting Plane are found as before; as is abundantly clear by the Figures here referred to, which repreſent all the four Caſes of the Situation of the Projecting Point, and are marked with the ſame Letters as the former.

GENERAL COROLLARY 2.

Fig. 125. N°. 1, 2, 3, 4, 5. When the Original of the given Line Ab is parallel to the Picture, but not to the Plane of the Projection; inſtead of the Oblique Seat T of the Projecting Point, the Point t muſt be uſed, where St drawn parallel to Ab, cuts Tt drawn parallel to EF; for then $A\beta$ drawn from t through A the Interſection of Ab with the Plane of the Projection $EFGH$, will be the Indefinite Projection of Ab, and Lines drawn through the Vaniſhing and Interſecting Points of the Projection $A\beta$, parallel to Ab or St, will be the Vaniſhing and Interſecting Lines of the Projecting Plane.

For tA is the Interſection of the Plane $EFGH$ with the Projecting Plane which paſſes through St and Ab, which two Lines are parallel, as well as their Originals which are parallel to the Picture.

And here t is the parallel Seat of S on the Plane $EFGH$ with reſpect to the Line Ab, and is alſo the *Focus* of the Projection $A\beta$, when the Projecting Point is at a moderate Diſtance before or behind the Directing Plane[1]; and becomes the Vaniſhing Point, as well as the *Focus* of the Projection, when the Point S is infinitely diſtant[2].

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[a] Theor. 15. B I.

[b] Caſe 3. Prob. 1. and Cor. 2.

[c] Cor. 1. Def. 18. B. I.

[d] Caſe 3. Prob 2.

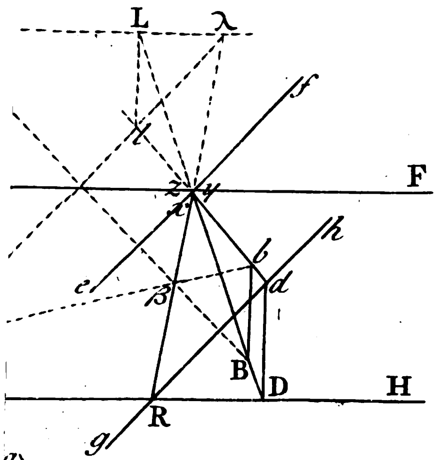
[e] Caſe 3. Prob. 1.

[f] Cor. 1. Theor. 15. B.I.

[1] Fig. 125. N°. 1, 2.

[2] Fig. 116. N°. 3, 4.

Plate 47. Book 5. Sect. 1.


124 N°.2.

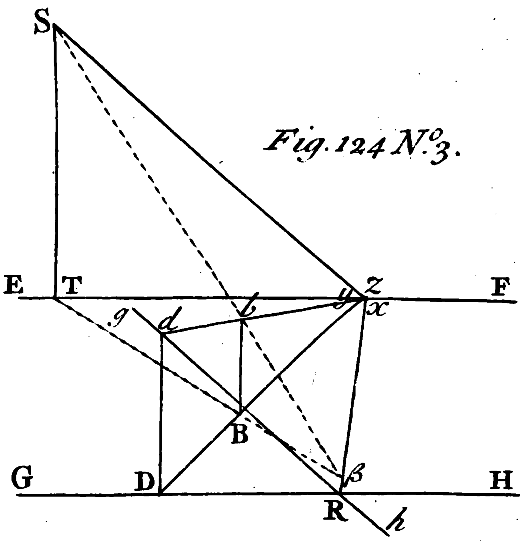


Fig. 124 N°.3.

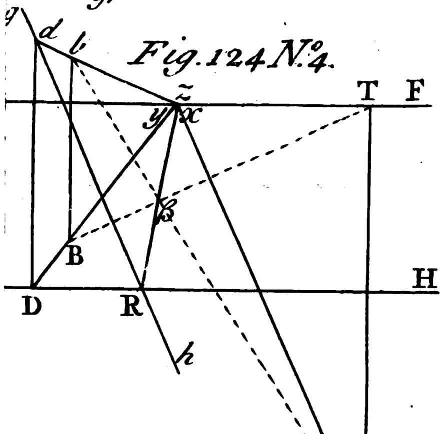


Fig. 124 N°.4.

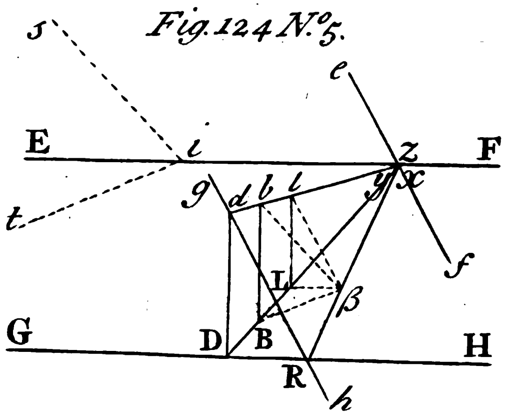
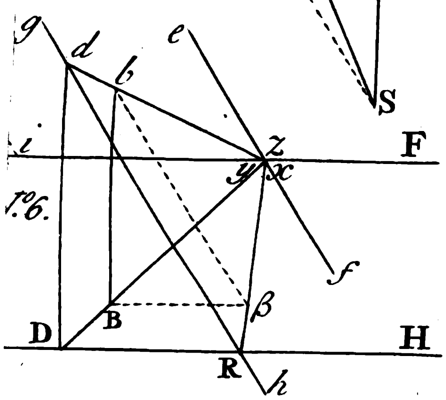


Fig. 124 N°.5.


N°.6.

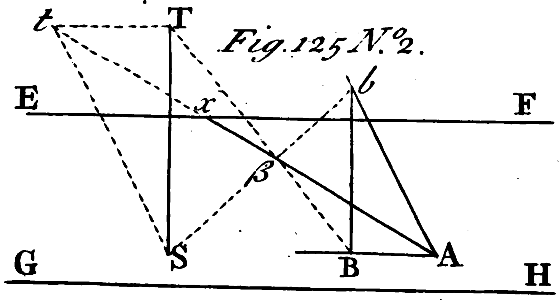


Fig. 125 N°.2.

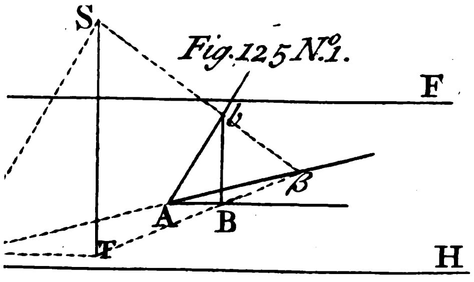


Fig. 125 N°.1.

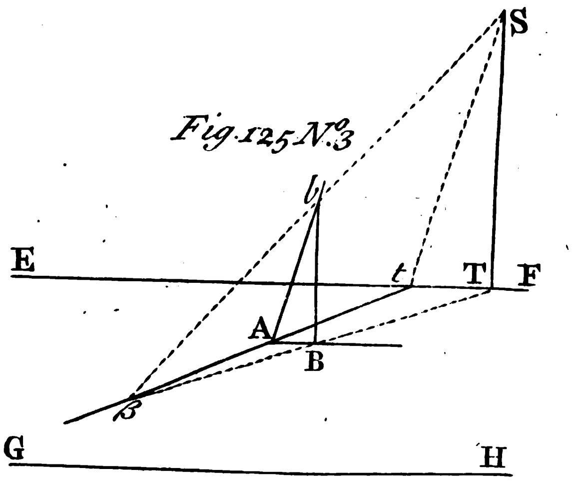


Fig. 125 N°.3.

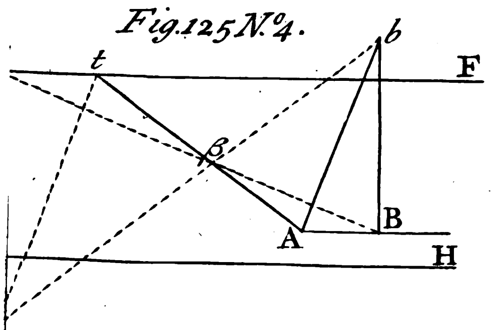


Fig. 125 N°.4.

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When the Projecting Point is at a moderate Diſtance in the Directing Plane[1], the Method is in effect the ſame; st drawn parallel to Ab giving ti the Direction of the Projection $A\beta$; and the *Focus* t of the Projection, being in this Caſe a Directing Point, the Projections of all Lines parallel to Ab will be parallel to the Direction ti; but the Direction si of the Projecting Lines continues the ſame whatever Line Ab parallel to the Picture be propoſed.

[1] Fig. 125. N°. 5.

And when the Projecting Point is at an infinite Diſtance in the Directing Plane[2], the Projection $A\beta$ is the ſame with AB the Oblique Seat of the propoſed Line on the Plane EFGH; and the *Focus* of that Projection being infinitely diſtant, the Projections not only of all Lines parallel to Ab, but of all other Lines whatſoever which are parallel to the Picture, will be parallel to $A\beta$, ſeeing their Projections and their Oblique Seats on the Plane EFGH are the ſame.

[2] Fig. 125. N°. 6.

If B the Oblique Seat of any Point b in the given Line be found, a Line drawn from T through B[3], or with the Direction Ti[4], will cut the Projection $A\beta$ in the ſame Point β where it is cut by $b\beta$ drawn through S, or with the Direction si; ſo that either way may be uſed as happens to be moſt convenient.

[3] Fig. 125. N°. 1, 2, 3, 4.

[4] Fig. 125. N°. 5.

GENERAL COROLLARY 3.

When the propoſed Line is parallel both to the Picture and to the Plane of the Projection, its Projection as well as its Seat on that Plane, are parallel to the Picture, and conſequently to EF; wherefore the Projection of any one Point of the propoſed Line being found, its intire Projection is thence determined, and the Vaniſhing and Interſecting Lines of the Projecting Plane may be found by the help of the Projection of the given Line, and the Support of the Projecting Point[a].

[a] Caſe 2. Prop. 46. B. IV.

PROB. IV.

An Original Plane parallel to the Picture, and the Indefinite Image of a Line, with its Seat on that Plane, being given; thence to find the Projection of that Line on the given Plane from any given Projecting Point, and the Vaniſhing and Interſecting Lines of the Projecting Plane.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Let O be the Center of the Picture, dz the given Line, and A its Interſection with the Plane of the Projection ſuppoſed parallel to the Picture; and let S be the Projecting Point, and T its Perpendicular Seat on that Plane.

Fig. 126. N°. 1, 2.

Draw zO, and from T draw Tr parallel to it, and having drawn Sz cutting Tr in r, draw rA, which will be the Projection deſired; and ef and gb drawn through z and d, the Vaniſhing and Interſecting Points of dz, parallel to rA, will be the Vaniſhing and Interſecting Lines of the Projecting Plane.

Dem. Becauſe of the Vaniſhing Line Oz, the Originals of SO and Sz which meet in S, are in the ſame Plane, and T being the Interſection of SO with the Plane of the Projection, Tr drawn parallel to Oz is the Interſection of that Plane with the Plane OzS[b], and r is therefore the Interſection of Sz with the Plane of the Projection; and becauſe of the Vaniſhing Point z, the Originals of Sz and dz are in the ſame Plane, and r being the Interſection of Sz, and A the Interſection of dz with the Plane of the Projection, rA is therefore the Interſection of that Plane, with the Plane which paſſes through Sz and dz; but this laſt is the Projecting Plane, rA is therefore the Projection of dz, and conſequently ef and gb drawn through z and d parallel to rA, are the Vaniſhing and Interſecting Lines of the Projecting Plane. *Q. E. I.*

[b] Caſe 4. Prop. 46. B. IV.

And here r is the *Focus* of the Projection rA[c].

[c] Def. 11.

In the firſt Figure, S is a Projecting Point before the Directing Plane, and in the ſecond Figure, it is a Projecting Point behind the Directing Plane[d].

[d] Schol. Caſe 1. Prob. 2.

CASE 2.

When the Projecting Point is at an infinite Diſtance behind the Directing Plane.

Here Sz is the Vaniſhing Line of the Projecting Plane, wherefore $A\beta$ drawn through A, the Interſection of the given Line dz with the Plane of the Projection,

Fig. 126. N°. 3.

parallel to Sz, is the Projection of dz, and gb drawn through d parallel to it, is the Interſecting Line of the Projecting Plane. *Q. E. I.*

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Fig. 126. N°. 4. Let O be the Center of the Picture, dz the given Line, and A its Interſection with the Plane of the Projection; and let T be the Seat of the Projecting Point on that Plane.

Draw zO and Tr parallel to it, and having drawn TO, draw zr parallel to it, cutting Tr in r, and rA will be the Projection required, whence ef and gb are found as before.

Dem. For TO is the Direction of the Projecting Lines[a], wherefore zr repreſents a Projecting Line having z for its Vaniſhing Point, and conſequently r, its Interſection with the Plane of the Projection[b], is the *Focus* of the Projection of dz, whence every thing elſe is found as before. *Q. E. I.*

[a] Cor. 2. Caſe 3. Prob. 2.

[b] Caſe 1.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

Fig. 126. N°. 5. Here, the Direction sO of the Projecting Lines being found, ef drawn through z parallel to sO, is the Vaniſhing Line of the Projecting Plane, whence the Projection Am and the Interſecting Line gb are determined. *Q. E. I.*

GENERAL COROLLARY 1.

If the propoſed Line be parallel to the Picture, it will alſo be parallel to the Plane of the Projection, and conſequently both to its Seat and its Projection on that Plane[c]; wherefore in the three firſt Caſes, if the Projection of any one Point of the given Line be found[d], its intire Projection is thence determined; but in the fourth Caſe there can be no Projection at all[e].

[c] 16 El. 11.

[d] Prob. 2.

[e] Caſe 4. Prob. 2.

Fig. 126. N°. 6. Thus let O be the Center of the Picture, bm the given Line, and AM its Seat on the Plane of the Projection; and let S be the Projecting Point, and T its Seat on that Plane.

Having found bA the Perpendicular Support of any Point b of the given Line on the Plane of the Projection, draw TA and Sb Interſecting in β the Projection of b[f], through which $\beta\mu$ being drawn parallel to bm, it will be the Projection required.

[f] Caſe 1. Prob. 2.

The Vaniſhing and Interſecting Lines of the Projecting Plane may be found in this manner:

Produce bA to its Interſecting Point D[g], and through O and D draw Oe and Dg parallel to TA, cutting Sb in e and g, through which draw ef and gb parallel to $\beta\mu$, and theſe will be the Vaniſhing and Interſecting Lines of the Projecting Plane.

[g] Cor. Caſe 2. Prop. 45. B. IV.

Dem. Becauſe of the Vaniſhing Point O, the Originals of SO and DO are in a Plane, in which the Projecting Line Sb alſo lies, and TA being the Interſection of this Plane with the Plane of the Projection, Oe and Dg parallel to TA are the Vaniſhing and Interſecting Lines of that Plane[h]; wherefore e and g are the Vaniſhing and Interſecting Points of the Projecting Line Sb, and conſequently Points in the Vaniſhing and Interſecting Lines of the Projecting Plane; and therefore ef and gb drawn through e and g parallel to $\beta\mu$ are the Vaniſhing and Interſecting Lines of the Projecting Plane. *Q. E. I.*

[h] Cor. 1. Theor. 15. B. I.

Fig. 126. N°. 7. When the Projecting Point S is at an infinite Diſtance behind the Directing Plane, Sf parallel to $\beta\mu$ is the Vaniſhing Line of the Projecting Plane, and the Interſecting Line gb of that Plane paſſes through g the Interſecting Point of the Projecting Line Sb as before.

Fig. 126. N°. 6. And when the Projecting Point is at a moderate Diſtance in the Directing Plane, the Vaniſhing and Interſecting Points e and g of the Projecting Line $b\beta$ are found as in the firſt Caſe; there being no difference whether $b\beta$ proceed from a real Projecting Point S, or whether it be only parallel to the Direction of the Projecting Lines.

GENERAL COROLLARY 2.

Fig. 123, 124, 125. When the Indefinite Projection of any Line is once found, the Projection β of any Point of that Line is determined, either by drawing a Line from S through b, or from T through its Seat B, or parallel to the Directions si or ti, which will both cut the Indefinite Projection in the ſame Point β; or the Projection β being given, the ſame Lines determine b or B, the Point projected or its Seat.

Plate 48 Book 5. Sect. 1.

7.125. N°5.

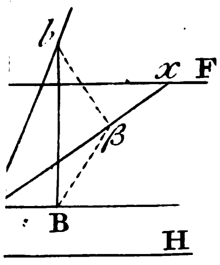
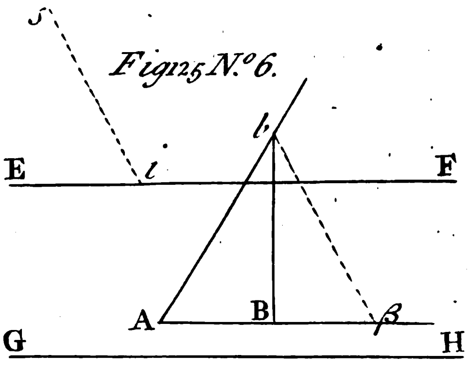


Fig125 N°6.



:N°1.

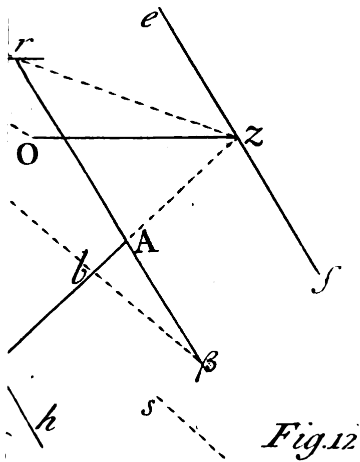
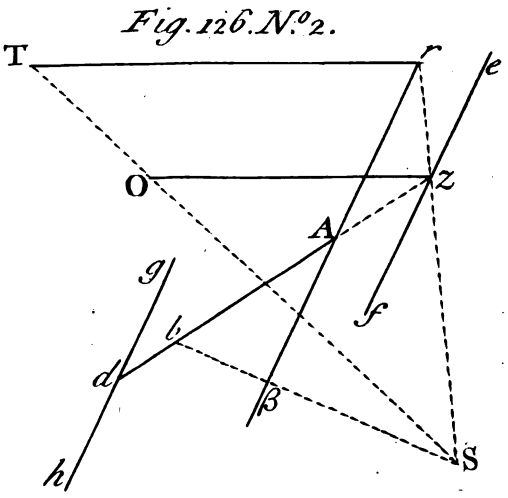


Fig. 126. N°2.



g.126. N°3.

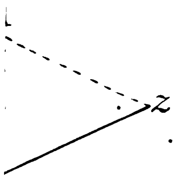


Fig.126. N°4

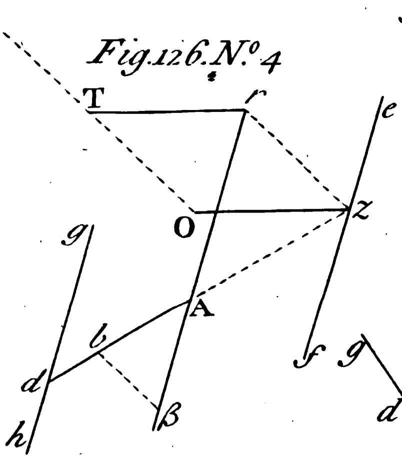
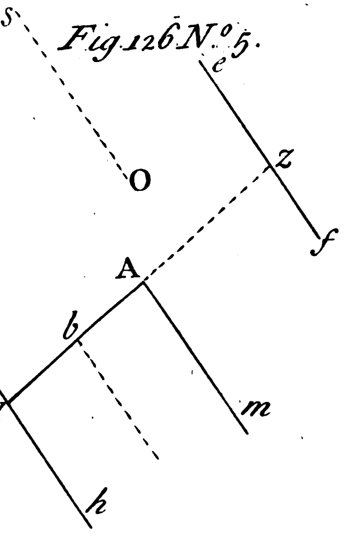


Fig.126 N°5.



6.

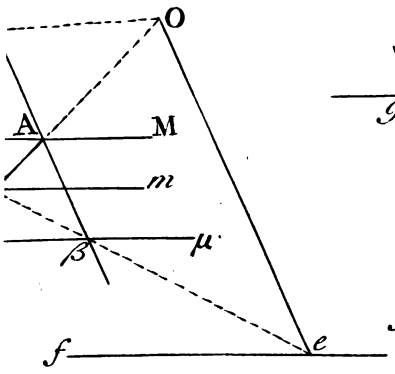
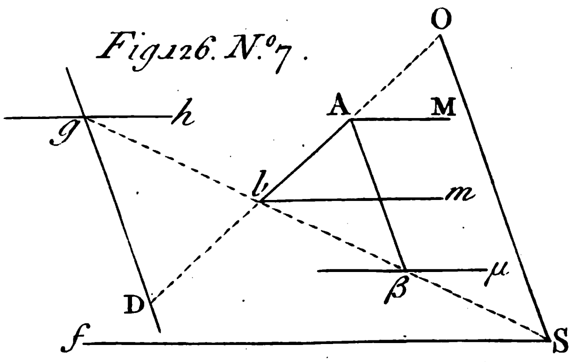


Fig126. N°7.



J. Mynde sc.

GENERAL COROLLARY 3.

When the Vaniſhing and Interſecting Lines of the Projecting Plane are found, theſe give the Projections of the propoſed Line on all Planes whatſoever; ſeeing there can but one Plane paſs through the given Line and the Projecting Point, which therefore continues the ſame, whatever be the Plane of the Projection; and the Interſection of the Projecting Plane with the Plane of the Projection, is always the Projection of the given Line; for finding of which Interſections in all different Poſitions of the Planes, ſufficient Rules have been already given [a]. [a] Prop. 46. B. IV.

SCHOL.

Although in the two laſt Problems, the Projection of a Line, and the Vaniſhing and Interſecting Lines of its Projecting Plane, are determined chiefly by the help of the Vaniſhing and Interſecting Points of the propoſed Line; yet if theſe Problems be compared with what was ſhewn at Prop. XL. Book IV. it will appear, that if any two Points of Relation of a Line to the Plane of the Projection, or if any two Points of the Projection of that Line be given, every thing elſe may be thence found; that Propoſition being in effect but a particular Caſe of theſe: for the Oblique Seat of a Line on a given Plane not parallel to the Picture, is the ſame with its Projection on that Plane from a Projecting Point at an infinite Diſtance in the Directing Plane, when the Direction of the Projecting Lines is perpendicular to the Vaniſhing Line; and the Perpendicular Seat of a Line on ſuch a Plane, is the ſame with its Projection on that Plane from a Projecting Point at an infinite Diſtance before or behind the Directing Plane, when that Point is the ſame with the Vaniſhing Point of Perpendiculars to the propoſed Plane; and the Perpendicular Seat of a Line on a Plane parallel to the Picture, is the ſame with its Projection on that Plane from a Point at an infinite Diſtance behind the Directing Plane, when the Tranſprojective Image of that Point falls in the Center of the Picture.

PROB. V.

A Triangle with it Seat on a Plane being given; thence to find the Projection of the Triangle on that Plane from any given Projecting Point.

Fig. 127. N°. 1, 2, 3, 4. Let EFGH be the Plane of the Projection, *abc* the given Triangle, and A, B, and C, the Seats of its angular Points on that Plane.

By the help of the given Supports *a*A, *b*B, and *c*C, find the Projections, α, β, γ, of the Points *a*, *b*, and *c* [b], which will give αβγ the Projection deſired. *Q. E. I.* [b] Prot. 1.

In Fig. N°. 1. the Projecting Point S is at a moderate Diſtance before the Directing Plane, its Seat being T on that Plane: in Fig. N°. 2. the Projecting Point S is infinitely diſtant behind the Directing Plane, and its Seat is T in the Vaniſhing Line EF: in Fig. N°. 3. the Projecting Point is at a moderate Diſtance in the Directing Plane, *si* and *ti* being the Directions of the Projecting Lines and their Seats: and in Fig. N°. 4. the Projecting Point is infinitely diſtant in the Directing Plane, the Direction of the Projecting Lines is *si*, and the Direction of their Seats is parallel to EF.

SCHOL.

By this general Method, the Projections of any Figures, or of any ſolid Bodies, on a given Plane may be found, the Seats of their angular Points being given; and which for the moſt part may be more conveniently done thus, than by finding the Projections of the Sides of the Figures, eſpecially when the Sides are many, or when they lie in different Planes, as thoſe of all ſolid Bodies do.

PROB. VI.

Any two Planes whoſe Vaniſhing Lines interſect, and a Line in one of them, being given; thence to find the Projection of that Line on the other Plane, from a Projecting Point whoſe Seat on this laſt Plane is given.

The Plane in which the given Line lies, ſhall be called the Original Plane, *and the other* the Plane of the Projection.

CASE

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 128. N°. 1, 2, 3, 4. Let $efgb$ be the Original Plane, in which dz is the given Line; and let EFGH be the Plane of the Projection, S the Projecting Point, T its Oblique Seat on that Plane, and t its Parallel Seat with reſpect to the Plane $efgb$[a].

[a] Def. 16. B. IV.

METHOD 1.

Fig 128. N°. 1, 2. Through any two Points b and c of the given Line draw Parallels to ef, cutting Dy the Interſection of the given Planes in B and C, and draw tB, tC, till they be cut by Sb and Sc in β and γ, and a Line drawn through β and γ will be the Projection deſired.

For the Originals of St, Bb, and Cc, are parallel to each other, and to the Picture[b]. *Q. E. I.*

[b] Gen. Cor. 2. Prob. 3.

COR.

After this manner the Projection of any Point b or c in the Plane $efgb$, or of any Line bB or Cc in that Plane, parallel to ef and conſequently to the Picture, may be found on the Plane EFGH; Cγ and Bβ being the Projections of Cc and Bb on that Plane, and both paſſing through t.

METHOD 2.

Fig. 128. N°. 3. From T or t draw TL parallel to EF, cutting Dy in L, and draw LT parallel to ef, till it be cut in T by ST drawn parallel to EF; then through T, and any two Points b and c of the given Line, draw Tb, Tc, cutting Dy in B and C, through which draw Bβ, Cγ, parallel to EF, till they be cut in β and γ by Sb and Sc, and $\beta\gamma$ will be the Projection ſought.

For the Originals of ST and Bβ being parallel to the Picture and to each other, they are in the ſame Plane with TB and Sb[c]; wherefore Sb cuts Bβ in β the Projection of b; after the ſame manner it is proved, that γ is the Projection of c, Cγ being parallel to ST. *Q. E. I.*

[c] 7 El. 11.

COR.

Here, the Point T is the Parallel Seat of S on the Plane $efgb$ with reſpect to the Plane of the Projection EFGH; and the Projections on that Plane, of all Lines in the Plane $efgb$ which paſs through T, are parallel to EF, and conſequently to the Picture, they being all parallel to ST.

METHOD 3.

Fig. 128. N°. 1, 2. Having drawn tL as before, draw Ll parallel to ef, cutting dz in l, and draw Sl; then through z and d the Vaniſhing and Interſecting Points of dz, draw zx, dR, parallel to Sl, cutting EF and GH in x and R, and Rx will be the Indefinite Projection of dz, and zx and dR will be the Vaniſhing and Interſecting Lines of the Projecting Plane.

For the Originals of St, Ll, and Sl are in a Plane parallel to the Picture[d]. *Q. E. I.*

[d] Caſe 1. Prob. 3.

METHOD 4.

Through t and y draw ty till it be cut in r by a Line Sz, and r will be the *Focus* of the Projection; by which and the Interſection A of dz with the Plane EFGH, the Indefinite Projection Rx is determined.

The Demonſtration of this is the ſame with that of Cor. 1. Caſe 1. Prob. III. there being no Difference whether zy be perpendicular or inclining to EF, ſo long as St is parallel to it. *Q. E. I.*

COR. 1.

The Line ty is the Interſection of the Plane EFGH with a Plane paſſing through S parallel to the Plane $efgb$; and is the Place of the *Foci* of the Projections of all Lines whatſoever in the Plane $efgb$ on the Plane EFGH.

For ef is the Vaniſhing Line of the Plane which paſſes through St and ty[e], and ty is the Interſection of that Plane with the Plane EFGH; and as the Line ty continues the ſame wherever the Vaniſhing Point z of the propoſed Line dz in the Plane $efgb$ falls, the Interſection of ty with a Line drawn from S through any Va-

[e] Cor. 1. Theor. 15. B. I.

127 N°1.

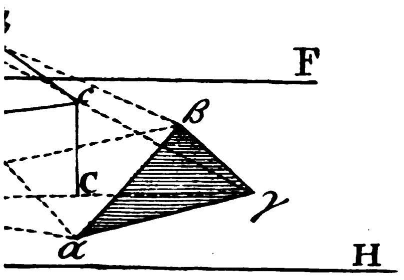
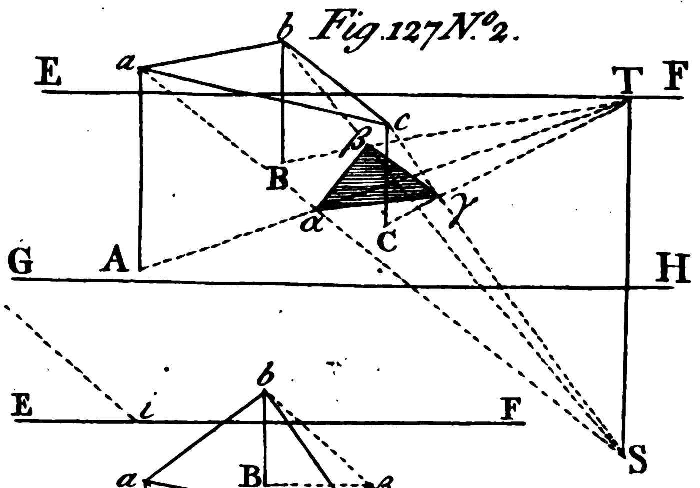


Fig. 127 N°2.



27 N°3.

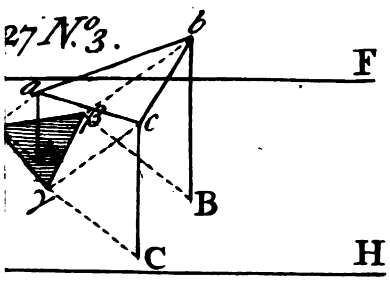
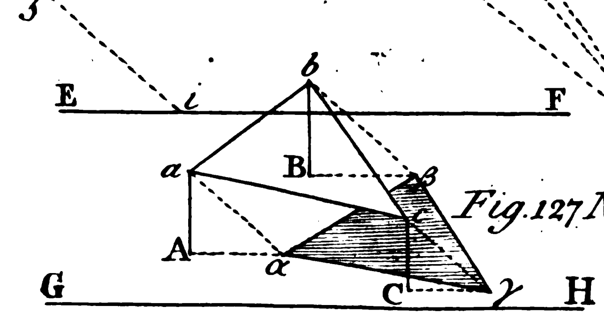


Fig. 127 N°4.



28 N°1.

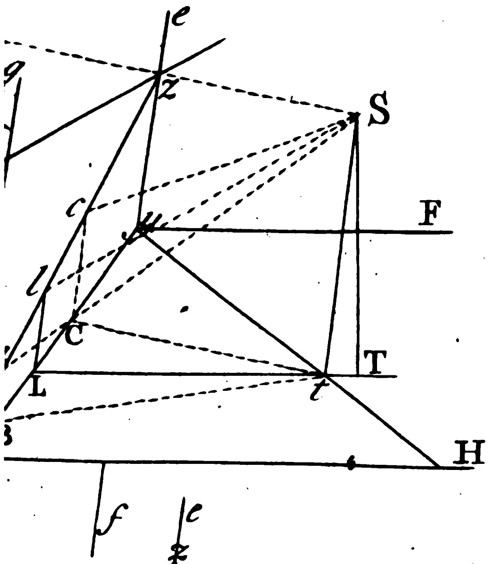


Fig. 128 N°2.

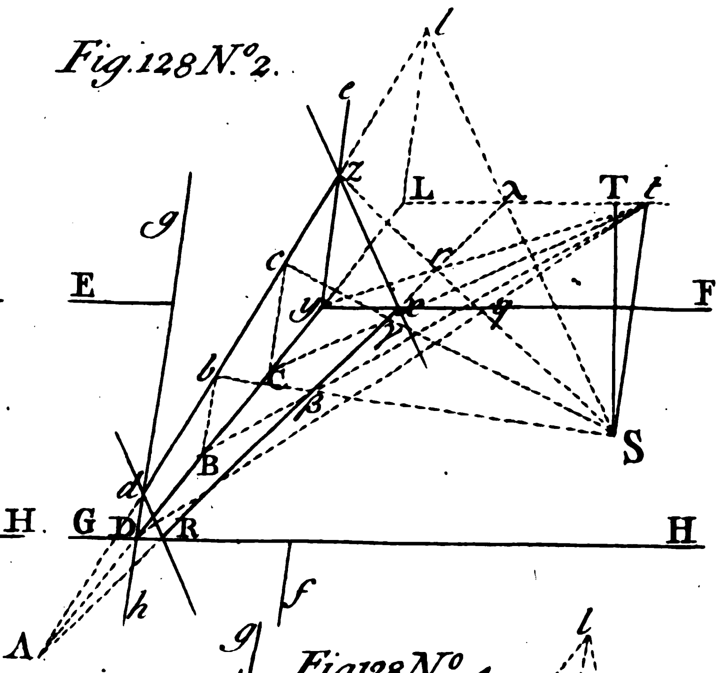


Fig 128 N°3.

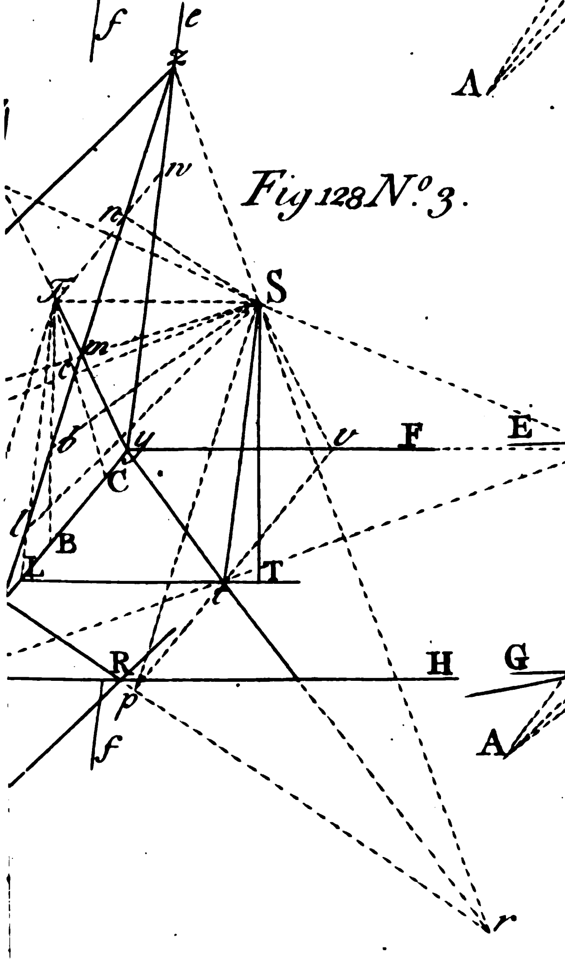
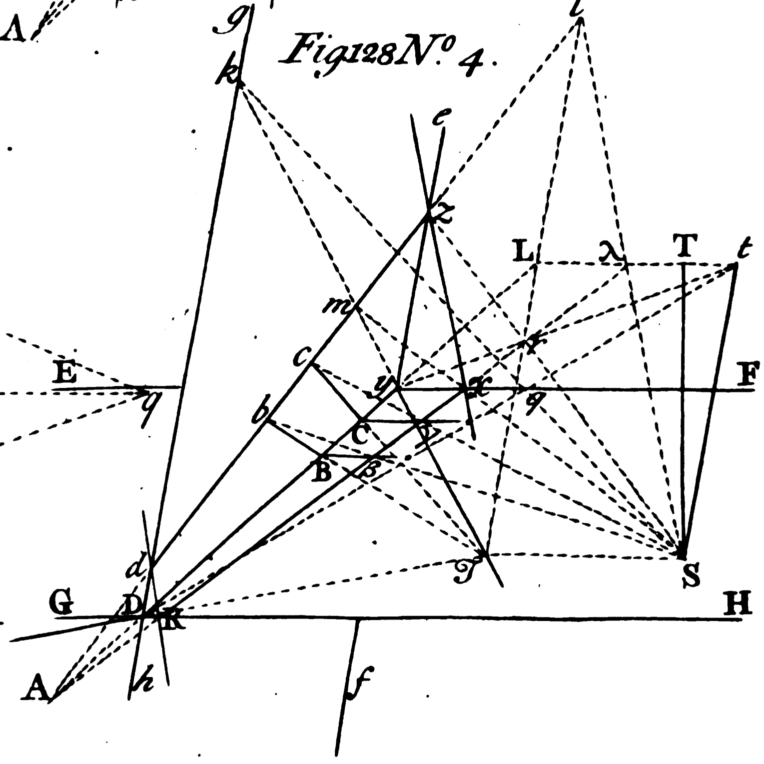


Fig 128 N°4.



J. Mynde f.

niſhing Point in ef is the *Focus* of the Projections of all Lines whatſoever to which that Vaniſhing Point belongs[a].

C O R. 2.

The Line ty is the imaginary Projection of the Vaniſhing Line ef of the Original Plane; ſeeing the imaginary Projection of every Vaniſhing Point of ef is in ty.

C O R. 3.

Fig. 128. N°. 1. If the Projection Aγ, of any determinate Part Ac of the Line dz be given, and Ac be anywiſe divided in the Points b and l, the Projection Aγ will be divided in β and λ the Projections of b and l in ſuch manner, that it ſhall repreſent a Line divided in the ſame Proportion with the Original of Ac, taking r the *Focus* of the Projection Aγ for its Vaniſhing Point.

For if Sr be taken as a Vaniſhing Line paſſing through the Vaniſhing Point z of the Original Line and the *Focus* r of its Projection, it is evident S may be alſo taken as a Vaniſhing Point in that Vaniſhing Line; and on this Suppoſition, Ar, Az, and the Projecting Lines from S, lying all in this imaginary Plane, the Projecting Lines from S will repreſent Parallels in that Plane, which will therefore divide the Originals of Ac and Aγ, conſidered as Lines in that Plane, proportionally[b].

M E T H O D 5.

Fig. 128. N°. 3, 4. Through T draw Ty cutting the given Line dz in m, and draw Sm, which will cut EF in x the Vaniſhing Point of the Projection, by which and any other Point of the Projection the whole of it is determined.

For Ty is the Interſection of the Plane $efgh$ with a Plane paſſing through ST, parallel to the Plane of the Projection EFGH, and Sm being a Line in that Plane, its Original is parallel to the Plane EFGH, and m being the Interſection of dz with Ty, Sm is the Projecting Line of the Point m of the Line dz; but the Original of Sm being parallel to the Plane EFGH, Sm can only meet that Plane in its Vaniſhing Point x[c], therefore x is the imaginary Projection of the Point m, and conſequently the Vaniſhing Point of the Projection of dz. *Q. E. I.*

C O R. 1.

The Interſection of Ty with any Line whatſoever in the Plane $efgh$, gives a Point correſponding to m, through which and the Projecting Point S a Line being drawn, it will cut EF in the Vaniſhing Point of the Projection of the propoſed Line on the Plane EFGH.

For wherever the Point m falls in Ty, the Original of Sm will be parallel to the Plane EFGH, and conſequently its Vaniſhing Point will be the imaginary Projection of m.

C O R. 2.

The imaginary Projection of the Line Ty coincides with EF the Vaniſhing Line of the Plane EFGH; ſeeing the imaginary Projection of every Point m in Ty is a Vaniſhing Point in EF.

C O R. 3.

Fig. 128. N°. 3, 4. If a Line TD be drawn, its Projection will coincide with GH the Interſecting Line of the Plane EFGH[d]; and that Part of the Plane $efgh$ which is bounded by TD and Dy, is therefore the whole of that Plane which can be projected on the Perſpective Part of the Plane EFGH from the Point S.

C O R. 4.

If from t a Line tD be drawn, cutting EF in q, Dq will be the Indefinite Projection of the Interſecting Line Dg of the Original Plane[e]; and that part of the Plane EFGH which is bounded by tD and Dy, is therefore the whole of that Plane on which the Projection of any Point in the Perſpective Part of the Plane $efgh$ can fall from the Point S: and if Ty be produced till it cut Dg in k, the imaginary Projection of k will fall in q the Vaniſhing Point of the Projection Dq[f]; and conſequently that part of the Plane $efgh$ which is bounded by Dk and kq, is the whole of the Perſpective Part of the Plane $efgh$ which can be projected on the Plane EFGH from the Point S.

M E T H O D 6.

Fig. 128. N°. 3. Through t draw tv in the Plane EFGH parallel to Dy, and from S draw Sp parallel

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[a] Cor. 1. Caſe 1. Prob. 3.

[b] Gen. Cor. Prob. 15. B. II.

[c] Cor. 3. Theor. 10. B. I.

[d] Cor. Meth. 2.

[e] Cor. Meth. 1.

[f] Cor. 1.

parallel to dz, cutting tv in p, and p will be a Point in the Projection required; by which and any other Point of the Projection the whole may be found.

For tv is the Interſection of the Plane EFGH with a Plane paſſing through St and tv, and the Original of St being parallel to the Picture and to ef, the Vaniſhing Line of the Plane Stv muſt paſs through v parallel to ef[a]; and tv and Dy being parallel, the Interſection of the Planes Stv and $efgb$ is their common Directing Line[b]; but Sp being a Line in the Plane Stv parallel to dz in the Plane $efgb$, theſe two muſt meet in their common Directing Point, wherefore Sp is the Projecting Line of the Directing Point of dz; and conſequently p the Interſection of Sp with the Plane EFGH, is the imaginary Projection of that Point, or rather its Perſpective Appearance on the Plane EFGH, as ſeen by an Eye at S; and therefore p is a Point in the Projection of dz. *Q. E. I.*

COR. 1.

The Line pv is the Place of the Projections of the Directing Points of all Lines whatſoever in the Plane $efgb$ on the Plane EFGH, or the Projection of the Directing Line of the Plane $efgb$.

For whatever Line in the Plane $efgb$ be propoſed, a Line drawn from S parallel to it, will cut pv in the Projection of the Directing Point of the propoſed Line.

COR. 2.

All Lines in the Plane $efgb$ whoſe Projections paſs through the ſame Point p in the Line tv, are parallel, thoſe Lines being all parallel to Sp.

COR. 3.

If the Projection of any Line on the Plane EFGH be parallel to pv, the Line in the Plane $efgb$ which produces that Projection muſt be parallel to Dy.

For the given Projection being parallel to pv, their Interſection p is infinitely diſtant; wherefore a Line drawn from S to that Interſection, muſt be parallel to pv, and being parallel to the Line which produces the Projection, that Line is therefore alſo parallel to pv, and conſequently to Dy.

METHOD 7.

Fig. 128. N°. 3.

Through T draw Tw in the Plane $efgb$ parallel to Dy, cutting the given Line dz in n, and draw Sn; then through any Point A, β or γ of the Projection ſought, draw Rx parallel to Sn, and Rx will be the Indefinite Projection deſired.

For Tw is the Interſection of the Plane $efgb$ with a Plane paſſing through ST and Tw, and the Original of ST being parallel to the Picture and to EF, the Vaniſhing Line of the Plane STw muſt paſs through w parallel to EF[c], and Tw and Dy being parallel, the Interſection of the Planes STw and EFGH is their common Directing Line[d]; but n being the Interſection of dz with Tn, the Projecting Line Sn is a Line in the Plane STw, which therefore can only cut the Plane EFGH in its Directing Point, which Interſection being the Projection of n, is therefore alſo the Directing Point of the Projection of dz, and conſequently the Image of that Projection is parallel to Sn[e]. *Q. E. I.*

COR. 1.

The imaginary Projection of Tw coincides with the Directing Line of the Plane EFGH; ſeeing the imaginary Projection of every Point n in the Line Tw is a Point in the Directing Line of the Plane EFGH; ſo that Tw hath no real Projection whatſoever, but only an imaginary, or rather impoſſible Projection, the Directing Line having no Image.

COR. 2.

All Lines in the Plane $efgb$ which paſs through the ſame Point n in the Line Tw, other than Tw itſelf, have parallel Projections; ſeeing thoſe Projections are all parallel to the ſame Line Sn.

COR. 3.

If any Line in the Plane $efgb$ be parallel to Tw, its Projection on the Plane EFGH will be parallel to Dy.

For a Line drawn from S parallel to the propoſed Line, will be alſo parallel to Tw, and conſequently to pv and Dy; and that Line being parallel to the Projection of the propoſed

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[a] Cor. 1. Theor. 15. B. I.

[b] Cor. Caſe 2. Prop. 46. B. IV.

[c] Cor. 1. Theor. 15. B. I.

[d] Cor. Caſe 2. Prop. 46. B. IV.

[e] Theor. 6. B. I.

propoſed Line, that Projection is therefore alſo parallel to D*y*.

This is the Converſe of Cor. 3. *Meth.* 6.

CASE 2.

When the Projecting Point is at an Infinite Diſtance before or behind the Directing Plane.

Let *efgh* be the Original Plane, in which *dz* is the given Line, and let EFGH be the Plane of the Projection, S the Projecting Point, T its Oblique Seat on the Vaniſhing Line EF, and *t* its Parallel Seat on that Line with reſpect to the Plane *efgh*[a]. Fig. 128. N°. 5, 6.

METHOD 1.

Through any two Points *b* and *c* of the given Line draw Parallels to *ef*, cutting D*y* in B and C, and the Interſections β and γ of *t* B, *t* C, with S*b*, S*c*, will give $\beta\gamma$ the Projection deſired. *Q. E. I.*

This is the ſame with Meth. 1. Caſe 1.

METHOD 2.

Draw S*T* parallel to EF, cutting *ef* in *T*, and from *T* through *b* and *c* draw *Tb*, *Tc*, cutting D*y* in *B* and *C*, through which draw *B*β, *C*γ parallel to EF, which will cut S*b* and S*c* in the ſame two Points β and γ as before. *Q. E. I.*

This is the ſame with Meth. 2. Caſe 1. and the Point *T* is the Parallel Seat of S on the Vaniſhing Line *ef* with reſpect to the Plane EFGH.

METHOD 3.

Through S and *z* draw S*z*, cutting EF in *x*, and through *d* draw *d*R parallel to it, cutting GH in R; and R*x* will be the Projection required, and S*z* and R*d* the Vaniſhing and Interſecting Lines of the Projecting Plane.

This is the ſame with the third, fourth, and fifth Methods Caſe 1. the Points marked L and *l* in the Figures of that Caſe, being here the ſame with *y* and *z*[b]; the *Focus r* coinciding with *x* the Vaniſhing Point of the Projection[c]; and the Point marked *m* coinciding with *z* the Vaniſhing Point of the propoſed Line[d]. *Q. E. I.*

METHOD 4.

Through *t* draw *tp* parallel to D*y*, and S*p* parallel to *dz*, and the Interſection *p* of *tp* and S*p* will be a Point in the Projection R*x*. Fig. 128. N°. 5.

This is the ſame with Meth. 6. Caſe 1. and needs no other Demonſtration. *Q. E. I.*

METHOD 5.

Through *T* draw *Tn* parallel to D*y*, cutting *dz* in *n*, and S*n* will be parallel to the Projection R*x*. Fig. 128. N°. 5.

This correſponds to Meth. 7. Caſe 1. the Point there marked *w* coinciding with *T*, and the Demonſtration is much the ſame, S*T* being here the Vaniſhing Line of the Plane S*Tn*, which has the ſame Directing Line with the Plane EFGH. *Q. E. I.*

In Fig. No. 6. the Lines Tn and tp are not drawn, their Interſections with S*n* *and* S*p being out of reach.*

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Let *efgh* and EFGH be the two given Planes, and *dz* the given Line in the Plane *efgh*. Fig. 128. N°. 7.

Any where a-part draw the Directing Planes *NRLM*, NRLM, of the given Planes, inclining to each other the ſame way, and in the ſame Angle as the Original Planes *&c*[e]; and let I be the Place of the Eye, and S the Place of the Projecting Point in the Directing Plane. Fig. 128. N°. 8.

METHOD 1.

From S draw S*t* parallel to *LM*, cutting LM in *t* the Parallel Seat of S on the Directing Line LM of the Plane EFGH with reſpect to the Plane *efgh*, and having drawn the Directors SI and *t*I of the Projecting Lines and their Parallel Seats on the Plane EFGH, transfer their Directions to *si* and *ti* in the Picture, meeting at any Point *i* in the Vaniſhing Line EF; then through any two Points *b* and *c* of the given Line *dz*, draw Parallels to *ef*, cutting D*y* in B and C, and draw Bβ, Cγ, parallel to the Fig. 128. N°. 8. Fig. 128. N°. 7.

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[a] Gen. Cor. 2. Prob. 3.

[b] Meth. 3.

[c] Meth. 4.

[d] Meth. 5.

[e] Caſe 3. Prob. 1.

the Direction ti, and $b\beta$ and $c\gamma$ parallel to the Direction si, which by their Interſections β ond γ will give $\beta\gamma$ the Projection deſired.

For the Originals of bB and cC are Lines parallel to the Picture and to St[a]. *Q. E. I.*

This Method correſponds to Meth. 1. Caſe 1.

METHOD 2.

Fig. 128. N°. 8. From S draw ST parallel to LM, cutting LM in T, and draw TI; then T is the Parallel Seat of S on the Directing Line of the Plane $efgh$ with reſpect to the Plane EFGH, and TI is the Director of the Seats of the Projecting Lines on the Plane $efgh$; having therefore transferred the Direction of TI to Ti in the Picture, through Fig. 128. N°. 7. any two Points b and c of the given Line, draw Parallels to Ti, cutting Dy in B and C, from whence draw Parallels to EF, which will cut $b\beta$ and $c\gamma$, drawn parallel to the Direction si, in the ſame two Points β and γ as before.

For the Originals of $B\beta$ and $C\gamma$ are parallel to ST. *Q. E. I.*

This correſponds to Meth. 2. Caſe 1.

METHOD 3.

Fig. 128. N°. 8. Fig. 128. N°. 7. From I draw Ip the Director of the given Line dz, cutting LM the Directing Line of the Plane $efgh$ in p its Directing Point, and draw Sp, which will be the Directing Line of the Projecting Plane; then zx and dR drawn through z and d parallel to Sp, will be the Vaniſhing and Interſecting Lines of the Projecting Plane[b], whence the Projection Rx is determined.

Or if from γ the Projection of any Point c of the given Line dz, a Line $C\gamma$ be drawn parallel to EF, cutting Dy in C, draw Cl parallel to ef, cutting dz in l, and $l\gamma$ will be parallel to zx[c]. *Q. E. I.*

This correſponds to Meth. 3. Caſe 1.

METHOD 4.

Through y and z draw yr and zr parallel to the Directions ti and si, and their Interſection r will be the *Focus* of the Projection[d]. *Q. E. I.*

This correſponds to Meth. 4. Caſe 1.

METHOD 5.

Fig. 128. N°. 7. Through y draw ym parallel to the Direction Ti, cutting dz in m, and draw mx parallel to the Direction si, which will cut EF in x, the Vaniſhing Point of the Projection Rx.

For ym is the Interſection of the Plane $efgh$ with a Plane paſſing through the Projecting Point parallel to the Plane EFGH, the Directing Line of which Plane is ST; and mx the Projecting Line of the Point m being a Line in this Plane, its Original is therefore parallel to the Plane of the Projection, and conſequently its Vaniſhing Point x is alſo the Vaniſhing Point of the Projection. *Q. E. I.*

This correſponds to Meth. 5. Caſe 1. and the Demonſtration is much the ſame.

METHOD 6.

Fig. 128. N°. 8. Through n the Interſection of Sp with LM draw nI, and its Direction being transferred to ni in the Picture, it will be parallel to the Projection Rx.

For Sp being the Directing Line of the Projecting Plane, n is the Directing Point of the Interſection of that Plane with the Plane FFGH, which Interſection is the Projection required, and nI being the Director of that Interſection, it is therefore parallel to its Image. *Q. E. I.*

SCHOL.

This correſponds to Meth. 6, and 7. Caſe 1. which here become the ſame; ſeeing a Plane paſſing through the Projecting Point and the Directing Line of either of the Planes $efgh$ or EFGH, muſt be the ſame with the Directing Plane, in which the Point S is here ſuppoſed to be; and n the Directing Point of the Projection of dz may be taken as the imaginary Projection of the Directing Point p of that Line; ſo that if p be given, Sp determines n, and conſequently nI the Director of the Projection; or if n be given, the ſame Line Sp by its Interſection with LM gives p the Directing Point, and conſequently pI the Director of the Line to be projected. And hence all Lines in the Plane $efgh$ which have parallel Images, and conſequently the ſame Directing Point, will alſo have parallel Projections, and *vice verſa.*

COR.

[a] Gen. Cor. 2. Prob. 3.

[b] Caſe 3. Prob. 3.

[c] Cor. Caſe 3. Prob. 3.

[d] Caſe 3. Prob. 3.

N°. 5.

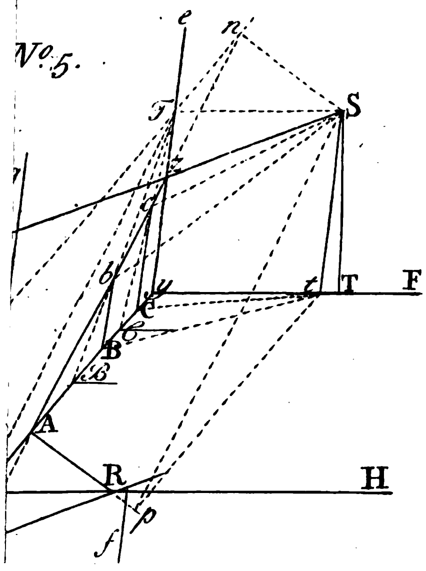


Fig. 128. N°. 6.

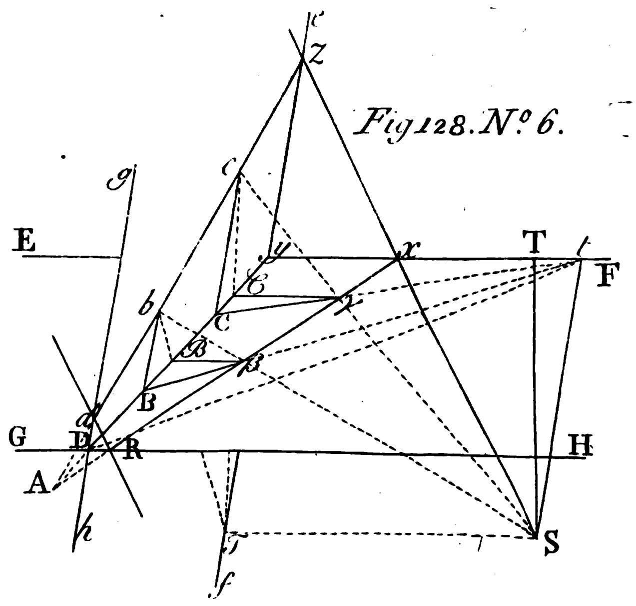


Fig. 128. N°. 8.

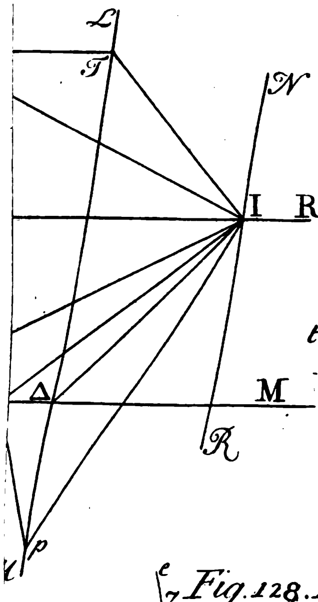


Fig. 128. N°. 7.

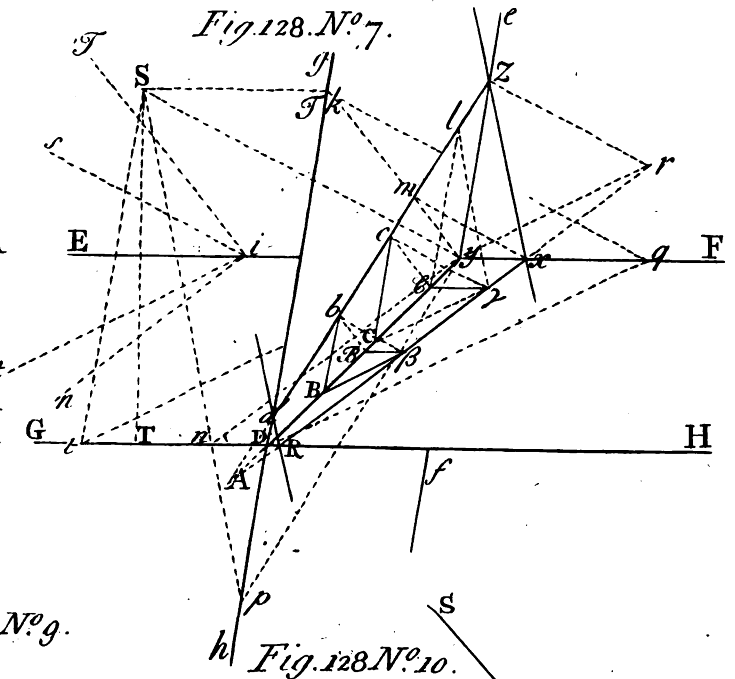


Fig. 128. N°. 9.

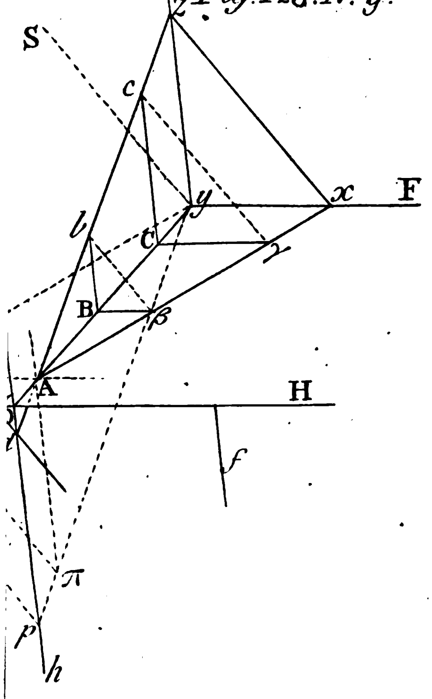
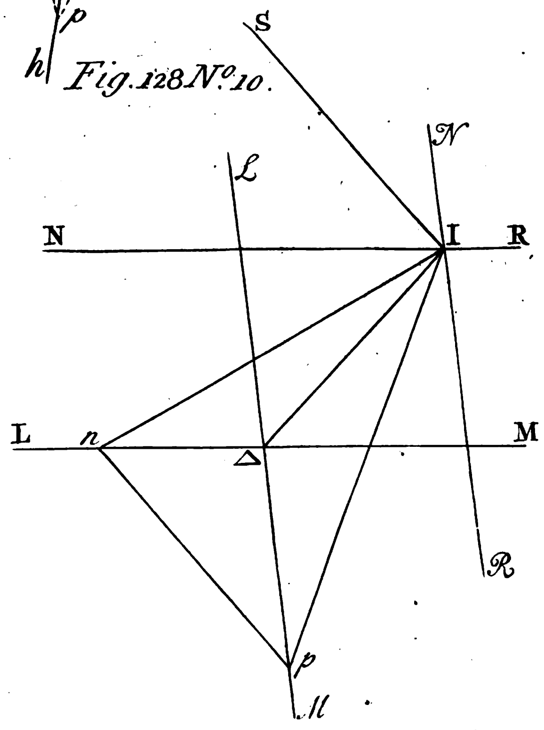


Fig. 128 N°. 10.



J. Mynde.

COR.

The Directing Plane may be here brought into the Picture, by placing the Point I at *y* in the Line EF, whereby the drawing of ſeveral Lines will be ſaved; for then *NR* and *LM* will coincide with *ef* and *gh*, NR and LM with EF and GH, and ΔI with D*y*: ſo that S being placed in the Picture, in the ſame Poſition with reſpect to *y* that the Projecting Point hath with regard to the Eye in the Directing Plane, S*t* parallel to *ef* cuts GH in *t*, S*T* parallel to EF cuts *gh* in *T*, *yp* parallel to *dz* cuts *gh* in *p*, and S*p* cuts GH in *n*; whence the Directions *Sy*, *ty*, *ny*, and *Ty* are found in the Picture, without the Trouble of transferring them from the ſeparate Directing Plane.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

Let *efgh* and EFGH be the given Planes, *dz* the given Line, and *si* the Direction of the Projecting Lines. Fig. 128. N°. 9.

METHOD 1.

Through any two Points *b* and *c* of the given Line draw Parallels to *ef*, cutting D*y* in B and C, through which draw Parallels to EF till they be cut in β and γ by Parallels to *si* drawn through *b* and *c*; and $\beta\gamma$ will be the Projection deſired, and *zx* and *d*R drawn through *z* and *d* parallel to *si*, will be the Vaniſhing and Interſecting Lines of the Projecting Plane[a].

This Method correſponds to the five firſt Methods of Caſe 1. all which here become the ſame; the Point L in the Figures of that Caſe, coinciding with *y* in this, the Points *l* and *m* with *z*, and the Point *r* with *x*; and the Seat *t* of the Projecting Point being at an infinite Diſtance in the Line EF, and the Seat *T* at an infinite Diſtance in *ef*, the Seats of the Projecting Lines on the Plane EFGH are parallel to EF, and their Seats on the Plane *efgh* are parallel to *ef*. *Q. E. I.*

METHOD 2.

Draw the ſeparate Directing Planes *NRLM*, NRLM, as in the laſt Caſe, and SI parallel to the Projecting Lines; and having drawn the Director I*p* of the Line *dz*, cutting *LM* in *p* its Directing Point, through *p* draw *pn* parallel to S*i*, cutting LM in *n*, and *pn* will be the Directing Line of the Projecting Plane, and *n* will be the Directing Point, and conſequently *n*I the Director of the Projection of *dz*. *Q. E. I.* Fig. 128. N°. 10.

This correſponds to Meth. 6. Caſe 3. which anſwers to Meth. 6, and 7. Caſe 1.

COR.

And here, as in the preceeding Caſe, the Directing Plane may be brought into the Picture, and the Direction *ny* thereby found as before[b].

But in this Situation of the Projecting Point, it is not neceſſary that GH ſhould be uſed as the Directing Line LM; but any Line Aν parallel to GH may be taken, cutting D*y* in any Point A, through which a Line Aπ being drawn parallel to *ef*, theſe two Lines will ſerve inſtead of GH and *gh*; for *yp* being drawn parallel to *dz*, cutting Aπ in π, and $\pi\nu$ being drawn parallel to the Direction S*y*, cutting Aν in ν, the Direction νy thereby found will coincide with *ny*.

For the Sides of the Triangle D*np* being reſpectively parallel to thoſe of the Triangle A$\nu\pi$, thoſe Triangles are Similar; wherefore, $Dp : A\pi :: Dn : A\nu$

And in the Similar Triangles D*py*, Aπy, $Dp : A\pi :: Dy : Ay :: Dn : A\nu$

And conſequently D*ny*, Aνy, are Similar Triangles, and the Lines *ny* and νy therefore coincide.

GENERAL COROLLARY.

The Corollaries to the ſeveral Methods of Caſe 1. of this Problem, are equally applicable to the correſponding Methods of the other Caſes.

SCHOL. 1.

If the Projecting Point S, when at a moderate Diſtance before the Directing Plane, be ſuppoſed to repreſent the Eye of a Spectator ſtanding on the Plane EFGH, and the Plane *efgh* be taken as the Repreſentation of a Picture expoſed to that Eye; the firſt, ſecond, fourth, and fifth Methods of this Problem will appear to be only ſeveral ways of putting the different Rules of *Stereography* into Perſpective. Fig. 128. N°. 1, 3.

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[a] Caſe 4. Prob. 3.

[b] Cor. Caſe 3.

Fig. 128. N°. 3. For on this Suppoſition, St will repreſent the Eye's Director, and t the Point of Station with reſpect to the Picture $efgh$, Dy will repreſent the Interſecting Line, ty the Directing Line, and $S\mathcal{T}$ the Radial of the Line tL in the Original Plane EFGH, and $\mathcal{T}$ will repreſent the Vaniſhing Point of tL, and $y\mathcal{T}$ the Vaniſhing Line of that Plane; ſeeing $y\mathcal{T}$, yD, and yt repreſent Parallels[a].

Fig. 128. N°. 1. If then $t\beta$, $t\gamma$, be conſidered as Lines in the Original Plane having the ſame Directing Point t, the Images of thoſe Lines being parallel to their Director St[b], which is here parallel to the Picture, Bb and Cc drawn from their Interſecting Points B and C parallel to St or ef, will repreſent the Indefinite Images of $B\beta$ and $C\gamma$; and conſequently b and c, where they are cut by $S\beta$ and $S\gamma$, repreſent the Images of β and γ[c].

Fig. 128. N°. 3. If βB and γC be taken as Original Lines, $S\mathcal{T}$ which is parallel to them, repreſents their Radial, and $\mathcal{T}$ their Vaniſhing Point; wherefore $B\mathcal{T}$ and $C\mathcal{T}$ repreſent the Indefinite Images of thoſe Lines[d], which muſt be cut by $S\beta$ and $S\gamma$ in b and c, the Images of β and γ[e].

Fig. 128. N°. 3. If Ax be conſidered as an Original Line, having A for its Interſecting Point, that Line produced to the Directing Line ty, cuts it in r its Directing Point; wherefore rS repreſents its Director, which being parallel to its Image[f], rS and Az muſt repreſent parallel Lines, and have therefore the ſame Vaniſhing Point z[g].

Laſtly, the Radial of the Original Line Ax being parallel to it, their Repreſentations muſt have the ſame Vaniſhing Point; wherefore Sx repreſents the Radial of Ax, and m where it cuts the Vaniſhing Line $\mathcal{T}y$, repreſents its Vaniſhing Point, wherefore Am repreſents the Indefinite Image of that Line[h].

Fig. 128. N°. 3. In this View, dz repreſents the Indefinite Image of Ax conſidered as a Line in the Original Plane EFGH, as it appears in the Picture $efgh$ expoſed to an Eye at S.

On the other hand, if $efgh$ be taken as the Original Plane, and EFGH as the Picture; then $S\mathcal{T}$ becomes the Eye's Director, $\mathcal{T}$ the Point of Station, and conſequently $\mathcal{T}y$ the Directing Line, and yt the Vaniſhing Line of the Original Plane $efgh$; and Rx then repreſents the Indefinite Image of the Original Line dz, as it appears in the Picture EFGH to an Eye at S.

What has been ſaid with reſpect to the Caſe when the Projecting Point is at a moderate Diſtance before the Directing Plane, is equally applicable when it is at a moderate Diſtance behind, or in that Plane.

SCHOL. 2.

When the Projecting Point is before, or in the Directing Plane[1], that part of the Projection of the propoſed Line dz which falls on the Perſpective Part of the Plane of the Projection EFGH, and on the contrary Side of the Original Plane $efgh$ from the Projecting Point S, is the only real part of the Projection, and all the reſt of it is imaginary; but when the Projecting Point is behind the Directing Plane[2], that Point being then repreſented by its Tranſprojection, its apparent Place is on the contrary Side of the given Planes to that where its Original lies, ſo that the real part of the Projection of dz falls on the ſame Side of the Plane $efgh$ with the apparent Place of the Projecting Point.

But in all Caſes, any Point of the Projection of dz, whether real or imaginary, is equally ſerviceable for finding the Indefinite Projection of that Line.

PROB. VII.

Any two Planes whoſe Vaniſhing Lines are either parallel, or coincide, being given, together with a Line in one of them; thence to find its Projection on the other Plane, from a Projecting Point whoſe Seat on either of the Planes is given.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 129. N°. 1, 2. Let $efgh$ and EFGH be the given Planes, dz the given Line in the Plane $efgh$, and S the Projecting Point, T its Seat on the Plane EFGH, and $\mathcal{T}$ its Seat on the Plane $efgh$.

METHOD

[a] Cor. 1. Def. 10. B. I.

[b] Cor. 1. Def. 18. B. I.

[c] Meth. 1.

[d] Theor. 4. B. I.

[e] Meth. 2.

[f] Cor. 1. Def. 18. B. I.

[g] Meth. 4.

[h] Meth. 5.

[1] Fig. 128. N°. 3, 5, 7, 9.

[2] Fig. 128. N°. 4, 6.

METHOD 1.

Through the Support ST of the Projecting Point, draw any ſubſtituted Plane $yy\Delta D$, cutting the given Planes in $y\Delta$ and yD[a]; then through any two Points b and c of the given Line draw Parallels to ef, cutting $y\Delta$ in B and C, and draw SB, SC, cutting yD in B and C, through which draw Parallels to EF, till they be cut by Sb, and Sc in β and γ; then β and γ will be the Projections of b and c, and $\beta\gamma$ the Indefinite Projection of dz.

For the Originals of Bb and $B\beta$ being parallel to the Picture and to each other, and in the ſame Plane with SB[b], $B\beta$ is the Interſection of that Plane with the Plane EFGH; and therefore $B\beta$ is the Projection of Bb on that Plane, wherefore β is the Projection of b: and in the ſame manner it is proved, that γ is the Projection of c, and conſequently $\beta\gamma$ the Projection deſired. *Q. E. I.*

[a] Caſe 2. Prop. 46. B. IV.

[b] 7 El. 11.

COR. 1.

And thus the Projection of any Point in the Plane $efgb$, or of any Line in that Plane, parallel to the Picture, may be found on the Plane EFGH.

COR. 2.

In Fig. N°. 1. the Projecting Point S is before the Directing Plane, and the Projection $C\gamma$ coinciding with the Interſecting Line GH of the Plane of the Projection, Cc is the neareſt Line in the Plane $efgb$ which can be projected within the Compaſs of the Perſpective Part of the Plane EFGH.

In Fig. N°. 2. the Projecting Point S is behind the Directing Plane, and the Line Cc coinciding with the Interſecting Line gb of the Original Plane, $C\gamma$ is the neareſt Line in the Plane EFGH, on which the Projection of any Point in the Perſpective Part of the Plane $efgb$ can fall.

METHOD 2.

Fig. 129. N°. 1, 2.

Find $A\delta$ the Seat of dz on the Plane EFGH, and thence b and c the Seats of any two Points b and c of that Line[c]; through T draw Tb, Tc, till they be cut by Sb and Sc in β and γ, which will be the Projections of b and c[d], whence the Projection $\beta\gamma$ is found. *Q. E. I.*

[c] Caſe 2. Prop. 47. B. IV.

[d] Prob. 1.

METHOD 3.

Fig. 129. N°. 1, 2.

From *T* draw *T*L parallel to ef, cutting dz in L, and draw SL; then through z and d draw zx, dR, parallel to SL, cutting EF and GH in x and R, and Rx will be the Indefinite Projection of dz, and zx and dR will be the Vaniſhing and Interſecting Lines of the Projecting Plane.

For SL is a Line in the Projecting Plane parallel to the Picture. *Q. E. I.*

METHOD 4.

Fig. 129. N°. 3, 4.

Having drawn any ſubſtituted Plane $yy\Delta D$ paſſing through ST, draw Sy cutting Dy in ϱ, and through ϱ draw ϱr parallel to EF; then Sz will cut ϱr in r, the *Focus* of the Projection Rx.

For ϱ is the Interſection of the Plane EFGH with a Line Sy paſſing through the Projecting Point S parallel to the Plane $efgb$; and therefore ϱr parallel to EF is the Interſection of the Plane EFGH, with a Plane paſſing through the Projecting Point S parallel to the Plane $efgb$[e], and conſequently r is the *Focus* of the Projection Rx, or the imaginary Projection of the Vaniſhing Point z of the given Line dz[f]. *Q. E. I.*

[e] Caſe 2. Prop. 46. B. IV.

[f] Cor. 1. Caſe 3. Prob. 3.

COR.

The Line ϱr is the Place of the *Foci* of the Projections of all Lines in the Plane $efgb$ on the Plane EFGH from the Point S, and is therefore the imaginary Projection of the Vaniſhing Line ef of the Original Plane.

For wherever the Point z falls in ef, Sz muſt cut ϱr in r the *Focus* of the Projection of the propoſed Line.

METHOD 5.

Fig. 129. N°. 3, 4.

Draw Sy cutting Δy in μ, and through μ draw μm parallel to ef, cutting dz in m; then Sm will cut EF in x the Vaniſhing Point of the Projection Rx.

For

For μ is the Interſection of the Plane $efgh$ with the Line Sy parallel to the Plane EFGH, and therefore μm is the Interſection of the Plane $efgh$ with a Plane paſſing through the Projecting Point S parallel to EFGH the Plane of the Projection; and conſequently the Original of the Projecting Line Sm being parallel to the Plane EFGH, its Vaniſhing Point x is alſo the Vaniſhing Point of the Projection Rx. *Q. E. I.*

COR.

The Interſection of μm with any Line whatſoever in the Plane $efgh$, gives a Point correſponding to m, through which a Line Sm being drawn, it will cut EF in the Vaniſhing Point of the Projection of the Line propoſed; and the imaginary Projection of μm therefore coincides with EF the Vaniſhing Line of the Plane of the Projection.

For wherever the Point m falls in μm, the Original of Sm will be parallel to the Plane EFGH, and conſequently the Projection of m will be a Vaniſhing Point in EF.

METHOD 6.

Fig. 129. N°. 3, 4.

Draw $S\pi$ parallel to $y\Delta$, cutting yD in π, and through π draw πp parallel to EF; then Sp drawn parallel to dz, will cut πp in p the Projection of the Directing Point of dz.

For yD being the Interſection of the Plane EFGH with a Plane paſſing through $y\Delta$ and the Point S, yD is therefore the Projection of $y\Delta$; and becauſe $S\pi$ and $y\Delta$ are parallel, and in the ſame Plane $yy\Delta D$, they have the ſame Directing Point [a]; wherefore $S\pi$ is the Projecting Line of the Directing Point of $y\Delta$, and π being the Interſection of $S\pi$ with the Plane EFGH, it is therefore the Projection of the Directing Point of $y\Delta$, and conſequently a Point in the Projection of the Directing Line of the Plane $efgh$ in which $y\Delta$ lies; which Directing Line being parallel to EF, its Projection is alſo parallel to EF [b], wherefore πp drawn through π parallel to EF, is the Projection of that Directing Line; and the Directing Point of dz being a Point in that Directing Line, its Projection is therefore at p, the Interſection of πp with Sp drawn parallel to dz, *Q. E. I.*

COR.

The Line πp is the Place of the Projections of the Directing Points of all Lines in the Plane $efgh$ on the Plane EFGH from the Point S; and is the Interſection of the Plane EFGH with a Plane paſſing through the Projecting Point and the Directing Line of the Original Plane $efgh$.

For whatever Line dz in the Plane $efgh$ be propoſed, a Line from S parallel to it, will cut πp in p the Projection of the Directing Point of the propoſed Line.

METHOD 7.

Fig. 129. N°. 3, 4.

Draw $S\nu$ parallel to yD, cutting $y\Delta$ in ν; and through ν draw νn parallel to ef, cutting the propoſed Line dz in n; then Sn being drawn, it will be parallel to the Projection Rx; and the imaginary Projection of the Point n will be at the Directing Point of Rx.

For $S\nu$ being the Projecting Line of the Point ν of the Line yD, and being parallel to its Projection yD [c], the imaginary Projection of ν is therefore at the Directing Point of yD [d], which is a Point in the Directing Line of the Plane EFGH; and this Directing Line being parallel to ef, the Interſection of the Plane $efgh$ with a Plane paſſing through that Directing Line and the Point S (and which is the Projecting Plane of that Line) muſt be parallel to ef [e]; wherefore νn being drawn through ν parallel to ef, its imaginary Projection coincides with the Directing Line of the Plane EFGH, and conſequently the imaginary Projection of the Point n of the Line dz, is at the Directing Point of its Projection Rx, which Projection is therefore parallel to Sn. *Q. E. I.*

COR.

The Interſection of νn with any Line dz in the Plane $efgh$ gives a Point n, whence a Line being drawn to S, it will be parallel to the Projection of the propoſed Line.

GENERAL COROLLARY.

Fig. 129. N°. 1, 2, 3, 4.

When the Vaniſhing Lines of the given Planes coincide, there is no Difference in the Practice of the firſt, ſecond, third, ſixth, and ſeventh Methods; and the ſame Figures

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[a] Cor. 5. Theor. 12. B. I.

[b] Theor. 15. B. I.

[c] Meth. 6.

[d] Cor. 5. Theor. 12. B. I.

[e] Theor. 15. B. I.

Plate 51 Book 5 Sect.1.

Fig.129 N°1.

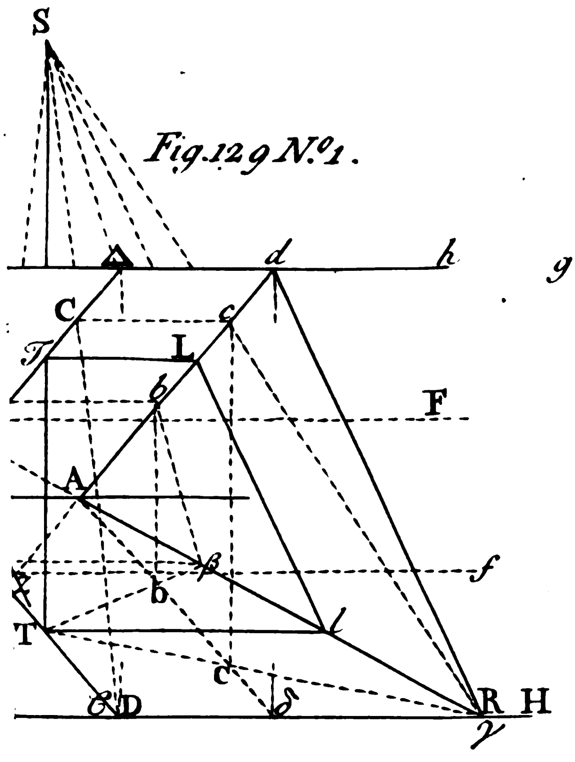
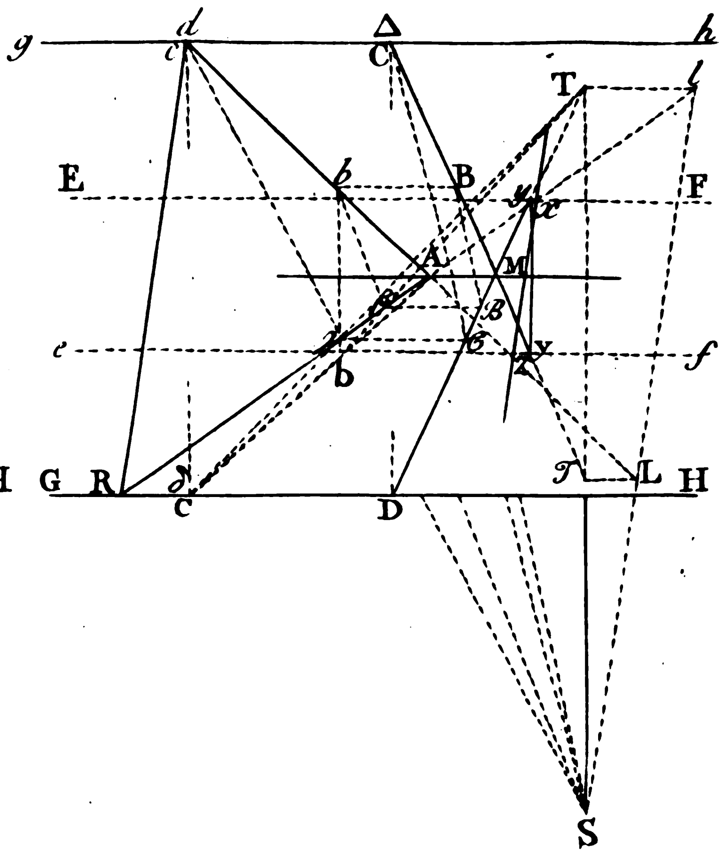


Fig.129 N°2.



129 N°3.

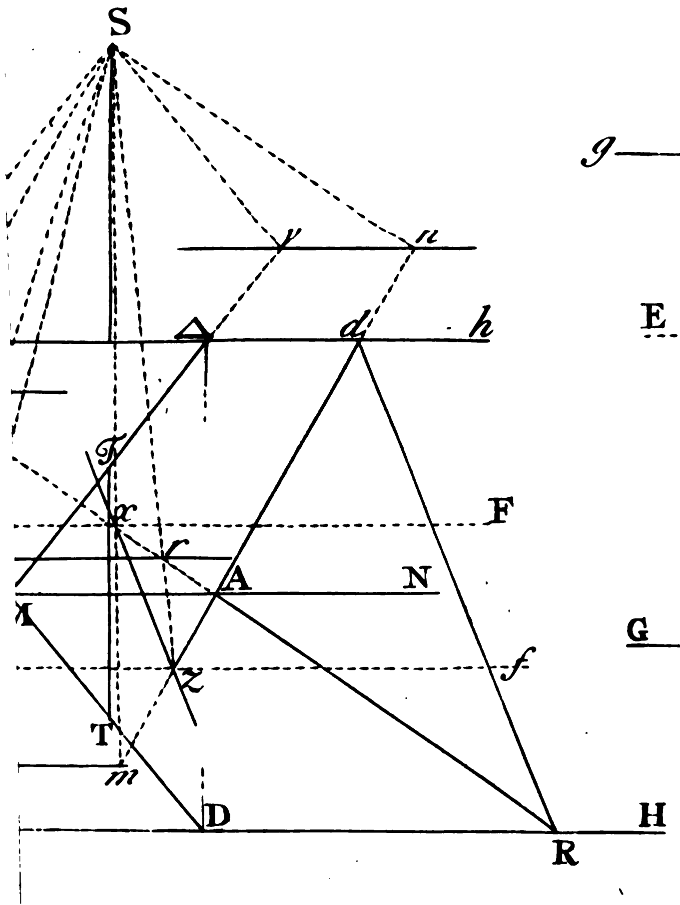
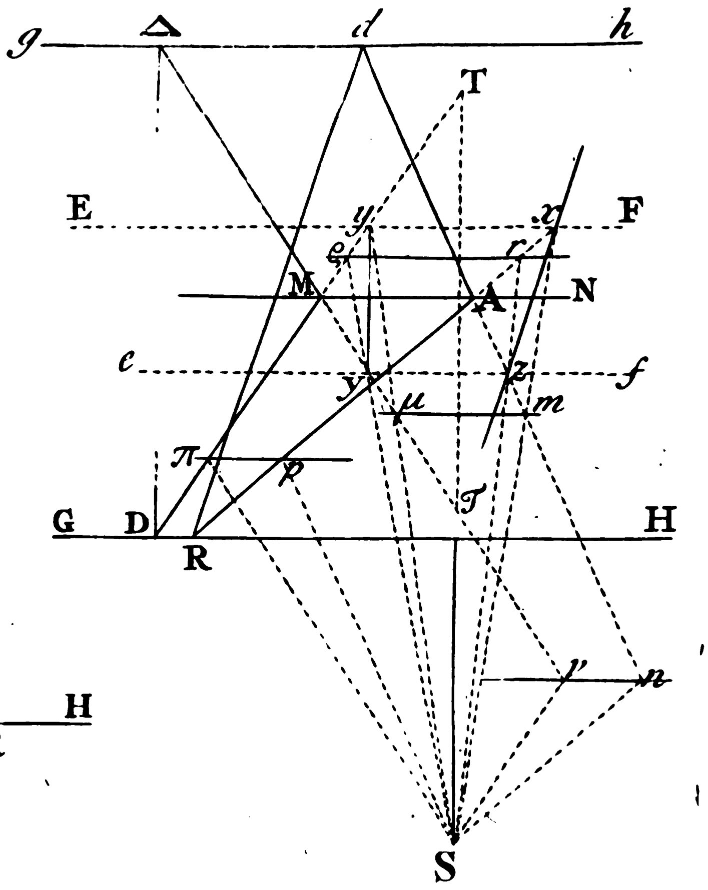


Fig.129 N°4.



Figures will ſerve, if MA be taken as the common Vaniſhing Line of the given Planes; in which Caſe the Points *z* and *x* coincide in A, and the Points *y* and y coincide in M.

Fig. 129. N°. 3, 4.

But the fourth and fifth Methods cannot in this Caſe be uſed; in regard that *y* and y coinciding in M, the Points ϱ and μ alſo coincide with them, the Lines ϱr and μm become the ſame with MA the common Vaniſhing Line of the given Planes, and the Points *r* and *m* coincide with A the common Vaniſhing Point of the propoſed Line and its Projection.

CASE 2.

When the Projecting Point is at an infinite Diſtance before or behind the Directing Plane.

Fig. 129. N°. 5, 6.

The firſt, ſecond, third, ſixth, and ſeventh Methods of the firſt Caſe, are all applicable to this; the only Difference being, that here ST is the Vaniſhing Line of the ſubſtituted Plane $\mathcal{T}$TΔD, $\mathcal{T}$ and T being the Seats of the Projecting Point S on the Vaniſhing Lines *ef* and EF: and the Point marked L in the former Figures, here coinciding with *z*, S*z* is the Vaniſhing Line of the Projecting Plane[a]; and the Points *y* and y being the ſame with T and $\mathcal{T}$, the Points ϱ and μ coincide with T and $\mathcal{T}$; ſo that the fourth and fifth Methods become the ſame with the third.

All this is ſufficiently evident by the Figures, the correſponding Points of which are marked with the ſame Letters as before, excepting where two or more Points coincide, the Letter belonging to the Principal of them being then only retained. *Q. E. I.*

SCHOL.

Theſe Figures, in which the Projecting Point is at an infinite Diſtance before the Directing Plane, and the Vaniſhing Lines are parallel, will alſo ſerve for the Caſe where the Vaniſhing Lines coincide, if MA be taken as the common Vaniſhing Line of the propoſed Planes; and then T and $\mathcal{T}$ will coincide in that Line, as will alſo *x* and *z* with A[b], but the Practice in either Caſe is in all other reſpects the ſame.

The Figures where the Projecting Point is at an infinite Diſtance behind the Directing Plane are not drawn, they being eaſily ſupplied by the Figures N°. 2. and 4. if the Seats T and $\mathcal{T}$ of the Projecting Point be ſuppoſed to lie in the Vaniſhing Lines EF and *ef*.

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Fig. 129. N°. 7.

Let *efgb* and EFGH be the given Planes, and *dz* the given Line in the Plane *efgb*.

Fig. 129. N°. 8.

Any where a-part draw the Directing Plane NRLM of the Plane EFGH, and draw *LM* parallel to NR at the ſame Diſtance, and on the ſame ſide of it, as *gb* is with reſpect to its Vaniſhing Line *ef*, and NR*LM* will be the Directing Plane of the Plane *efgb*, the ſame Line NR being the Parallel of the Eye to both the Directing Planes[c]; let I be the Place of the Eye, and S the Place of the Projecting Point, $\mathcal{T}$ its Seat on *LM* the Directing Line of the Plane *efgb*, and T its Seat on LM the Directing Line of the Plane EFGH; draw the Directors SI, $\mathcal{T}$I, and TI, and transfer their Directions to *si*, $\mathcal{T}$*i*, and T*i* in the Picture, placing the Point *i* in either of the given Vaniſhing or Interſecting Lines as may be moſt convenient, it not being material where it is placed, ſo as the Directions be parallel to their reſpective Directors.

Fig. 129. N°. 7.

This Preparation being made, draw any Line D*y* in the Plane EFGH parallel to T*i* the Direction of the Seats of the Projecting Lines on that Plane, and having drawn *y*y perpendicular to EF, compleat the ſubſtituted Plane *y*yΔD, which will be a Plane having the Support ST of the Projecting Point for its Directing Line, ſeeing the Seat T of the Projecting Point is the Directing Point of D*y*; and Δy will be the Interſection of that Plane with the Plane *efgb*, and conſequently parallel to $\mathcal{T}$*i* the Direction of the Seats of the Projecting Lines on that Plane.

METHOD 1.

Fig. 129. N°. 7.

Through any two Points *b* and *c* of the given Line *dz*, draw Parallels to *ef*, cutting Δy in B and C, from whence draw Parallels to *si* the Direction of the Projecting Lines, cutting D*y* in *B* and *C*, and having drawn *B*β, *C*γ, parallel to EF, the Lines *b*β and *c*γ drawn parallel to *si*, will cut them in β and γ the Projections of *b* and *c*, whence the Indefinite Projection $\beta\gamma$ is found. *Q. E. I.*

This correſponds to Meth. 1. Caſe 1. and is demonſtrated in the ſame manner.

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[a] Meth. 3.

[b] Gen. Cor. Caſe 1.

[c] Cor. 2. Theor. 14 B. I.

METHOD 2.

Find Aδ the Seat of dz on the Plane EFGH, and thence b and c the Seats of b and c, and $b\beta$, $c\gamma$, drawn parallel to si, will cut bβ, cγ, drawn parallel to Ti, in the ſame two Points β and γ[a]. *Q. E. I.*

[a] Caſe 3. Prob. 1.

METHOD 3.

Fig. 129. N°. 8. Find Ip, the Director of the given Line dz, and draw Sp, and that will be the Directing Line of the Projecting Plane, whence the Vaniſhing and Interſecting Lines of that Plane are determined[b]. *Q. E. I.*

[b] Meth. 3. Caſe 3. Prob. 6.

METHOD 4.

Fig. 129. N°. 7. Through y draw yϱ parallel to si, cutting Dy in ϱ, and through ϱ draw ϱr parallel to EF, and zr drawn parallel to si, will cut ϱr in r the *Focus* of the Projection. *Q. E. I.*

METHOD 5.

Through y draw $y\mu$ parallel to si, cutting Δy in μ, through μ draw μm parallel to ef, cutting dz in m, and mx parallel to si will cut EF in x the Vaniſhing Point of the Projection. *Q. E. I.*

Theſe two laſt Methods correſpond to the fourth and fifth of Caſe 1. and are demonſtrated after the ſame manner.

METHOD 6.

Fig. 129. N°. 8. If Sp be produced till it cut LM in n, the Director In will be parallel to the Projection of the given Line.

For Sp being the Directing Line of the Projecting Plane, it muſt, if produced, cut LM the Directing Line of the Plane EFGH in n the Directing Point of the common Interſection of thoſe Planes, which is the Projection required[c]. *Q. E. I.*

[c] Meth. 6. Caſe 3. Prob. 6.

GENERAL COROLLARY.

If the given Planes have the ſame Vaniſhing Line, the General Corollary of Caſe 1. is here alſo applicable; ſave that if MA be taken as the common Vaniſhing Line of the given Planes, the Directing Planes, and conſequently the Directions of the Projecting Lines and their Seats, muſt be varied, ſeeing the Directing Lines LM and *LM* muſt be at an equal Diſtance reſpectively from NR, as gh and GH are from MA[d].

[d] Cor. 3. Def. 18. B. I.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

Fig. 129. N°. 9. Find the Seat Aδ of the given Line dz on the Plane EFGH, and thence the Seats b and c of any two Points b and c of that Line[e], through b and c draw Parallels to EF till they be cut in β and γ by $b\beta$, and $c\gamma$ drawn parallel to si the Direction of the Projecting Lines; then β and γ will be the Projections of b and c, and $\beta\gamma$ the Indefinite Projection of dz; and zx and dR drawn through z and d parallel to si will be the Vaniſhing and Interſecting Lines of the Projecting Plane[f]. *Q. E. I.*

[e] Prop. 47. B. IV.

[f] Caſe 4. Prob. 3.

This correſponds to the firſt five Methods of Caſe 1. which are all reduced to this, and the Director of the Projection of dz may be found, as at Caſe 4. Prob. VI.

The Method is the ſame when the given Planes have the ſame Vaniſhing Line, if AM be taken as that Vaniſhing Line; z and x then coinciding with A.

GENERAL COROLLARY.

The Corollaries to the ſeveral Methods of Caſe 1. of this Problem, are applicable to the correſponding Methods of the other Caſes.

SCHOL.

The Methods propoſed in this Problem for finding the Projection of a Line by the help of a ſubſtituted Plane $yy\Delta$D, when the Vaniſhing Lines of the given Planes are parallel; are alſo applicable when the Vaniſhing Lines incline to each other ſo obliquely, that their Interſection is out of reach, or at an inconvenient Diſtance.

We ſhall give one Example, when the Projecting Point is at a moderate Diſtance before the Directing Plane, by which it will be eaſy to apply the ſame Methods to all the other Caſes of the Situation of the Projecting Point.

Fig. 130. Let $efgh$ and EFGH be the given Planes, the Interſection X of their Vaniſhing Lines

Lines ef and EF being ſuppoſed out of reach; and let dz be the given Line in the Plane $efgh$, S the Projecting Point, and T its Oblique Seat on EFGH the Plane of the Projection.

METHOD 1.

Draw any ſubſtituted Plane yyΔD paſſing through ST, cutting the given Planes in yD and yΔ, and by the help of any other ſubſtituted Plane FfhH, find MN the common Interſection of the given Planes; then through any two Points b and c of the given Line dz, draw bB, cC, parallel to MN, cutting yΔ in B and C, from S through B and C draw SB, SC, cutting Dy in B and C, from whence draw $B\beta$, $C\gamma$, parallel to MN, which will be cut by Sb and Sc in β and γ the Projections of b and c.

For the Parallels Bb and MN being Lines in the Plane $efgh$, they have the ſame Directing Point, and the Parallels MN and $B\beta$ being Lines in the Plane EFGH they have the ſame Directing Point[a], wherefore Bb and $B\beta$ have the ſame Directing Point, and conſequently are in the ſame Plane with SB, in which Plane the Projecting Line Sb alſo lies; wherefore β is the Projection of b; and in the ſame manner it may be ſhewn, that γ is the Projection of c, and conſequently $\beta\gamma$ the Projection of dz. *Q. E. I.*

METHOD 2.

The ſecond Method is exactly the ſame as in the Problem. *Q. E. I.*

METHOD 3.

Through t the Interſection of ST with the Plane $efgh$, draw tL parallel to ef, cutting dz in L, and SL will be parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane; and if Tl be drawn parallel to EF, SL will cut it in l the Projection of L.

For it is evident STl is in a Plane parallel to the Picture, the Interſection of which with the Plane $efgh$ is tL. *Q. E. I.*

METHOD 4.

The Points ϱ is found in the ſame manner as in the Problem, but ϱr muſt here be drawn tending to the Interſection X of the given Vaniſhing Lines[b].

For ϱr is the Interſection of the Plane EFGH with a Plane paſſing through the Projecting Point parallel to the Plane $efgh$[c], which two laſt Planes having the ſame Vaniſhing Line ef, X the Interſection of ef with EF is therefore the Vaniſhing Point of ϱr. *Q. E. I.*

METHOD 5.

The Point μ is alſo found as in the Problem, and μm muſt be drawn tending to the Vaniſhing Point X, for the ſame reaſon as before. *Q. E. I.*

METHOD 6.

The Point π is alſo found as in the Problem, but $p\pi$ muſt here be drawn parallel to MN the common Interſection of the given Planes.

For $p\pi$ is the Interſection of the Plane EFGH, with a Plane paſſing through the Projecting Point and the Directing Line of the Plane $efgh$[d]; and the Directing Point of MN being the Interſection of the Directing Lines of the given Planes[e], it is therefore alſo the Directing Point of $p\pi$, and conſequently the Images $p\pi$ and MN muſt be parallel[f]. *Q. E. I.*

METHOD 7.

The Point ν is alſo found as in the Problem, but $n\nu$ muſt be drawn parallel to MN.

For $n\nu$ is the Interſection of the Plane $efgh$ with a Plane paſſing through the Projecting Point and the Directing Line of the Plane EFGH[g]. *Q. E. I.*

PROB. VIII.

Two Planes, the one parallel and the other inclining to the Picture, being given, together with a Line in the parallel Plane; thence to find its Projection on the other Plane, from a Projecting Point whoſe Seat on either of the Planes is given.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Let

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[a] Cor. 5. Theor. 12. B. I.

[b] Prob. 18. B. II.

[c] Meth. 4. Caſe 1.

[d] Cor. Meth. 6. Caſe 1.

[e] Theor. 16. B. I.

[f] Cor. 4. Theor. 12. B. I.

[g] Meth. 7. Caſe 1.

Fig. 131. N°. 1. Let EFGH be the Plane of the Projection, and MA its Interſection with an Original Plane parallel to the Picture; and let A*b* be the given Line in the parallel Plane, S the Projecting Point, and T its Seat on the Plane EFGH.

METHOD 1.

Through any two Points *b* and *c* of the given Line, draw *b*B, *c*C, perpendicular to EF, cutting MA in B and C; from T through B and C draw TB, TC, till they be cut in β and γ by S*b* and S*c*, then β and γ will be the Projections of *b* and *c*, and $\beta\gamma$ the Indefinite Projection of A*b*[a]. *Q. E. I.*

This correſponds to Meth. 1. Prob, VI. and VII.

COR.

In this manner the Projection of any Point *b* or *c* in the Original Plane on the Plane EFGH may be found.

METHOD 2.

Let *o* be the Center of the Vaniſhing Line EF, and by its help find *T* the Oblique Seat of S on the Original Plane[b]; from *T* through *b* and *c* draw *Tb*, *Tc*, cutting MA in *B* and *G*, and from *o* through *B* and *C* draw *oB*, *oC*, which will be cut by S*b*, and S*c* in the ſame two Points β and γ.

For the Originals of *To* and *Bo* being parallel, they are in the ſame Plane with *TB*, in which Plane S*b* alſo lies, S*b* therefore cuts *Bo* in β the Projection of *b*; and for the like reaſon, S*c* cuts *Co* in γ the Projection of *c*. *Q. E. I.*

This correſponds to Meth. 2. Prob. VI. and VII.

METHOD 3.

Through T draw TL parallel to EF, and draw SL parallel to the given Line A*b*, cutting TL in L, through L and A the Interſection of A*b* with the Plane EFGH, draw R*x*, and that will be the Projection ſought; and Lines drawn through *x* and R parallel to SL will be the Vaniſhing and Interſecting Lines of the Projecting Plane[c].

This Method correſponds to the third, fourth, and ſixth of Prob. VI. and VII. for SL is a Line in the Projecting Plane parallel to the Picture[d]; TL is the Interſection of the Plane EFGH with a Plane paſſing through the Projecting Point parallel to the Original Plane, wherefore TL cuts the Projection R*x* in L its *Focus*, through which Point the Projections of all Lines in the Original Plane parallel to A*b* muſt paſs[e]; and TL may be alſo taken as the Interſection of the Plane EFGH with a Plane paſſing through the Projecting Point, and the Directing Line of the Original Plane, which here is infinitely diſtant, wherefore SL parallel to A*b* cuts TL in L a Point of the Projection[f]. *Q. E. I.*

METHOD 4.

Through *T* draw *Tm* parallel to EF until it be cut by A*b* in *m*, and draw S*m* which will cut EF in *x* the Vaniſhing Point of the Projection.

For *Tm* is the Interſection of the Original Plane with a Plane paſſing through the Projecting Point parallel to the Plane EFGH[g]. *Q. E. I.*

METHOD 5.

Through S draw S*v* parallel to *o*T, cutting the Original Plane in *v*, and through *v* draw *vn* parallel to EF until it be cut by A*b* in *n*; then S*n* will be parallel to the Projection R*x*.

For *vn* is the Interſection of the Original Plane with a Plane paſſing through the Projecting Point and the Directing Line of the Plane EFGH[h]. *Q. E. I.*

CASE 2.

When the Projecting Point is at an infinite Diſtance before or behind the Directing Plane.

METHOD 1. *and* 2.

Fig. 131. N°. 2. Here S and T being both Vaniſhing Points, and the Point *T* being at an infinite Diſtance in the Line TS, the firſt and ſecond Methods become the ſame; *b*B and *c*C perpendicular to EF alſo repreſenting *TB* and *TC* in the other Figure. *Q. E. I.*

METHOD 3. *and* 4.

The third Method is the ſame as before, ſave that TL coinciding with EF, L coincides with *x*, and S*x* is the Vaniſhing Line of the Projecting Plane; and *T* being

infinitely

[a] Prob. 1.

[b] Caſe 3. Prop. 52. B. IV.

[c] Gen. Cor. 2. Prob. 3.

[d] Meth. 3.

[e] Meth. 4.

[f] Meth. 6.

[g] Meth. 5. Prob. 6. and 7.

[h] Meth. 7. Prob 6. and 7.

Fig. 130.

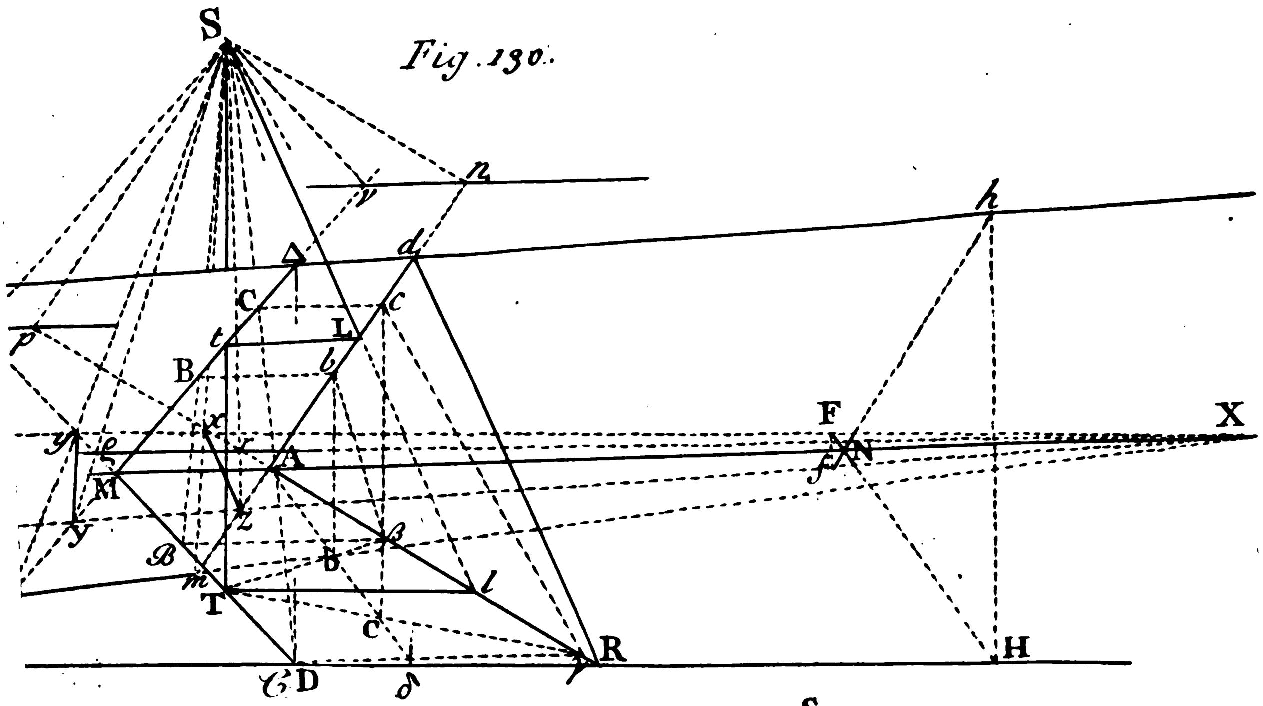


Fig. 131. N^o 1.

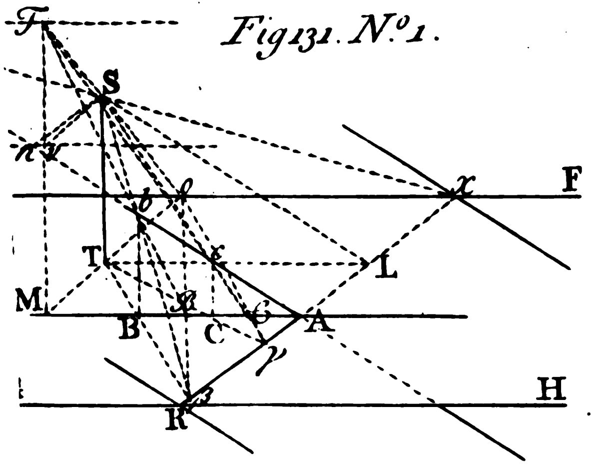


Fig. 131. N^o 2.

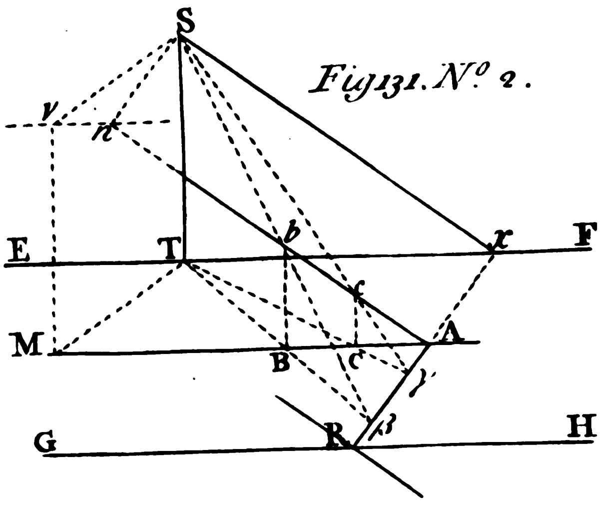


Fig. 131. N^o 3.

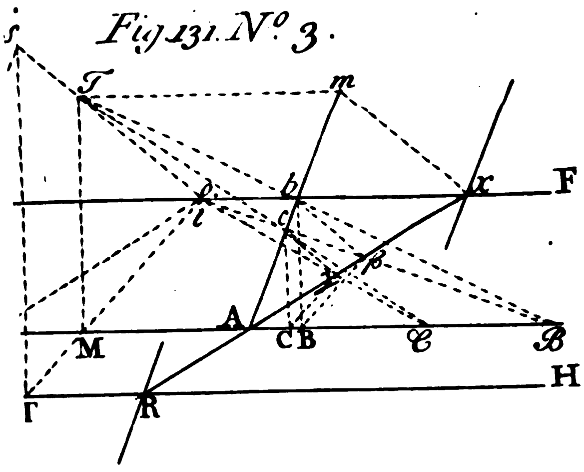


Fig. 132. N^o 1.

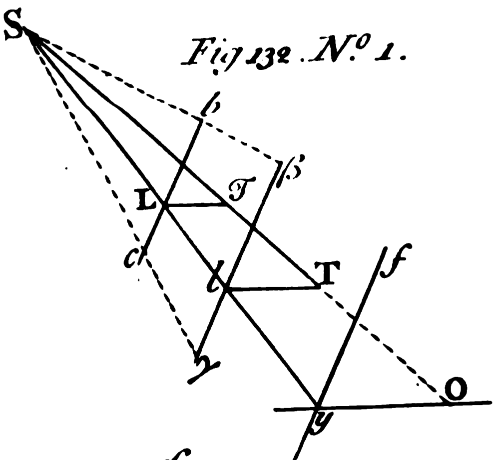


Fig. 132. N^o 2.

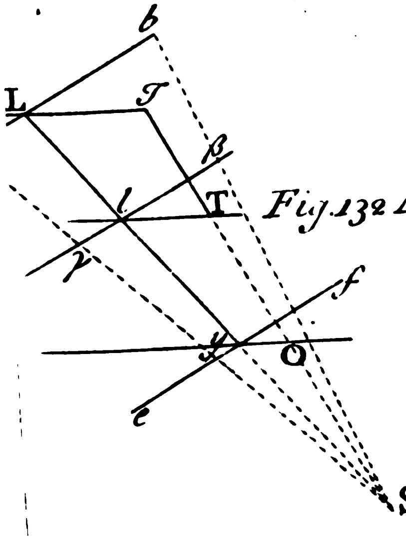
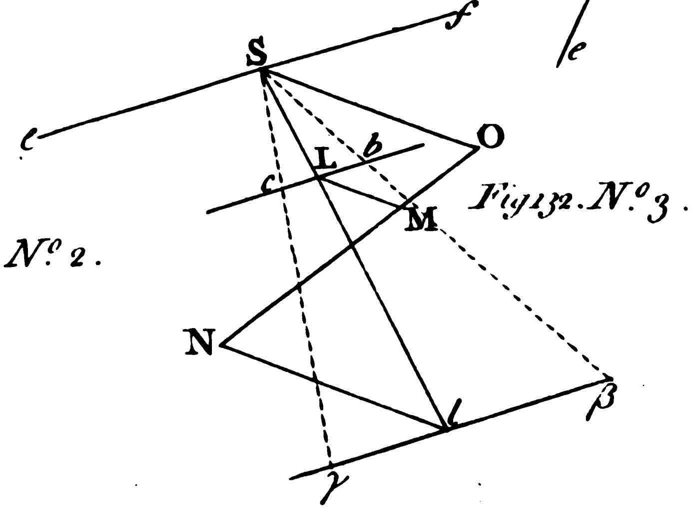


Fig. 132. N^o 3.



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infinitely diſtant, Tm and conſequently m the Interſection of Ab with that Line are alſo infinitely diſtant, wherefore Sm becomes parallel to Ab, and therefore coincides with Sx; ſo that the fourth Method becomes the ſame with the third. *Q. E. I.*

METHOD 5.

The fifth Method is the ſame as before; only that inſtead of oM, any Line TM may be drawn from T, cutting AM in M; and $S\nu$ drawn parallel to TM, will cut $M\nu$ parallel to ST in ν, through which a Parallel to EF being drawn cutting Ab in n, the Line Sn will be parallel to the Projection Rx. *Q. E. I.*

SCHOL.

Both the Figures here referred to, repreſent the Caſe when the Projecting Point is before the Directing Plane and beyond the Original Plane, which throws the real part of the Projection towards the Eye; but the Methods being the ſame, when the Projecting Point is behind the Directing Plane, or between the Eye and the Original Plane, the Figures are not drawn.

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

The ſame things being ſuppoſed as before, let s be the Projecting Point, and T its Seat on the Directing Line of the Plane EFGH, the Directing Plane being brought into the Picture, and the Point i made to coincide with o the Center of the Vaniſhing Line EF[a].

Fig. 131, N°. 3.

[a] Cor. 2. Caſe 3. Prob. 1.

METHOD 1.

Having drawn the Directions si and Ti of the Projecting Lines and their Seats on the Plane EFGH, through any two Points b and c of the given Line Ab, draw Perpendiculars to EF, cutting MA in B and C; then Lines drawn from B and C parallel to the Direction Ti, will cut $b\beta$ and $c\gamma$ drawn from b and c parallel to the Direction si, in β and γ the Projections of b and c[b]. *Q. E. I.*

[b] Caſe 3. Prob. 1.

METHOD 2.

From M, the Interſection of To with MA, draw MT parallel to sT, cutting so in T, and T will be the Oblique Seat of the Projecting Point on the Original Plane[c]; therefore from T through b and c draw Tb, Tc, cutting MA in B and C, from whence draw Bo, Co, which will be cut by $b\beta$, $c\gamma$, drawn parallel to si, in the ſame two Points β and γ as before. *Q. E. I.*

[c] Cor. 2. Caſe 3. Prob. 2. and Schol.

METHOD 3.

From s draw sL parallel to Ab cutting GH in L, and the Direction Li will be parallel to the Projection Rx[d], and sL will be parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane. *Q. E. I.*

[d] Gen. Cor. 2. Prob. 3.

METHOD 4.

Through T draw Tm parallel to EF, cutting Ab in m, and mx drawn parallel to the Direction si, will cut EF in x the Vaniſhing Point of the Projection[e]. *Q. E. I.*

[e] Meth. 4. Caſe 1.

SCHOL.

The fifth Method cannot be here uſed, ſeeing a Plane paſſing through the Projecting Point and the Directing Line of the Plane FFGH, is the ſame with the Directing Plane, and cannot therefore cut the Original Plane to which it is parallel.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

In this Caſe, the Projection of Ab is the ſame with MA the Interſection of the Original Plane with the Plane EFGH, and the Projection of any Point of that Line is the Interſection of MA with a Line drawn from the propoſed Point parallel to the Direction si[f]. *Q. E. I.*

[f] Gen. Cor. 2 Prob. 3.

PROB. IX.

Two Planes, the one parallel and the other inclining to the Picture, being given, together with a Line in the inclining Plane; thence to find its Projection on the Parallel Plane, from a Projecting Point whoſe Seat on either of the Planes is given.

This Problem may be ſolved by any of the Methods of the preceeding, with a ſmall Variation in the Order of drawing the neceſſary Lines.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 131. N°. 1. Let EFGH be the Original Plane, and R*x* a given Line in that Plane, whoſe Projection is ſought on a Plane parallel to the Picture, cutting the Plane EFGH in MA.

METHOD 1.

Through any two Points β and γ of the given Line, draw Tβ, Tγ, cutting MA in B and C, from whence draw the Perpendiculars B*b*, C*e*, which will be cut by Sβ, Sγ, in *b* and *e*, the Projections of β and γ on the Parallel Plane, whereby the Indefinite Projection A*b*, either real or imaginary, is found. *Q. E. I.*

METHOD 2.

Having drawn βo, γo, cutting MA in *B* and *C*, draw *TB*, *TC*, which will be cut by Sβ, Sγ, in *b* and *c*, the Projections of β and γ as before. *Q. E I.*

METHOD 3.

Draw TL parallel to EF, cutting R*x* in L, and draw SL, then A*b* parallel to SL is the Projection ſought. *Q. E. I.*

METHOD 4.

Through *T* draw *Tm* parallel to EF, cutting S*x* in *m*, and *m*A will be the Projection deſired. *Q. E. I.*

METHOD 5.

Laſtly, draw S*v* parallel to *o*T, cutting the Parallel Plane in *v*, through which draw *vn* parallel to EF; then S*n* drawn parallel to R*x* will cut *vn* in a Point *n*, through which and the Point A the Indefinite Projection A*b* muſt paſs. *Q. E. I.*

All this is evident from the Conſtructions in the preceeding Problem; and after the like manner, the ſame Methods may be applied to the other Caſes of the Situation of the Projecting Point as is ſufficiently obvious.

SCHOL.

In the Figure here referred to, A*b* is only the imaginary Projection of RA, ſuch part of the Original Line R*x* as lies on the hither Side of the Plane of the Projection, and oppoſite to the Projecting Point S; or more properly, it is the Repreſentation of the Image of RA on the Parallel Plane, taken as a Picture expoſed to an Eye at S; but *b*A being indefinitely produced the contrary way beyond A, it will become the real Projection of the part AL of the Original Line, which lies beyond the Plane of the Projection, on the ſame Side of it with S; and the Projection of L will be infinitely diſtant, SL and *b*A being parallel.

PROB. X.

Two Planes being propoſed, both parallel to the Picture, and a Line in one of them being given; thence to find its Projection on the other Plane, from a Projecting Point whoſe Seats on both Planes are given.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 132. N°. 1, 2. Let O be the Center of the Picture, S the Projecting Point, *T* its Seat on the Original Plane, and T its Seat on the Plane of the Projection, and let *bc* be the given Line, whoſe Projection is deſired.

Through *T* draw any Line *T*L, cutting *bc* in L, and through T draw TL parallel to it, till it be cut by SL in *l*, through which draw $\beta\gamma$ parallel to *bc*, and $\beta\gamma$ will be the Projection deſired: through O draw O*y* parallel to *T*L, till it be cut by SL in *y*, and *ef* drawn through *y* parallel to *bc* will be the Vaniſhing Line of the Projecting Plane, the Interſecting Line of which Plane muſt be drawn parallel to it through the Interſecting Point of SL, found as already ſhewn [a].

[a] Cor. Caſe 2. Prop. 45. B. IV.

For *T* being a Point in the Original Plane in which *bc* lies, *T*L is a Line in that Plane, and T being a Point in the Plane of the Projection, T*l* is a Line in that Plane parallel to *T*L; wherefore *T*L and T*l* are in the ſame Plane with ST, in which Plane the Projecting Line S*l* alſo lies; *l* is therefore the Projection of the Point L of the given Line *bc*, and conſequently *βγ* drawn through *l* parallel to *bc* is the Projection of that Line[a]; which muſt be parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane[b]. *Q. E. I.*

CASE 2.

When the Projecting Point is at an Infinite Diſtance before or behind the Directing Plane.

Fig. 132. N°. 3.

In this Caſe, the Projecting Point having no Seat on either of the given Planes, ſome Line NO muſt be drawn having O for its Vaniſhing Point, and the Interſections N and M of that Line with the given Planes muſt be found[c]: this being done, draw SO, and from M the Interſection of NO with the Original Plane, draw ML parallel to SO, cutting *bc* in L, and through N draw N*l* alſo parallel to SO till it be cut by SL in *l*; and *βγ* drawn through *l* parallel to *bc* will be the Projection ſought, and *ef* drawn through S parallel to *bc* will be the Vaniſhing Line of the Projecting Plane.

The Demonſtration of this is in effect the ſame with that of Caſe 1. for it is evident that ML and N*l* are the Interſections of the given Planes with a Plane whoſe Vaniſhing Line is SO, in which Plane the Lines ON and SL lie, and that therefore *l* is the Projection of L. *Q. E. I.*

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Fig. 132. N°. 1.

The only Difference in this from the firſt Caſe is, that L*l* muſt be drawn parallel to *T*O the Direction of the Projecting Lines[d]. *Q. E. I.*

CASE 4.

When the Projecting Point is at an Infinite Diſtance in the Directing Plane.

In this Caſe, no Line in the Original Plane can be projected on the other Plane, they being both parallel to the Picture and to the Projecting Lines[e].

METHOD 2.

Fig. 132. N°. 4.

This Problem may alſo be ſolved by the help of any ſubſtituted Plane EFGH cutting both the given Planes in ML and N*l*[f]; for the Projection L*l* of the given Line *bc* on the Plane EFGH being found[g], a Line *lβ* drawn from *l* the Interſection of L*l* with the Plane of the Projection, parallel to *bc*, will be the Projection of *bc* on that Plane; and if L*l* be produced to its Vaniſhing and Interſecting Points *y* and *d*, *ef* and *gh* drawn through *y* and *d* parallel to *bc*, will be the Vaniſhing and Interſecting Lines of the Projecting Plane.

For it is evident, *l* is a Point of the Interſection of the Projecting Plane with the Plane of the Projection here ſuppoſed to be parallel to the Picture. *Q. E. I.*

GENERAL COROLLARY.

In all the Caſes of the foregoing Problems of this Section, if the Projection of any Line be conſidered as the Original Line, the Original Line will be its Projection, either real or imaginary, on the Original Plane taken as the Plane of the Projection; the Original Line and its Projection on any Plane being reciprocal, as they are the Interſections of the given Planes by the ſame Projecting Plane[h]; ſo that the Rules for finding the Projection of a given Line, are equally applicable to the finding the Original Line in a propoſed Plane, whoſe Projection on any other Plane is given, or for finding the Projection of a Line on one Plane from its given Projection on another Plane; and it will always be eaſy to diſtinguiſh, in the Indefinite Projection of any Line, what part of it is real and what imaginary, by the Poſition of the Projecting Point with reſpect to the Eye and the given Planes[i].

PROB. XI.

Any two Planes whoſe Vaniſhing Lines interſect, being given, toge-

ther

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[a] Gen. Cor. 1. Prob. 4.

[b] Theor. 15. B. I.

[c] Prop. 45. B. IV.

[d] Cor. 2. Caſe 3. Prob. 2.

[e] Caſe 4. Prob. 2.

[f] Prop. 38. B. IV.

[g] Prob. 8.

[h] Gen. Cor. 3. Prob. 4.

[i] Schol. Prob. 6.

ther with a Line out of thoſe Planes; thence to find the Projections of that Line on both the given Planes, from a Projecting Point whoſe Seat on either of the Planes is given.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 133. N°. 1. Let EFGH and *efgb* be the two given Planes, *dz* the given Line, S the Projecting Point, and T its Oblique Seat on the Plane EFGH.

METHOD 1.

Having found $\delta\pi$ the Parallel Seat of *dz*, and *t* the Parallel Seat of S on the Plane EFGH with reſpect to the Plane *efgb*[a], find the Parallel Seats B and C of any two Points *b* and *c* of the given Line, and draw *t*B, *t*C, cutting D*y* the Interſection of the given Planes in *B* and *C*, from whence draw *B*b, *C*c, parallel to *ef*; from S draw S*b*, cutting *B*b and *tB* in b and β, alſo draw S*c*, cutting *C*c and *tC* in c and γ; then b and c will be the Projections of *b* and *c* on the Plane *efgb*, and β and γ will be their Projections on the Plane EFGH, by which the Indefinite Projections cb and $\gamma\beta$ are determined.

For the Originals of S*t*, *b*B, and b*B* being parallel, they are in the ſame Plane with S*b* and *t*B, the Interſections of which Plane with the Planes *efgb* and EFGH are b*B* and *tB*; wherefore b is the Projection of *b* on the Plane *efgb*, and β its Projection on the Plane EFGH: and in the ſame manner it is proved, that c and γ are the Projections of *c* on the ſame two Planes reſpectively. *Q. E. I.*

SCHOL.

It is no wiſe material whether the Projections of the Points thus found be real or imaginary, or whether they be in or out of Sight with reſpect to the given Planes, they being alike ſerviceable in either Caſe for determining the Indefinite Projections required[b].

COR.

After this manner, the Projection b of any Point *b* which can fall on the Plane *efgb* from the Point S, may be found by the Parallel Seats B and *t* of the Points *b* and S on any other Plane EFGH with reſpect to the Plane *efgb*.

METHOD 2.

Produce T*t* till it cut $\delta\pi$ in L, and find the Point *l* in *dz*, whoſe Parallel Seat is L, and draw S*l*; then through *z* and *d* draw *zx*, *d*Q, parallel to S*l*, and theſe will be the Vaniſhing and Interſecting Lines of the Projecting Plane[c], and conſequently Q*v* and R*x*, the Interſections of the Planes *efgb* and EFGH with the Plane *zxd*Q, are the Indefinite Projections ſought. *Q. E. I.*

COR.

The ſame Point *l*, and conſequently S*l*, may be found by the Oblique Seat of *dz* on the Plane EFGH, as well as by its Parallel Seat; or by the Interſection of the Plane EFGH wirh any Plane whatever paſſing through *dz*; ſo as L*l* and S*t* be drawn parallel to the Vaniſhing Line of that Plane, by which means they will always repreſent Lines parallel to the Picture.

METHOD 3.

Draw *t*π and S*z* meeting in *r*, and through *q*, where *t*π and D*y* interſect, draw *q*ϱ parallel to *ef*, cutting S*z* in ϱ; then *r* will be the *Focus* of R*x* the Projection of *dz* on the Plane EFGH, and ϱ will be the *Focus* of its Projection Q*v* on the Plane *efgb*, by which *Foci* and any one other Point in either of the Projections, both of them may be determined; ſeeing either of the Indefinite Projections R*x* or Q*v* being found, its Interſection a, with the given Planes, gives a Point in the other Indefinite Projection.

That *r* is the *Focus* of the Projection of *dz* on the Plane EFGH is evident, ſeeing *r* is the Interſection of that Plane with the Projecting Line S*z*[d]: in the next place *q* being a Point in the Interſection of the Plane *efgb* with a Plane paſſing through the Projecting Point parallel to the Plane *z*$\pi\delta$*d*, whoſe Vaniſhing Line *z*π is

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[a] Def. 15, and 16. B. IV.

[b] Schol. 2. Prob. 6.

[c] Meth. 3. Prob. 6.

[d] Cor. 1. Caſe 1. Prob 3.

32 No. 4.

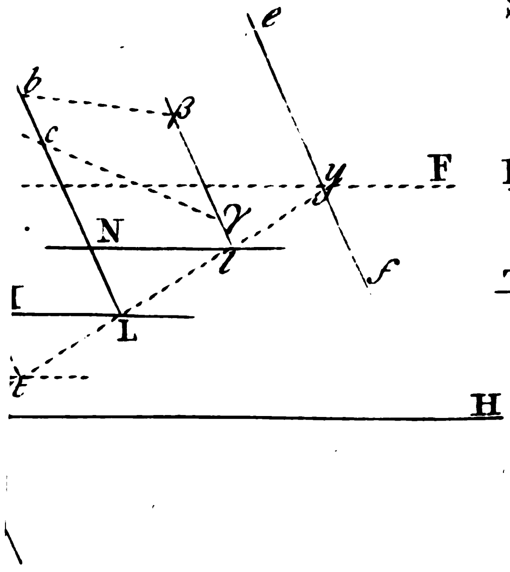


Fig 133 No. 1.

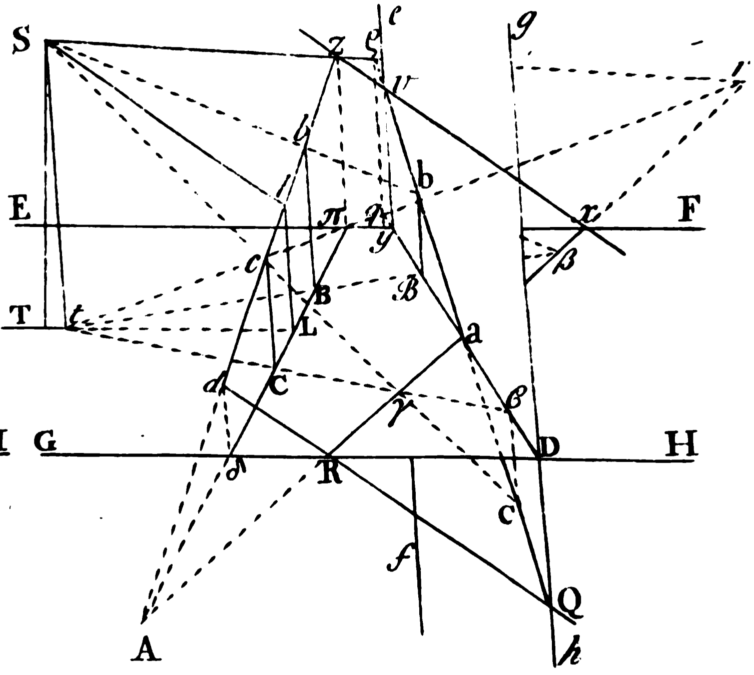


Fig. 133 No. 2.

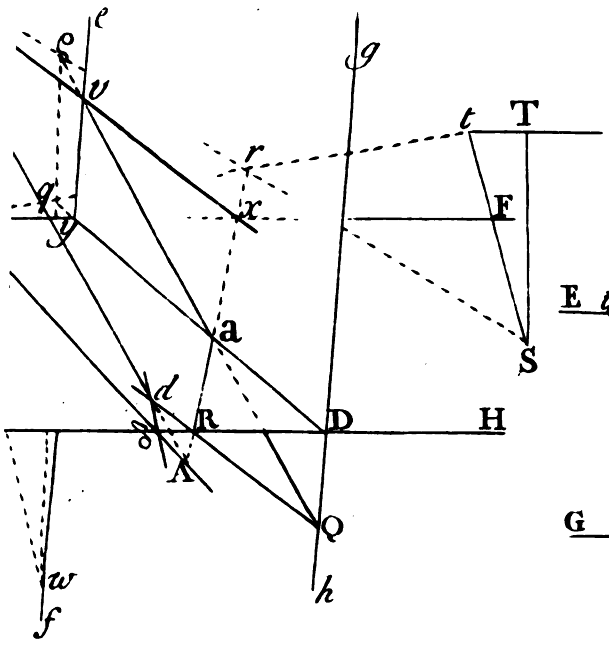


Fig. 133 No. 3.

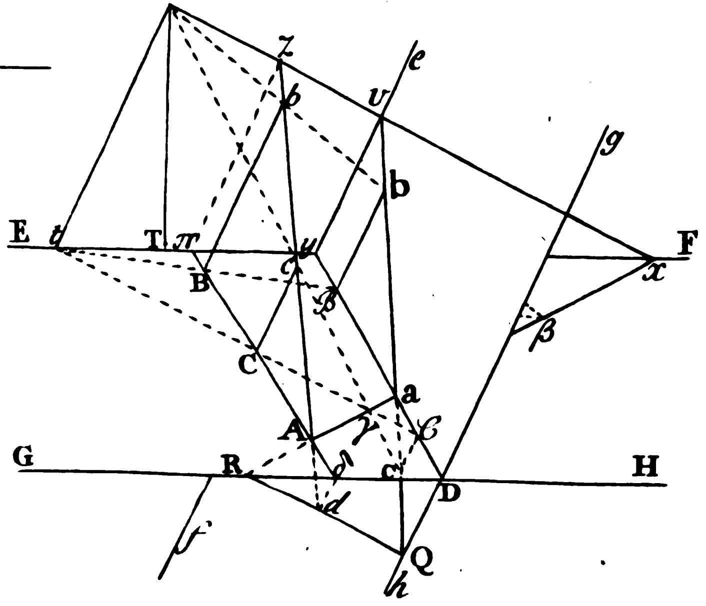


Fig. 133 No. 4.

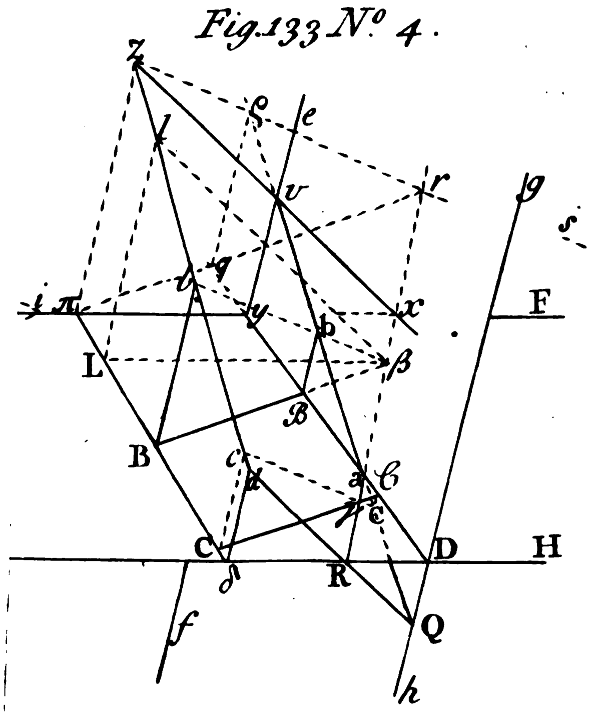
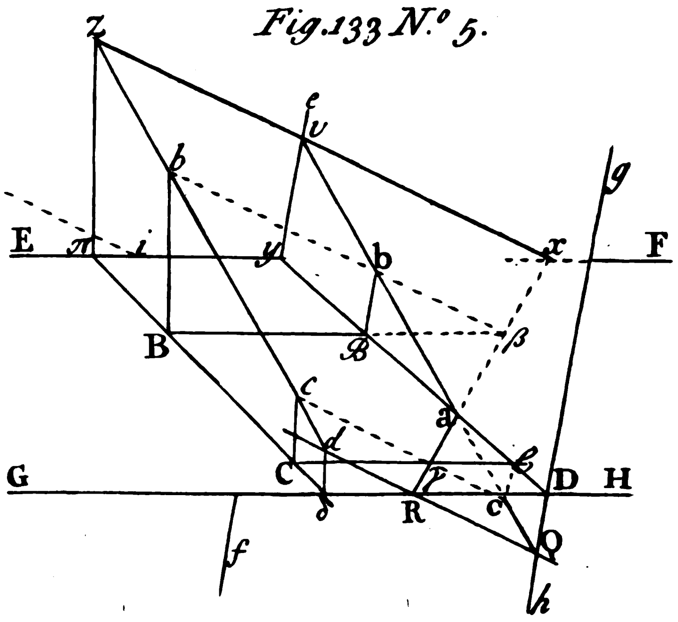


Fig. 133 No. 5.



J. Mynde fc.

is parallel to ef, $q\varrho$ parallel to ef is the Indefinite Image of the Interſection of thoſe two Planes [a], and is therefore the Line of the *Foci* of the Projections of all Lines in the Plane $z\pi d\delta$ on the Plane $efgb$ [b]; wherefore ϱ, the Interſection of $q\varrho$ with Sz, is the *Focus* of the Projection of dz on that Plane. *Q. E. I.*

COR. 1.

Fig. 133. N°. 2. It is not neceſſary, in order to find the *Foci* of the Projections, that the Parallel Seat of the given Line dz on the Plane EFGH with reſpect to the Plane $efgb$, ſhould be uſed; but the Line dz may be reduced into any other Plane $z\pi d\delta$, whoſe Vaniſhing Line $z\pi$ may cut ef in any Point w; only in ſuch Caſe, the Parallel Seat t of the Projecting Point with reſpect to the aſſumed Plane $z\pi d\delta$, muſt be found; and then $t\pi$ will be the Line of the *Foci* of the Projections of all Lines in the Plane $z\pi d\delta$ on the Plane EFGH, and conſequently r will be the *Focus* of the Projection of dz on that Plane; and a Line drawn from w through q, the Interſection of $t\pi$ with Dy, will be the Line of the *Foci* of the Projections of all Lines in the Plane $z\pi d\delta$ on the Plane $efgb$, and conſequently ϱ will be the *Focus* of the Projection of dz on that Plane; ſeeing w is the Vaniſhing Point, and q is another Point of the Interſection of the Plane $efgb$, with a Plane $z\pi St$ paſſing through the Projecting Point parallel to the Plane $z\pi d\delta$.

COR. 2.

If $t\pi$ and Dy ſhould happen to be parallel, then the Point q which marks the Interſection of thoſe Lines will be infinitely diſtant, that is, it will be their common Directing Point [c], which will alſo be the Directing Point of the Line of the *Foci* of the Projections of Lines in the Plane $z\pi d\delta$ on the Plane $efgb$; which Line muſt therefore be drawn through its Vaniſhing Point w, parallel to Dy or $t\pi$.

And if the Vaniſhing Lines $z\pi$ and ef be parallel, as well as $t\pi$ and Dy, then the Interſection of the Planes $z\pi St$ and $efgb$ being their common Directing Line [d], that Line, which is alſo the Line of the *Foci* of the Projections on the Plane $efgb$, can have no Repreſentation; and in this Caſe, the Projection of any Line dz in the Plane $z\pi d\delta$ on the Plane $efgb$ will be parallel to Sz, ſeeing the *Focus* of the Projection of dz on that Plane is then the Directing Point of Sz.

CASE 2.

When the Projecting Point is at an infinite Diſtance before or behind the Directing Plane.

Fig. 133. N°. 3. Here, the Projections bc and $\beta\gamma$ of the given Line dz on the Planes $efgb$ and EFGH, are found as in the firſt Method of the laſt Caſe; and the ſecond and third Methods become the ſame, Sz being the Vaniſhing Line of the Projecting Plane; and the *Foci* of the Projections Qv and Rx coincide with their Vaniſhing Points v and x, the Lines of the *Foci* $t\pi$ and $q\varrho$ coinciding with EF and ef. *Q. E. I.*

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Fig. 133. N°. 4. All the Difference between this and Caſe 1. is, that here in the firſt Method, $b\beta$ and Bβ muſt be drawn parallel to si and ti, the Directions of the Projecting Lines and their Parallel Seats on the Plane EFGH with reſpect to the Plane $efgb$ [e].

In the ſecond Method, zx and dQ, the Vaniſhing and Interſecting Lines of the Projecting Plane, are determined either by finding the Directing Line of that Plane, or by the help of a Point β of the Projection, and the Triangle lLβ [f].

And in the third Method, the *Focus* r of the Projection Rx is found by the Interſection of zr and πr, drawn parallel to the Directions si and ti; the Point q is found by the Interſection of πr with Dy, and $z\pi$ and ef being parallel, $q\varrho$ drawn parallel to them, by its Interſection with zr, determines ϱ the *Focus* of the Projection Qv. *Q. E. I.*

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

Fig. 133. N°. 5. Here the Projecting Lines, as alſo the Vaniſhing and Interſecting Lines of the Projecting Plane, are parallel to si, and the Seats of the Projecting Lines are parallel to the Picture [g]; wherefore it is not neceſſary that $\pi\delta$ ſhould be the Parallel Seat of dz on the Plane EFGH with reſpect to the Plane $efgb$, but it may be its Oblique Seat on the Plane EFGH, or the Interſection of that Plane with any Plane paſſing through

[a] Theor. 15. B. I.

[b] Cor. 1. Meth. 4. Caſe 1. Prob. 6.

[c] Cor. 5. Theor. 12. B. I.

[d] Cor. Caſe 2. Prop. 46. B. IV.

[e] Caſe 3. Prob. 6.

[f] Caſe 3. Prob. 3. and Cor.

[g] Caſe 4. Prob. 6.

dz; obſerving only that bB and cC muſt always be drawn parallel to $z\pi$ the Vaniſhing Line of the Plane which paſſes through dz, and bB, Cc, muſt be drawn parallel to ef. *Q. E. I.*

GENERAL COROLLARY.

Fig. 133. N°. 6. If the propoſed Line Ab be parallel to the Picture; then if the Projecting Point be at a moderate Diſtance before or behind the Directing Plane, the Projection Ax of the given Line on the Plane EFGH, is found by t the Parallel Seat of the Projecting Point with reſpect to Ab, and A the Interſection of that Line with the Plane EFGH, as in the Figure[a]; and the given Line being parallel to the Vaniſhing and Interſecting Lines of its Projecting Plane[b], xv parallel to Ab, by its Interſection with ef, gives v the Vaniſhing Point of its Projection on the Plane $efgb$, by which, and the Point a, the Projection vb is found, and thereby b, the Interſection of Ab with that Plane, and as t is the *Focus* of the Projection of Ab on the Plane EFGH, ſo if St be produced to its Interſection $\mathcal{T}$ with the Plane $efgb$ (found by tL and L$\mathcal{T}$ drawn parallel to EF and ef) $\mathcal{T}$ will be the Parallel Seat of S, as well as the *Focus* of the Projection of Ab on that Plane.

[a] Gen. Cor. 2. Prob. 3.

[b] Theor. 15. B. I.

Fig. 133. N°. 7. When the Projecting Point is at a moderate Diſtance in the Directing Plane, the Directions ty and $\mathcal{T}y$ of the Projections Ax and bv are found by uſing the given Planes inſtead of their Directing Planes, and taking y as the Place of the Eye[c]; for then st drawn through s the Repreſentation of the Projecting Point, parallel to Ab, gives t and $\mathcal{T}$ its Parallel Seats on GH and gb taken as the Directing Lines of thoſe Planes, whence the Projections are determined as in the Figure[d].

[c] Cor. 2. Caſe 3. Prob. 1.

[d] Gen. Cor. 2. Prob. 3.

When the Projecting Point is at an infinite Diſtance before or behind the Directing Plane, the *Foci* and Vaniſhing Points of the Projections coincide, and the two laſt Methods propoſed at Caſe 1. become the ſame[e].

[e] Caſe 2.

And laſtly, when the Projecting Point is at an infinite Diſtance in the Directing Plane, the Projections are parallel to the Picture, and conſequently to the Vaniſhing Lines of the reſpective Planes, the ſame with AB and Bb in the Figure.

PROB. XII.

Any two Planes whoſe Vaniſhing Lines are either parallel, or coincide, being given, together with a Line out of thoſe Planes; thence to find the Projections of that Line on both the given Planes, from a Projecting Point whoſe Seat on either of the Planes is given.

CASE 1.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane.

Fig. 134. N°. 1. Let $efgb$ and EFGH be the given Planes, dz the given Line, S the Projecting Point, and T its Seat on the Plane EFGH.

METHOD 1.

Draw zy perpendicular to EF, and compleat the ſubſtituted Plane zydδ paſſing through dz, and cutting the given Planes in yδ and yd the Seats of dz on thoſe Planes; and with the ſame Vaniſhing Line zy compleat another ſubſtituted Plane zyΔD paſſing through ST the Support of the Projecting Point, whereby $\mathcal{T}$ the Seat of that Point on the Plane $efgb$ is found: from T draw TL parallel to EF, cutting yd in L, and having drawn Ll parallel to zy, cutting dz in λ, draw Sλ which will be parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane, whence the Projecting Plane zvQR is found, the Interſections Rx and Qv of which Plane with the given Planes are the Indefinite Projections required.

For it is evident Sλ is a Line in the Projecting Plane parallel to the Picture. *Q. E. I.*

COR.

The ſame Point λ, and conſequently Sλ, may be found by $\mathcal{T}l$ drawn from $\mathcal{T}$ parallel to ef, cutting yδ in l; ſeeing l is the Seat of the ſame Point λ of the given Line dz, on the Plane $efgb$.

METHOD

Fig. 133. No. 6.

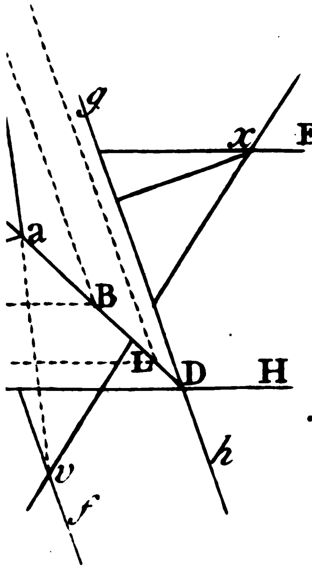


Fig. 133. No. 7.

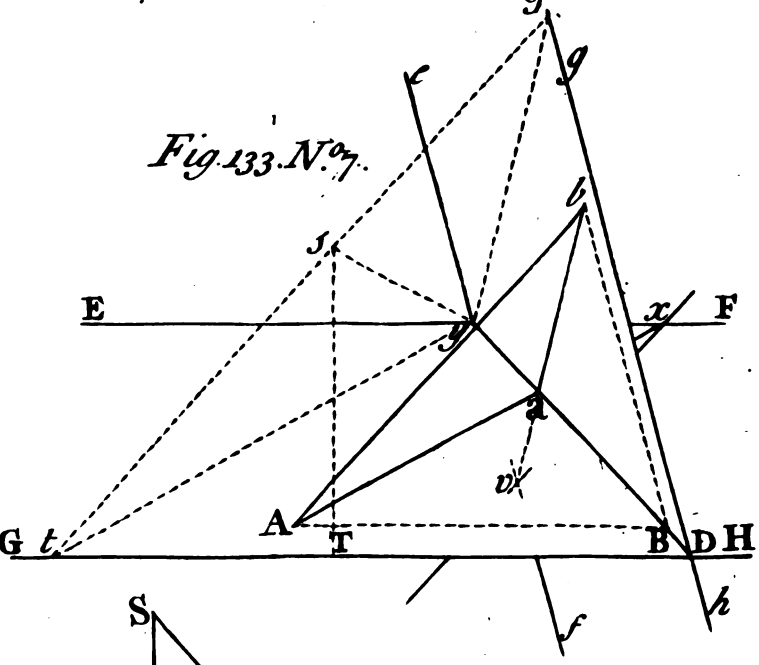
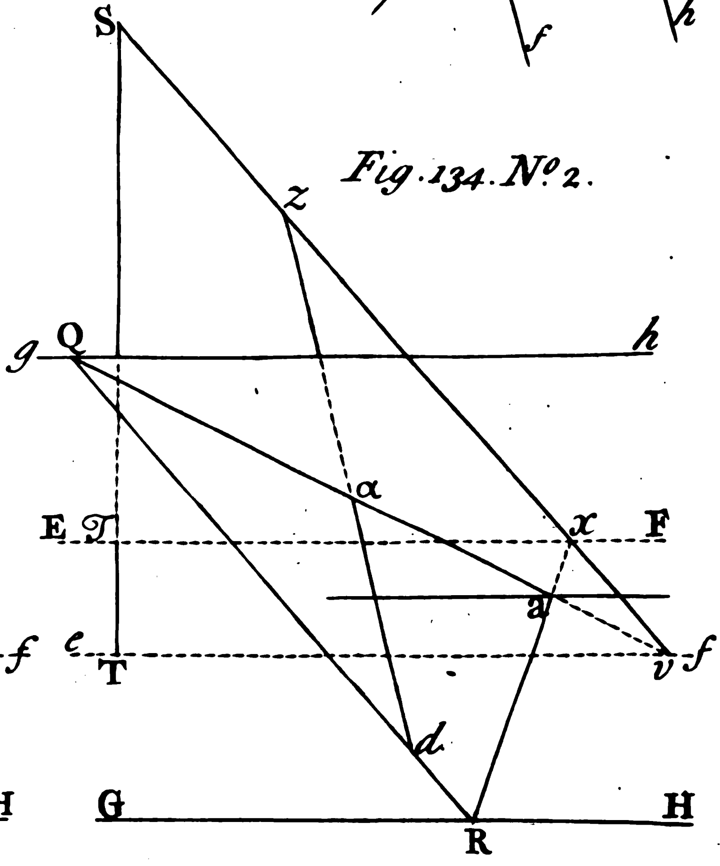


Fig. 134. No. 2.



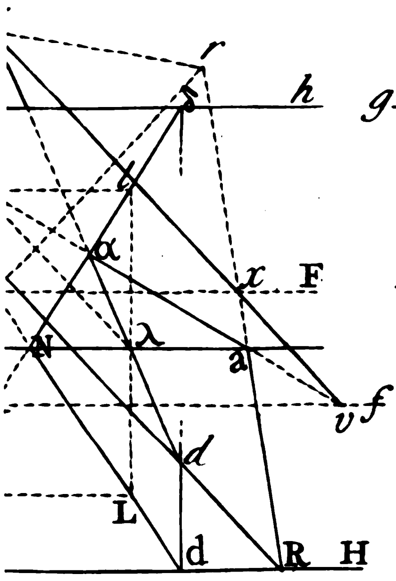
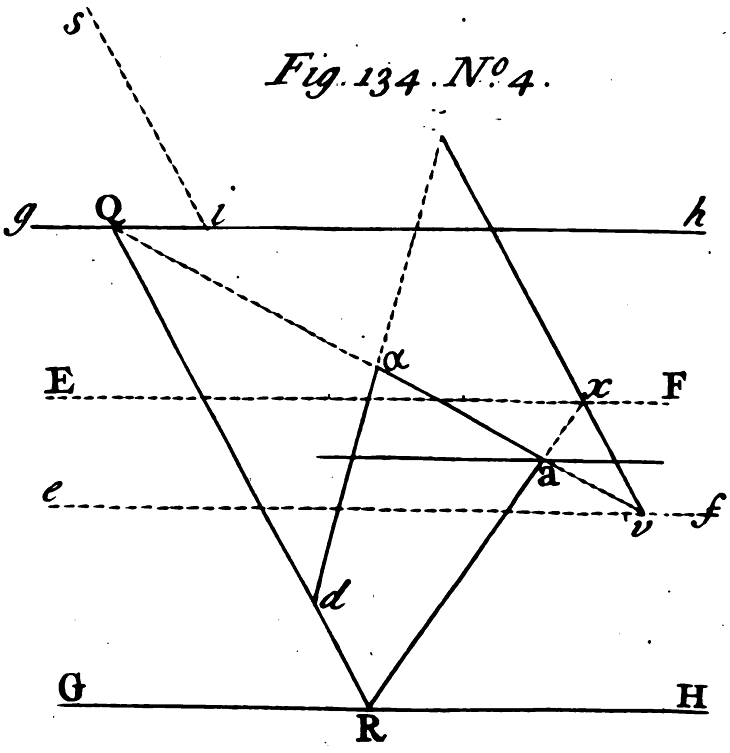
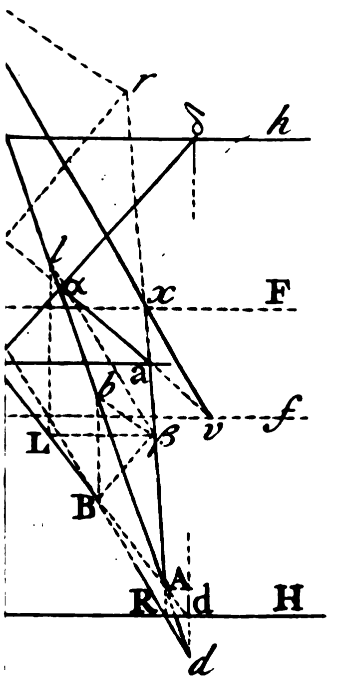


Fig. 134. No. 4.





J Mynde sc

METHOD 2.

Through S and z draw Sz, cutting yD and $y\Delta$ in r and ϱ, and r will be the *Focus* of the Projection Rx, and ϱ the *Focus* of the Projection Qv.

For r is the Interſection of the Plane $EFGH$ with Sz, a Line paſſing through the Projecting Point and the Vaniſhing Point of dz, whoſe Originals are therefore parallel, and ϱ is the Interſection of the ſame Line with the Plane $efgh$. *Q. E. I.*

COR.

Here, $y\Delta$ and yD are the Lines of the *Foci* of the Projections of all Lines in the Plane $zyd\delta$, or in any other Plane whoſe Vaniſhing Line is zy, on the Planes $efgh$ and $EFGH$ reſpectively; they being the Interſections of thoſe Planes with a Plane paſſing through the Projecting Point parallel to the Planes zy.

METHOD 3.

The required Projections may alſo be found by the Projections of any two Points of the given Line on both the given Planes; which is ſufficiently obvious, without incumbering the Figure with drawing the Lines. *Q. E. I.*

SCHOL.

It is not neceſſary that the ſubſtituted Plane $zyd\delta$ ſhould be the Plane of the Seats of dz on the given Planes; for any other ſubſtituted Plane paſſing through dz will equally ſerve the purpoſe, provided the ſubſtituted Plane $zy\Delta D$ be made to paſs through the Parallel Support of the Projecting Point with reſpect to the other ſubſtituted Plane.

CASE 2.

When the Projecting Point is at an infinite Diſtance before or behind the Directing Plane.

Fig. 134. N°. 2. Here, the two firſt Methods of the preceeding Caſe become the ſame, Sx being itſelf the Vaniſhing Line of the Projecting Plane, and the *Foci* of the Projections coinciding with their Vaniſhing Points; and the Indefinite Projections Rx and Qv are found without the help of any ſubſtituted Planes, which are not in this Caſe neceſſary, unleſs thoſe Projections be required to be found by the help of the Projections of two Points of the given Line on the propoſed Planes. *Q. E. I.*

In the Figures here referred to, the Projecting Point is before the Eye, but it is ſo eaſy to apply the ſame Rules when the Projecting Point is behind the Eye, that the Figures are not drawn.

CASE 3.

When the Projecting Point is at a moderate Diſtance in the Directing Plane.

Fig. 134. N°. 3. Having found the Directions sy and Ty of the Projecting Lines and their Seats on the Plane $EFGH$, find β the Projection of any Point b of the given Line dz on that Plane, and thence $l\beta$ the Parallel to the Vaniſhing and Interſecting Lines of the Projecting Plane [a], by which thoſe Lines, and conſequently the Indefinite Projections Rx and Qv are determined: and if the Directions $y\mathrm{T}$ and yT of the Seats of the Projecting Lines on both the given Planes, be ſo placed as to form a ſubſtituted Plane $yyT\mathrm{T}$ parallel to $zyd\delta$ the Plane of the Seats of the given Line, as in the Figure [b], $y\mathrm{T}$ will be the Line of the *Foci* of the Projections on the Plane $EFGH$ of all Lines in the Plane $zyd\delta$, and conſequently r, where it is cut by zr drawn parallel to the Direction sy, will be the *Focus* of the Projection Rx; and in like manner yT will be the Line of the *Foci* of the Projections on the Plane $efgh$, and its Interſection with zr (if within reach) will be the *Focus* of the Projection Qv. *Q. E. I.*

COR.

The Direction of the Vaniſhing and Interſecting Lines of the Projecting Plane may be likewiſe had by finding the Directing Line of that Plane [c].

The Scholium at the End of the firſt Caſe is alſo applicable here.

CASE 4.

When the Projecting Point is at an infinite Diſtance in the Directing Plane.

Fig. 134. N°. 4. Here, the Vaniſhing and Interſecting Lines of the Projecting Plane are parallel to the Direction si of the Projecting Lines, and the Indefinite Projections Rx and Qv are found as in the Figure, which needs no farther Explanation. *Q. E. I.*

SCHOL.

[a] Caſe 3. Prob. 3. and Cor.

[b] Caſe 3. Prob. 7.

[c] Meth. 3. Caſe 3. Prob. 6.

SCHOL.

In all the Figures uſed in this Problem, the Vaniſhing Lines of the given Planes are parallel, and fall between their Interſecting Lines, but the Methods are in every reſpect the ſame which ever way thoſe Lines fall; and when the Vaniſhing Lines of the given Planes coincide, the Practice is ſtill eaſier, the Vaniſhing Points of the Projections on both the given Planes being the ſame: all which is ſufficiently evident without multiplying Figures, which any one may eaſily draw to ſatisfy himſelf in all the Variety of Caſes that can happen.

PROB. XIII.

Any two Planes, both parallel to the Picture, being propoſed, and a Line out of thoſe Planes being given; thence to find the Projections of that Line on both the given Planes, from a Projecting Point whoſe Seat on either of the Planes is given.

METHOD 1.

This is done by firſt finding the Projection of the given Line on either of the propoſed Planes[a], and uſing that Projection as a given Line in that Plane, and thence finding its Projection on the other Plane, which will alſo be the Projection of the Line firſt propoſed[b]. *Q. E. I.*

[a] Prob. 4.

[b] Prob. 10. and Gen. Cor.

METHOD 2.

Or, it may be more conveniently done, by the help of any ſubſtituted Plane taken at pleaſure, and cutting both the given Planes and the Picture[c]; for the Projection of the given Line on the ſubſtituted Plane being found[d], and uſed as if it were the Line propoſed, its Projection on either or both of the Parallel Planes may be thence determined[e]. *Q. E. I.*

[c] Meth. 2. Prob. 10.

[d] Prob. 3.

[e] Prop. 9.

GENERAL COROLLARY.

If two Planes be given, the one inclining and the other parallel to the Picture, the Projections of any given Line out of thoſe Planes, may be found on both of them, by the Method laſt propoſed.

For the Projection of the propoſed Line on the inclining Plane being found[f], its Projection on the Parallel Plane is thence determined[g].

[f] Prob. 3.

Prob. 9.

PROB. XIV.

Any Original Plane, and in it the Image of a Parallelogram any wiſe ſubdivided by Lines parallel to its Sides, being given; thence to find its Projection on any two or more Planes, from a Projecting Point whoſe Seat on any one of the propoſed Planes is given.

CASE 1. *and* 3.

When the Projecting Point is at a moderate Diſtance.

Fig. 135. No. 1.

Let $efgh$ be the Original Plane, and $abcd$ the given Image of a Parallelogram in that Plane, and let $EFGH$ and $\varepsilon\phi gh$ be two other Planes, on which the Projection of $abcd$ is required from a Projecting Point S at a moderate Diſtance before the Eye, whoſe Seat T on the Plane $EFGH$ is given; and let Dy and $\Delta\zeta$ be the Interſections of the Plane $EFGH$ with the Planes $efgh$ and $\varepsilon\phi gh$.

METHOD 1.

Having found t the Parallel Seat of S on the Plane $EFGH$ with reſpect to the Original Plane $efgh$, draw ty cutting $\Delta\zeta$ in q, and through y the Interſection of the Vaniſhing Lines ef and $\varepsilon\phi$, and the Point q, draw yq; produce the Sides ab and bc of the given Parallelogram to their Vaniſhing Points z and x, and draw Sz, Sx, cutting ty and yq reſpectively in v, w, v, and w; then v and w will be the *Foci* of the Projections on the Plane $EFGH$ of all Lines in the given Parallelogram whoſe Vaniſhing Points are z and x; and v and w will be the *Foci* of the Projections of the ſame Lines reſpectively on the Plane $\varepsilon\phi gh$; ty being the Line of the *Foci* of the Projections of all Lines in the Plane $efgh$ on the Plane $EFGH$[h], and yq being the Line of the *Foci* of the Projections of the ſame Lines on the Plane $\varepsilon\phi gh$[i].

[h] Cor. 1. Meth. 4. Caſe 1. Prob. 6.

[i] Cor. 1. Meth. 3. Caſe 1. Prob. 11.

This being done, find α the Projection of any convenient angular Point a of the Paral-

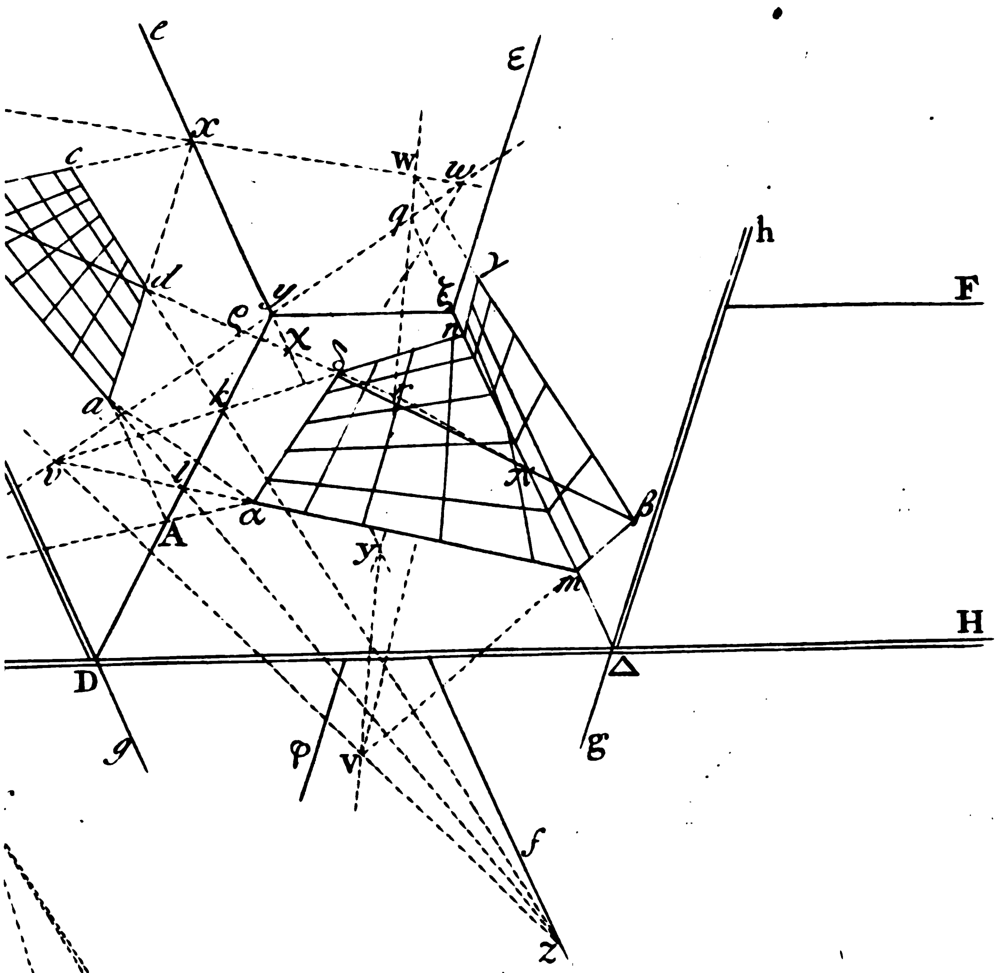
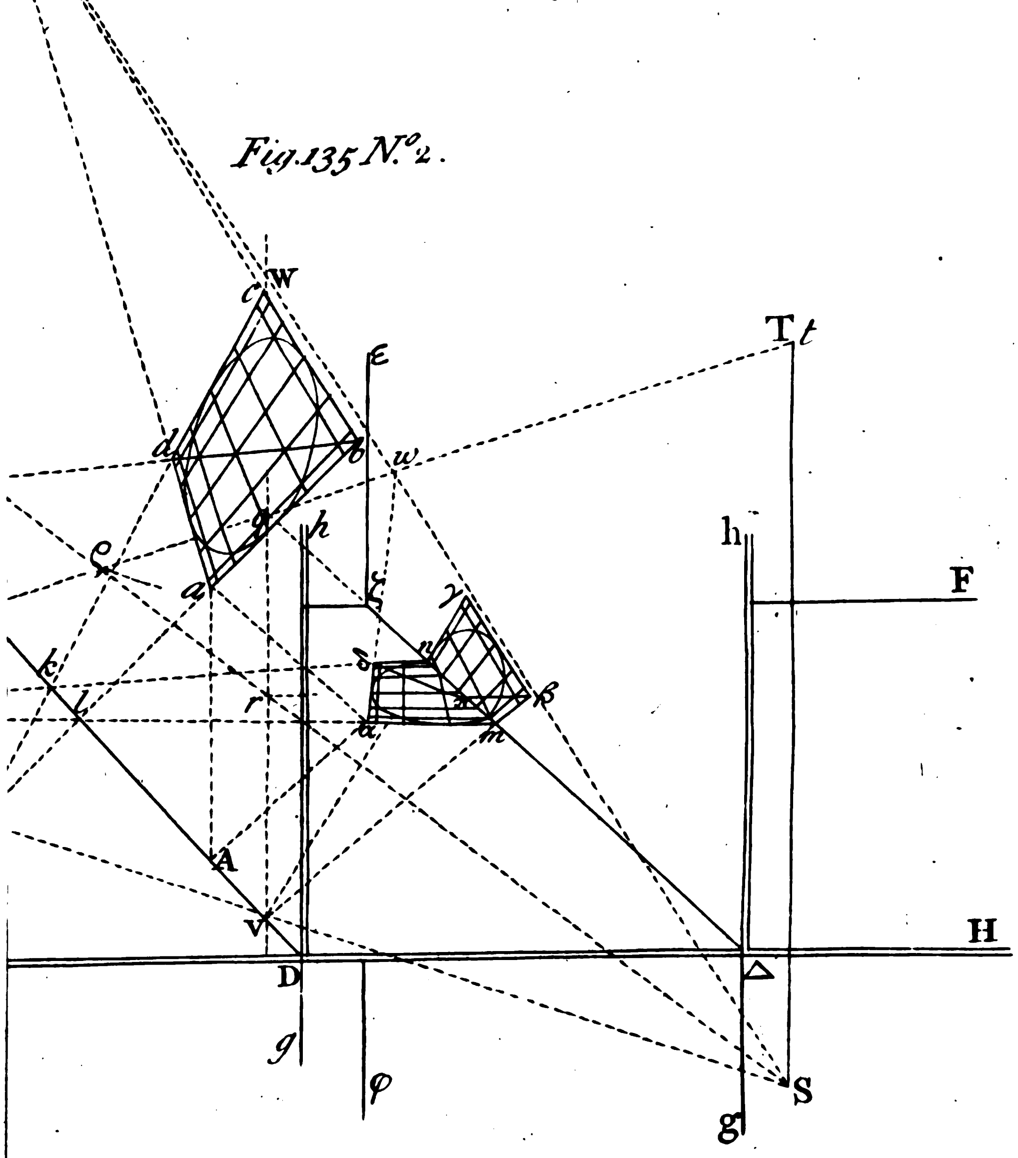


Fig. 135 N°. 2.



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Parallelogram on the Plane EFGH[a], by the help of which and of the *Foci* v and w, the Indefinite Projections αm and $\alpha\delta$ of its Sides ab and ad on that Plane are determined, and δ the Projection of d being found in $\alpha\delta$[b], $v\delta$ gives the Indefinite Projection δn of the Side dc; and the Points m and n where αm and δn cut $\Delta\zeta$, being alſo Points of the Projections of ab and dc on the Plane $\epsilon\phi$gh, theſe by the help of the *Focus* v, give $m\beta$ and $n\gamma$ the Indefinite Projections of ab and cd on that Plane; and the Projection β or γ of either of the Points b or c being found, a Line drawn from thence to the *Focus* w gives $\beta\gamma$ the Projection of bc on the ſame Plane: and thus the Projection $\alpha\beta\gamma\delta$ of the propoſed Parallelogram is compleated, of which the part $\alpha mn\delta$ falls on the Plane EFGH, and the Remainder $m\beta\gamma n$ on the Plane $\epsilon\phi$gh.

Laſtly, the Projections of the Diviſions of ab being found on $\alpha m\beta$, theſe will be Points in the Projections of the Diviſions of the Parallelogram which are parallel to the Side ad; wherefore ſuch of them as fall on the Plane EFGH muſt tend to the *Focus* w, and thoſe which fall on the Plane $\epsilon\phi$gh muſt be drawn to w; obſerving that wherever the Projection of any of theſe Lines cuts $\Delta\zeta$, it is a Point common to its Projection on both the Planes, by which and its proper *Focus* on either of the Planes, the Remainder of its Projection which falls on that Plane is determined.

In like manner, the Projections of the Diviſions of bc being found in $\beta\gamma$, theſe by the help of the *Focus* v, give ſo much of the Projections of the Diviſions parallel to ab as fall on the Plane $\epsilon\phi$gh, and their Interſections with $\Delta\zeta$ by the help of the *Focus* v, give the Remainder of the Projections of thoſe Lines which fall on the Plane EFGH. *Q. E. I.*

Fig. 135. N°. 2.

In Fig. N°. 2. $abcd$ repreſents a Square circumſcribing a Circle, and ſubdivided according to Meth. 1. Prob. XXIV. Book II. the Projecting Point S is at a moderate Diſtance behind the Eye, and the Vaniſhing Line ef of the Original Plane being perpendicular to EF, the Parallel Seat t of the Projecting Point S on the Plane EFGH with reſpect to the Plane $efgh$, coincides with T its Oblique Seat on that Plane; alſo the Vaniſhing Lines $\epsilon\phi$ and ef being parallel, vw the Line of the *Foci* of the Projections on the Plane $\epsilon\phi$gh is drawn through q parallel to ef[c]; but the reſt of the Practice is the ſame as before. *Q. E. I.*

Fig. 135. N°. 3.

In Fig. N°. 3. $abcd$ alſo repreſents a Square circumſcribing a Circle; the Projecting Point is at a moderate Diſtance in the Directing Plane, and si and ti are the Directions of the Projecting Lines and their Parallel Seats on the Plane EFGH with reſpect to the Original Plane $efgh$, which are here the ſame with their Oblique Seats on that Plane, the Vaniſhing Lines ef and EF being perpendicular: beſides the Planes EFGH and $\epsilon\phi$gh, there is alſo a third Plane RLNB parallel to the Picture, on which part of the Projection falls, the Interſections of which Plane with the three other Planes are RL, LN, and NB; vw the Line of the *Foci* of the Projections on the Plane EFGH is drawn through y parallel to the Direction ti[d]; the Interſection of vy with $\Delta\zeta$ gives q, by which, and the Interſection y of the Vaniſhing Lines ef and $\epsilon\phi$, the Line vw of the *Foci* of the Projections on the Plane $\epsilon\phi$gh is found, as in Fig. N°. 1. laſtly mn the Line of the *Foci* of the Projections on the Parallel Plane RLNB is found by drawing it parallel to ef, either through M the Interſection of LN with vw, or through μ the Interſection of BN with vw.

For M is the Point where LN the common Interſection of the Planes RLNB and EFGH cuts the Plane efv, which paſſes through the Projecting Point parallel to the Plane $efgh$; and μ is the Point where NB the common Interſection of the Planes RLNB and $\epsilon\phi$gh cuts the ſame Plane efv, vw and vw being both Lines in this laſt Plane; wherefore M and μ are both of them Points in the Interſection of the Planes RLNB and efv, and conſequently mn drawn through M or μ parallel to ef is the common Interſection of thoſe Planes[e], and is therefore the Line of the *Foci* of the Projections on the Plane RLNB.

The Lines of the *Foci* of the Projections on all the three Planes being thus determined, the particular *Foci* are found by the Interſections of thoſe Lines reſpectively with Lines drawn through z and x parallel to the Direction si[f], by the help of which *Foci*, the intire Projection $\alpha\pi\delta\gamma k\beta p$ is found on the three propoſed Planes, by the like Proceſs as before; of which the part $\alpha\pi Np$ falls on the Plane EFGH, having v and w for its *Foci*, the part $pNk\beta$ falls on the Plane $\epsilon\phi$gh, and hath v and w for its *Foci*, and the part $\pi\delta\gamma k$N falls on the Parallel Plane RLNB, of which part n and m are the *Foci*. *Q. E. I.*

[a] Cor. Meth. 1 or Meth. 4. Caſe 1. Prob. 6.

[b] Gen. Cor. 2. Prob. 4.

[c] Meth. 3: Caſe 1. Prob. 11.

[d] Meth. 4. Caſe 3. Prob. 6.

[e] Cor. Theor. 3. B. I.

[f] Meth. 4. Caſe 1. and 3. Prob. 6.

METHOD 2.

Fig. 135. N°. 1, 2.

If either of the Diagonals *bd* of the given Figure be produced to its Vaniſhing Point χ, and the *Foci* ϱ and *r* of its Projections on the propoſed Planes be found, by the Interſections of Sχ with *vw* and vw; thoſe *Foci*, with the help of the *Foci* of the Projections of the Sides of the Figure, will be ſufficient to determine its intire Projection on the propoſed Planes, the Projection $\alpha\delta$ of any one of its Sides *ad* being given.

For $\varrho\delta$ gives $\delta\pi$ ſo much of the Projection of the Diagonal as falls on the Plane EFGH, and $r\pi$ gives $\pi\beta$ the Remainder of it which falls on the Plane $\varepsilon\varphi$gh, which alſo determines β in the Line $m\beta$, and conſequently $\beta\gamma$: and if the Diviſions of the propoſed Figure be proportional, ſo as to paſs through the ſame correſponding Points of the Diagonal[a] (as in Fig. N°. 2.) the Projections of the Diviſions parallel to the Side *ab* being found, their Interſections with the Projection of the Diagonal will be Points of the Projections of the correſponding Diviſions which are parallel to *ad*, by which and their reſpective *Foci* they may be found. *Q. E. I.*

After this manner, the Projection of any given Triangle *abd* may be determined, having the Projection of any one of its Sides given.

COR. 1.

Fig. 135. N°. 1.

If the Point *q* ſhould be inconvenient for determining vw the Interſection of the Planes $\varepsilon\varphi$gh and *efSt*, as it will be, when it falls too cloſe to y the Interſection of the Vaniſhing Lines $\varepsilon\varphi$ and *ef*; the Interſection of the Plane $\varepsilon\varphi$gh with S*t*, or with any other Line parallel to it in the Plane *efSt*[b], will give another Point in vw, whereby it may be determined: or if none of the Methods here propoſed, for finding the Interſections of the Plane *efSt* with any of the other Planes, ſhould be convenient, thoſe Interſections may be found by ſome or other of the Methods before ſhewn[c].

COR. 2.

Fig. 135. N°. 1, 2.

If either of the Sides *ab* of the given Figure be parallel to *ef* and conſequently to the Picture, the *Focus* of the Projections of that Side and its Parallels on either of the propoſed Planes is in the Interſection of S*t* with the Line of the *Foci* of the Projections on that Plane.

For the Lines of the *Foci* of the Projections on all the propoſed Planes being the Interſections of thoſe Planes with the Plane *efSt*[d], they are all Lines in this laſt Plane, and conſequently in the ſame Plane with S*t*, which muſt therefore, if produced, cut them all, if it be parallel to none of them; which Interſections being the Interſections of the propoſed Planes with a Line S*t* paſſing through the Projecting Point parallel to the propoſed Line *ab* (here ſuppoſed parallel to the Picture) they are therefore the *Foci* of the Projections of *ab* and its Parallels, on the reſpective Planes propoſed[e].

COR. 3.

When the Projecting Point is at a moderate Diſtance before or behind the Directing Plane, the *Foci* of the Projections of all Lines in the propoſed Figure which are parallel to the Picture, have real Images, to which the Projections tend; except only when the Line of the *Foci* of the Projections on either of the propoſed Planes happens to be parallel to S*t*, in which Caſe that particular *Focus* becomes infinitely diſtant, and the Projections of the propoſed Lines on that Plane are then alſo parallel to S*t*.

COR. 4.

When the Projecting Point is at a moderate Diſtance in the Directing Plane, the Projections of all Lines in the propoſed Figure which are parallel to the Picture, on any Plane whatſoever, are parallel to the reſpective Lines of the *Foci* of the Projections on thoſe Planes.

For in this Caſe, S*t* lying wholly in the Directing Plane[f], it can cut the Lines of the *Foci* only in their Directing Points, which Interſections are the imaginary *Foci* of the Projections on the ſeveral Planes[g]; wherefore theſe Projections are parallel to the reſpective Lines of the *Foci*, ſeeing their Directing Points are the ſame[h]; and when the Line of the *Foci* of the Projections on any Plane is parallel to *ef* or S*t*, their Interſection being infinitely diſtant, the Projections of the propoſed Lines on that Plane muſt be parallel to *ef*, as in the preceding Corollary.

COR.

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[a] Cor. Caſe 3. Prob. 20. B. II.

[b] Gen. Cor. Prob. 11.

[c] Prop. 46. B. IV.

[d] Cor. 1. Meth. 4. Caſe 1. Prob. 6.

[e] Def. 11.

[f] Caſe 3. Prob. 6.

[g] Cor. 2.

[h] Cor. 4. Theor. 12. B. I.

C O R. 5.

If the Vaniſhing Line of any Plane on which the Projection is deſired, be parallel to *ef*, or if any ſuch Plane be parallel to the Picture, in either Caſe the Line of the *Foci* of the Projections of all Lines in the Plane *efgb* on that Plane, will be parallel to *ef*[a], and conſequently to S*t*.

Fig. 135. N°. 2, 3.

C A S E 2. *and* 4.

When the Projecting Point is infinitely diſtant.

In theſe Caſes, the *Foci* and Vaniſhing Points of the Projections of the Sides of the propoſed Figure coinciding[b], the Lines of the *Foci* are the ſame with the Vaniſhing Lines of the Planes on which the Projections are required; ſo that ſo much of the Operation as relates to the finding the Lines of the *Foci* when the Projecting Point is moderately diſtant, is ſaved when that Point is at an infinite Diſtance, and the remaining Part of the Operation is in all other reſpects the ſame as before.

Fig. 135. N°. 4, 5, 6.

It will therefore be ſufficient to explain the Figures here referred to, in which ſome farther Varieties in the Poſition of the Planes are introduced.

In Fig. N°. 4. the Projecting Point S is at an infinite Diſtance before the Eye; the Original Plane *efgb*, in which the given Figure ***abcd*** lies, is parallel to the Plane εφgh, their Vaniſhing Lines *ef* and εφ being the ſame; and the latter of theſe Planes is cut off in the Line hy, ſo as to let part of the Projection fall beyond it on the Plane EFGH; v and w are the Vaniſhing Points, as well as the *Foci* of the Projections on the Plane εφgh, the ſame with *z* and *x* the Vaniſhing Points of the Sides of the propoſed Figure ***abcd***; and *v* and *w*, where S*z* and S*x* cut EF, are the Vaniſhing Points and *Foci* of the Projections of the ſame Sides on the Plane EFGH.

Fig. 135. N°. 4.

Here it will be convenient, firſt to find Rξ the Projection of the Line hy on the Plane EFGH[c], which will be the Boundary of the Projection of the given Figure ***abcd*** on that Plane, as hy is the Boundary of its Projection on the Plane εφgh; the Indefinite Projection of the Side ***ab*** on the Plane EFGH is found by drawing *v*β from *v* through *l* the Interſection of ***ab*** with that Plane[d], and its Indefinite Projection on the Plane εφgh, is had by drawing v*m* from v through *p* the Interſection of *v*β with that Plane; and by the like Proceſs as before, the whole Projection of the given Figure and its Subdiviſions may be thence determined, obſerving that where the Projection of any Line of the given Figure on the Plane EFGH cuts Rξ, a Line drawn from thence to S will cut hy in the Projection of the ſame Point on the Plane εφgh, and ſo *vice verſa*, by which and its proper Vaniſhing Point, the Projection of that Line on either of the Planes may be found by its Projection on the other *Q. E. I.*

In Fig. N°. 5. the Projecting Point S is at an infinite Diſtance behind the Eye; the Vaniſhing Lines εφ and EF of the Planes εφgh and EFGH are parallel; and the Plane εφgh is cut off in the Line hy, ſo as to let part of the Projection fall on the lower Plane EFGH; the Projection Rξ of the Line hy, and the Vaniſhing Points v, w, *v*, and *w* of the Projections of the Sides of the given Figure ***abcd*** on the propoſed Planes, are found as before, whence the entire Projection is obtained by the Methods already mentioned. *Q. E. I.*

Fig. 135. N°. 5.

In Fig. N°. 6. the Projecting Point is infinitely diſtant in the Directing Plane, ***si*** being the Direction of the Projecting Lines; the Planes εφgh and EFGH are parallel, and are joined together by a third Plane vwΔh; ***v*** and ***w*** found in the Vaniſhing Line EF, by ***zv*** and ***xw*** drawn parallel to ***si***, are the common Vaniſhing Points and *Foci* of the Projections of the Sides of the propoſed Figure ***abcd*** on both the Planes EFGH and εφgh[e], and the ſame Lines ***zv*** and ***xw*** determine v and w the Vaniſhing Points of the Projections on the Plane vwΔh. The reſt of the Operation needs not to be explained, it appearing ſufficiently by the Figure. *Q. E. I.*

Fig. 135. N°. 6.

GENERAL COROLLARY 1.

When the propoſed Parallelogram is ſubdivided into ſmaller, in order to find the Projection of any Curve, or otherwiſe irregular Figure incloſed in it, on two or more Planes; the Projection of the Parallelogram and its Subdiviſions being broken by falling partly on one and partly on another Plane, is thereby rendered unfit for determining the Projection of the incloſed Figure with ſufficient Exactneſs.

To remedy this Inconveniency, conſider the common Interſections of the Planes on which the Projections are deſired, as Projections of Lines in the Plane of the Figure,

and

[a] Theor. 15. and Cor. Theor. 3. B. I.

[b] Caſe 2. and 4. Prob. 3.

[c] Meth. 3. Caſe 2. Prob. 6.

[d] Meth. 4. Caſe 1. and Meth. 3. Caſe 3. Prob. 6.

[e] Caſe 4. Prob. 6.

and find the Lines in that Plane, of which the others are the Projections[a]; and theſe will cut the incloſed Figure in the ſame Points, where its Projection is broken by paſſing from one Plane to another; by the help of which, its Projection may be found to any deſirable Degree of Exactneſs.

Fig. 135. Nº. 6. Thus if two Lines $h\chi$ and $k\chi$ be found in the Plane $efgh$, whoſe Projections may coincide with $h\zeta$ and $\Delta\zeta$ the common Interſections of the Planes on which the Projections are required; the Figure $abcd$ will be thereby divided into three parts in the correſponding Places where its Projection is broken, and the part cpq will have its Projection γMN all on the Plane $\epsilon\phi$gh, the part $pq\mu\nu$ will have its entire Projection MNmn on the Plane vwΔh, and the Projection $mn\alpha$ of the Remainder $\mu\nu a$ will fall wholly on the Plane EFGH; and all ſuch Points of the incloſed Figure as lie in pq or $\mu\nu$ will have their Projections in hζ or $\Delta\zeta$.

The ſame Rule is applicable to all the other Caſes of this Problem.

GENERAL COROLLARY 2.

By the Help of this Problem, the Projection of any propoſed Figure in a given Plane, may be found on any Number of other Planes: and it may alſo be applied to the finding the Projection of any ſolid Body on ſeveral Planes, by firſt finding the Projection of that Body on ſome one convenient ſubſtituted Plane[b], and uſing that Projection as a propoſed Object in the ſubſtituted Plane[c], and thence finding its Projection on all the other Planes propoſed.

GENERAL COROLLARY 3.

Fig. 135. Nº. 1, 2, 3, 4, 5, 6. It is evident, that if the Parallelogram $abcd$ be conſidered as an Opake Object in the Plane $efgh$, and all the reſt of that Plane be tranſparent, the Projection of $abcd$ is the ſame with its Shadow on the propoſed Planes; but if $abcd$ be taken as an Aperture in the Plane $efgh$, tranſmitting the Light from the Projecting Point, whilſt the Remainder of that Plane is Opake, the ſame Projection then repreſents the Shape and Bounds of the Light which falls through that Aperture on the propoſed Planes, and is therefore alſo the whole Space on thoſe Planes, where the Shadow of any Object, expoſed to that Aperture, from the Projecting Point can fall.

SCHOL.

The Figures uſed in ſeveral of the Propoſitions of this Section, appear the more intricate, by reaſon that the neceſſary Lines are drawn in each of them, for attaining the propoſed End by ſeveral different Methods, with intent only to ſhew their mutual correſpondency; but it will be of Service to the Learner to draw new Figures for his own uſe, and adapt them to each particular Method ſingly, which will make the Figures more intelligible, and at the ſame time ſhew which of the Methods is the moſt convenient, according to the Poſition and Circumſtances of the ſeveral *Data* in the Figure, which he may chuſe at pleaſure; and thereby alſo the Facility of the Practice of every ſingle Method by itſelf, will be the more apparent.

SECTION II.

Of the Reflection of Light from a Poliſhed Plain Surface.

1. WHEN the Light proceeding from any Luminous Point falls on a Reflecting Poliſhed Plane, every Ray is reflected back with the ſame Angle of Inclination to that Plane with which it falls on it, but with a contrary Direction; that is, the Angle of Incidence is equal to the Angle of Reflection; and the incident Ray and its Reflection are always in a Plane paſſing throngh the Luminous Point perpendicular to the Reflecting Plane.

Fig. 136. Let QR be a Reflecting Plane, S a Luminous Point at a moderate Diſtance, and P its Perpendicular Seat on that Plane, and let PD be the Interſection of that Plane with a Plane LM perpendicular to it, paſſing through SP; then every Ray of Light SA, SB, SC, in the Plane LM, proceeding from S, and meeting the Reflecting Plane QR in A, B, and C, will be thence reflected back into Aa, Bb, Cc, in the ſame Plane LM, and the Angle SAP, made by any incident Ray SA with the Reflecting Plane, will

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[a] Gen. Cor. Prob. 10.

[b] Schol. Prob. 5.

[c] Gen. Cor. Prob. 10.

Fig. 135. N°3.

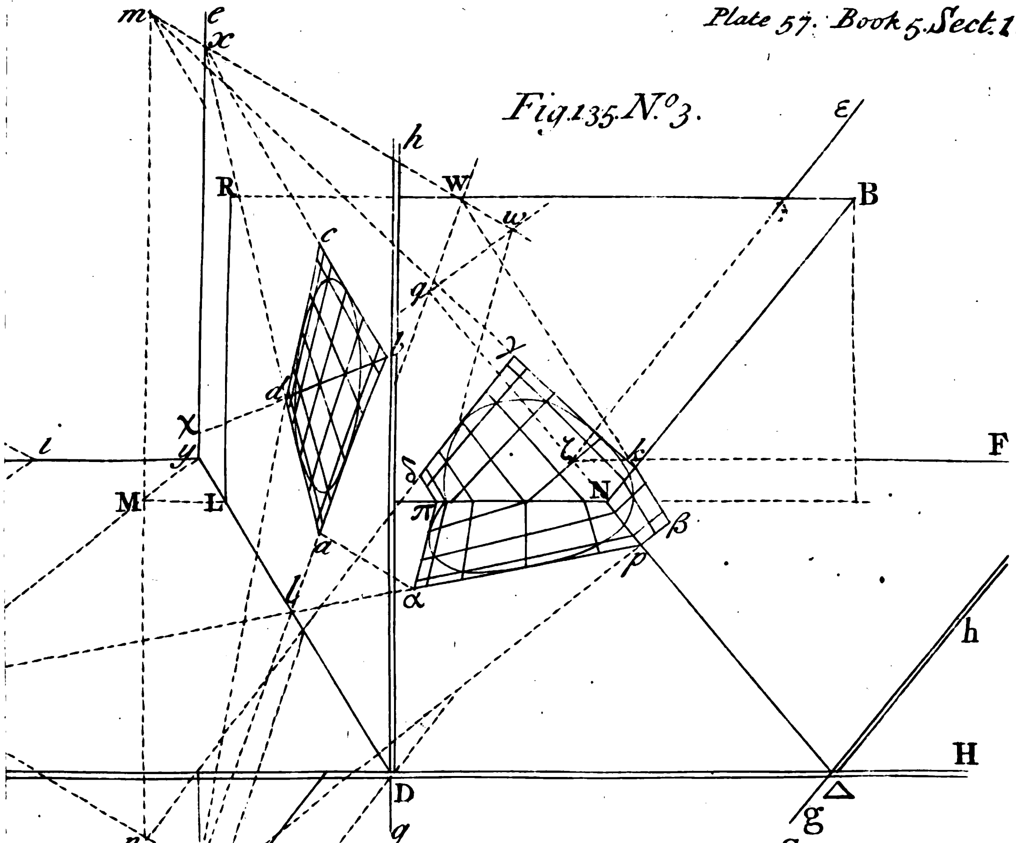
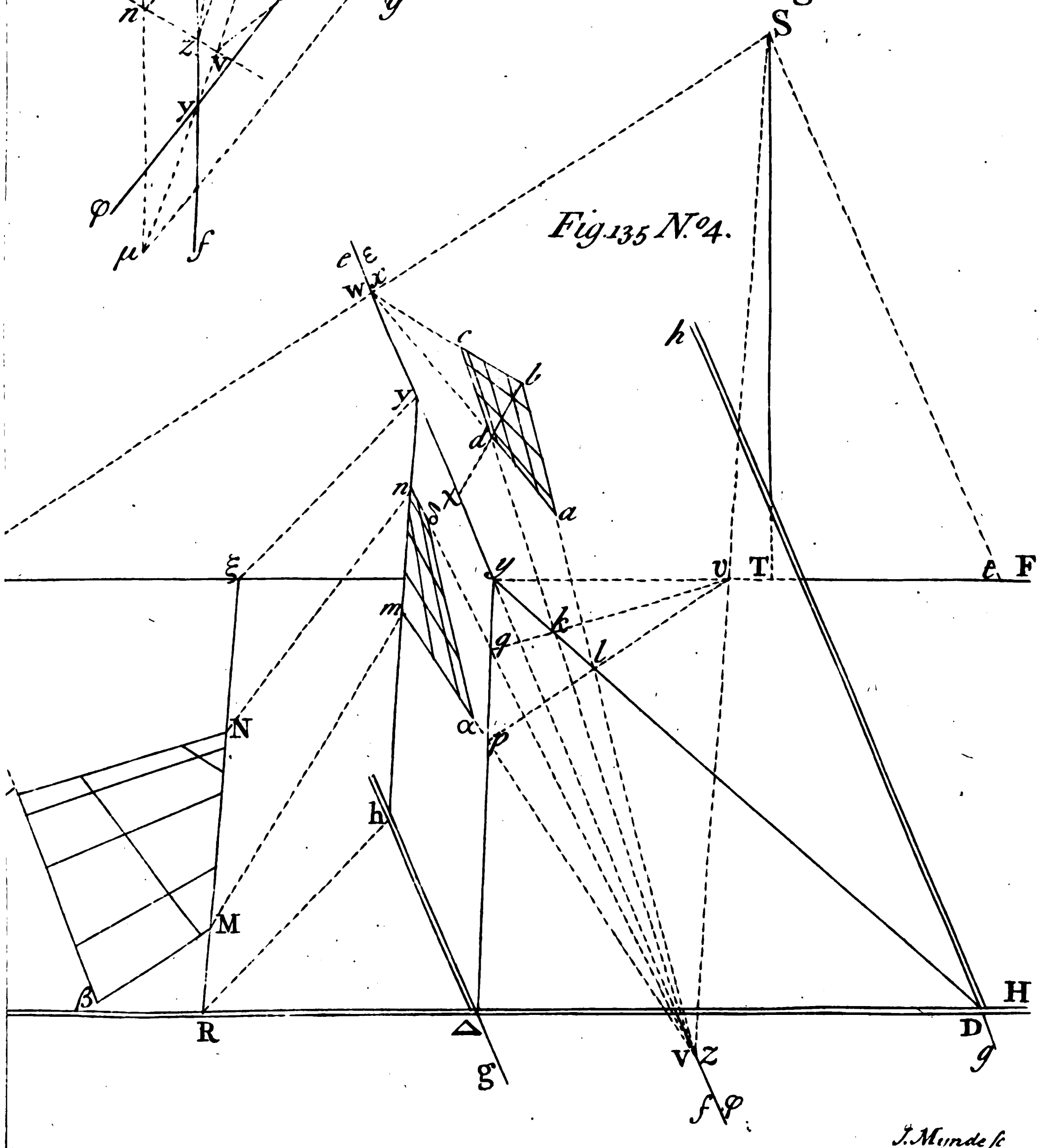


Fig. 135 N°4.



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will be equal to *a* A D the Angle made by the reflected Ray A*a* with that Plane; and the Ray S P, which is perpendicular to that Plane, will therefore coincide with its Reflection, or may be ſaid to be reflected into itſelf.

2. If the Perpendicular Support S P of the Luminous Point be produced beyond P to *s* on the contrary Side of the Reflecting Plane, until P S and P*s* be equal; the reflected Rays A*a*, B*b*, C*c*, being alſo produced the ſame way, will all meet in *s*.

For the Angles S A P, *a* A D, *s* A P being equal, the Rectangular Triangles S A P, *s* A P, are ſimilar and equal, and conſequently P*s* is equal to P S; and after the ſame manner it may be proved, that the Reflected Rays B*b* and C*c* meet in the ſame Point *s*.

3. Hence it is, that the Reflected Light ſtrikes on any Objects expoſed to it, in the ſame manner as the direct Light would do, were the Luminous Point removed from its firſt Situation, in a Line perpendicular to the Reflecting Plane, to an equal Diſtance from it on the contrary Side, the Reflecting Plane being then ſuppoſed tranſparent, ſo as to let the Light paſs through it, from this new Situation of the Luminous Point, without Refraction or Interruption.

4. From theſe Principles it follows, that the way to find the Appearance of the Light reflected by any determinate part of a Reflecting Plane, is the ſame with that of finding the Projection of the given part of the Reflecting Plane, taken as a Figure in an Original Plane, from a Projecting Point placed as far perpendicularly behind that Plane, as the Luminous Point is really before it; and therefore if a Point in this Situation with reſpect to the Light and the Reflecting Plane, with its Seat on any propoſed Original Plane, be found, and that Point be uſed as a Projecting Point, the Reflection will then be determined by the common Rules of Projection.

5. The ſame Reaſoning alſo holds when the Luminous Point S is at an infinite Diſtance; for then the incident Rays S A, S B, S C, being parallel, they have all the ſame Angle of Inclination to the Reflecting Plane, and the Reflected Rays A*a*, B*b*, C*c*, inclining the contrary way to that Plane in the ſame Angle, are alſo parallel, and conſequently the Point *s* becomes infinitely diſtant on the contrary Side of the Reflecting Plane; ſo that the incident and Reflected Rays will in this Caſe appear to proceed from two Vaniſhing Points, one on each Side of the Vaniſhing Line of the Reflecting Plane, and inclining in equal Angles to it.

DEF. 12.

The Image of the Point *s*, whether it be a Point at a moderate Diſtance, or a Vaniſhing Point, is called *the Tranſpoſed Place of the Luminous Point.*

PROB. XV.

The Center and Diſtance of the Picture, and the Vaniſhing and Interſecting Lines of a Reflecting Plane which inclines to the Picture, being given, together with the Image of a Luminous Point, and its Seat on a given Original Plane; thence to find the Reflection of the Light on the Original Plane from any given determinate part of the Reflecting Plane, when the Vaniſhing Lines of the Original and Reflecting Planes interſect.

The Luminous Point may be either at a moderate or infinite Diſtance, before, behind, or in the Directing Plane, as has been explained in the laſt Section with reſpect to a Projecting Point.

CASE 1. *and* 3.

When the Luminous Point is at a moderate Diſtance before, behind, or in the Directing Plane. Fig. 137. No. 1, 2.

Let E F G H be an Original Plane, *efgh* a Reflecting Plane, S a Luminous Point, and T its oblique Seat on the Plane E F G H; and let it be required to find the Reflection of the Light on that Plane from a given determinate Part *abcd* of the Reflecting Plane.

METHOD 1.

Having found *x* the Vaniſhing Point of Perpendiculars to the Reflecting Plane[*],

[*] Prop. 20. B. IV.

and t the parallel Seat of S on the Plane EFGH with reſpect to the Reflecting Plane[a], by their help find p the Perpendicular Seat of S on that Plane[b]; then in the Line xS, whoſe Vaniſhing Point is x, make sp repreſent a Line equal to the Original of Sp, and s will be the tranſpoſed place of the Luminous Point S; and the Parallel Seat τ of the Point s being found, by the Interſection of $s\tau$ drawn parallel to ef, with tq, proceed to find the Projection $\alpha\beta\gamma\delta$ of the Figure $abcd$ on the Plane EFGH from a Projecting Point s, whoſe Parallel Seat on that Plane with reſpect to the Plane $efgh$ is τ[c], and $\alpha\beta\gamma\delta$ thus found will be the Reflection deſired.

Dem. For the Original of xS being a Line paſſing through the Luminous Point S perpendicular to the Reflecting Plane $efgh$, which it cuts in p, and ps and pS repreſenting equal Lines, s repreſents a Point in that Perpendicular, as far behind the Reflecting Plane as the Point S is before it, and is therefore the tranſpoſed Place of the Luminous Point: the reſt is evident from the Introduction to this Section. *Q. E. I.*

METHOD 2.

From t to q the Parallel Seat of x on the Vaniſhing Line EF with reſpect to the Reflecting Plane $efgh$, draw tq cutting Dy in a, and make aτ and at in the Line tq repreſent equal Lines; then τs drawn parallel to ef, will cut Sx in s the tranſpoſed Place of the Luminous Point.

Dem. For xS being Harmonically divided in x, s, p, and S[d], St, pa, $s\tau$, and xq, are Harmonical Parallels, wherefore tq is alſo Harmonically divided in t, a, τ, and q[e], and conſequently τa and at repreſent equal Lines[f], and the Point s is therefore thus rightly determined. *Q. E. I.*

COR. 1.

The Luminous Point and its tranſpoſed Place are reciprocal; that is, as s is the tranſpoſed Place of the Luminous Point S, ſo if s be taken as the Luminous Point, its tranſpoſed Place will be S; and the Projection of $abcd$ on the Plane EFGH from S, will be the ſame with the Reflection of $abcd$ on that Plane from s, the Reflecting Side of the Plane $efgh$ being ſuppoſed to be turned towards s.

COR. 2.

Fig. 137. N°. 1, 2.

If px be biſected in m, then if either of the Points S or s fall between m and p, the other of them will fall on the outſide of p; and *vice verſa*, if either of thoſe two Points fall on the outſide of p, the other will fall between p and m: in either Caſe, the Luminous Point S, and its tranſpoſed Place s, will lie both on the ſame Side of the Directing Plane.

For Sx being Harmonically divided in the Points S, p, s, and x[g], if s fall between m and p, the part sp being leſs than sx, sp muſt be the middle part[h]; and conſequently the Point S which compleats the Harmonical Diviſion of that Line, muſt fall on the outſide of p: and the Points S and s being both on the ſame ſide of the Vaniſhing Point x, their Originals are both on the ſame ſide of the Directing Plane.

Thus in Fig. N°. 1. the Luminous Point S and its tranſpoſed Place s are both of them Points at a moderate Diſtance before the Directing Plane; and in Fig. N°. 2. they are both Points at a moderate Diſtance behind that Plane.

COR. 3.

Fig. 137. N°. 3, 4.

If either of the Points S or s fall between m and x, the other of them will fall on the outſide of x; and *vice verſa*, if either of thoſe two Points fall on the outſide of x, the other will fall between m and x: in either Caſe, the Luminous Point S and its tranſpoſed Place s will fall on the contrary Sides of the Directing Plane, and Sx and xs will repreſent equal parts of the Line ps, taking p for its Vaniſhing Point[i].

This is demonſtrated after the ſame manner as the laſt Corollary, the Line Sx being ſtill Harmonically divided in the Points p, s, x, and S[k].

Thus in Fig. N°. 3. the Luminous Point S and its tranſpoſed Place s falling on the contrary Sides of the Vaniſhing Point x, the firſt is a Point at a moderate Diſtance before the Directing Plane, and the other a Point at a moderate Diſtance behind it; and in Fig. N°. 4. the Luminous Point is behind, and its tranſpoſed Place before the Directing Plane.

COR. 4.

Fig. 137. N°. 5, 6, 7, 8.

If either of the Points S or s biſect xp, the other of them will be infinitely diſtant;

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[a] Def. 15. B. IV.

[b] Meth. 2. Prop 52. B. IV.

[c] Prob. 14.

[d] Cor. 1. Lem. 8. B. III.

[e] Def. 3. and Lem. 3. B. III.

[f] Cor. 6. Lem. 8. B. III.

[g] Cor. 1. Lem. 8. B. III.

[h] Cor. 1. Lem. 1. B. III.

[i] Cor. 6. Lem. 8. B. III.

[k] Cor. 1. Lem. 8. B. III.

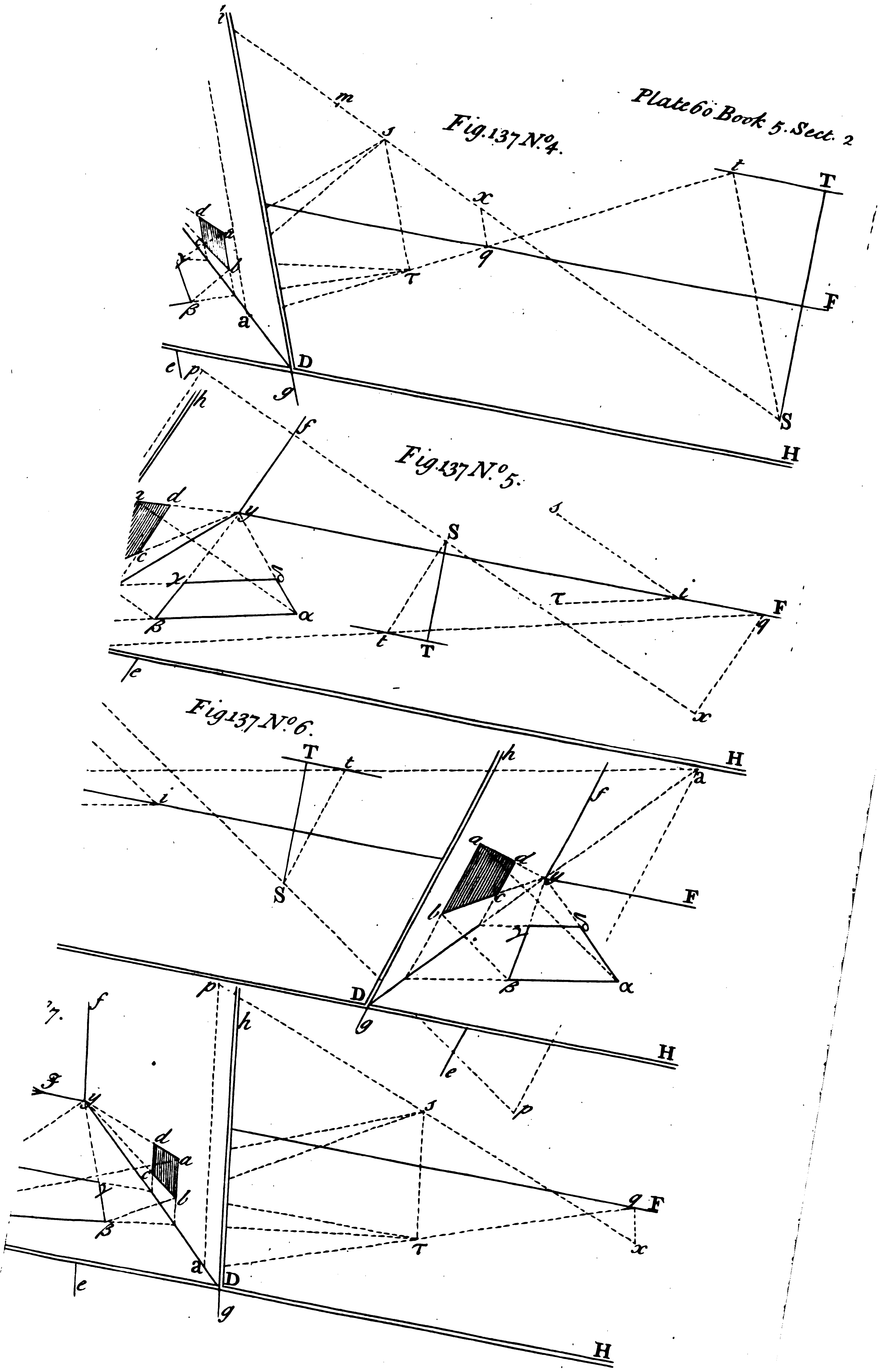

Fig. 137 N.º 4.

Fig. 137 N.º 5.

Fig. 137 N.º 6.

7.

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ſtant; that is, its Original will be at the Directing Point of xp[a]: the Direction of the Rays proceeding from that Point will be parallel to xp, and the Direction of their Parallel Seats on the Plane EFGH with reſpect to the Plane $efgb$, will be parallel to qa.

For as on this Suppoſition, either the Luminous Point or its tranſpoſed Place is at the Directing Point of xp, ſo the Parallel Seat of that Directing Point muſt be at the Directing Point of qa the Parallel Seat of xp, and conſequently the Directions of all Lines which proceed from thoſe two Directing Points are reſpectively parallel to xp and qa[b].

Fig. 137. N°. 5, 6.

Thus in Fig. N°. 5. the Luminous Point S is at a moderate Diſtance before the Directing Plane; and in Fig. N°. 6. it is at a moderate Diſtance behind it; in both it biſects xp, ſo that its tranſpoſed place is at the Directing Point of xp; the Direction si of the reflected Rays is therefore drawn parallel to xp, and the Direction τi of their Parallel Seats is drawn parallel to qa.

Fig. 137. N°. 7, 8.

In Fig. N°. 7, 8. the Luminous Point is at a moderate Diſtance in the Directing Plane, $S\mathcal{J}$ being the Direction of the Rays of Light, and $t\mathcal{J}$ the Direction of their Parallel Seats; xp is therefore drawn parallel to $S\mathcal{J}$, and qa to $t\mathcal{J}$, and s biſects xp: in Fig. N°. 7. s is at a moderate Diſtance before the Directing Plane, and in Fig. N°. 8. it is at a moderate Diſtance behind it; but it cannot be a Point in that Plane, ſeeing the Line xp can cut that Plane only in one Point, which by Suppoſition is the Luminous Point itſelf.

COR. 5.

If a Perpendicular from the Luminous Point to the Reflecting Plane, cut it in its Directing Line; then the Luminous Point and its tranſpoſed Place being on the oppoſite Sides of the Directing Plane, their Images will fall on the oppoſite Sides of the Vaniſhing Point x, and at an equal Diſtance from it[c], and the Point p will be infinitely diſtant, its Original being a Directing Point.

Fig. 137. N°. 9.

Thus in Fig. N°. 9. the Line tq which joins the Parallel Seats t and q of the Luminous Point S and of the Point x, being parallel to Dy the Interſection of the Original and Reflecting Planes, the Lines tq and Dy have the ſame Directing Point, and conſequently the Planes $xqSt$ and $efgb$ have the ſame Directing Line[d]; wherefore tq and xS cut the Plane $efgb$ in that Line, and the Points marked a and p in the other Figures are here infinitely diſtant; Ss is therefore biſected by x, and $t\tau$ by q.

CASE 2. and 4.

When the Luminous Point is at an infinite Diſtance before, behind, or in the Directing Plane.

In theſe Caſes the Luminous Point being a Vaniſhing Point, its tranſpoſed Place, and likewiſe its Perpendicular Seat on the Reflecting Plane are alſo Vaniſhing Points[e].

Fig. 138. N°. 1, 2.

Let then EFGH and $efgb$ be the Original and Reflecting Planes as before, and let S be the Luminous Point at an infinite Diſtance.

METHOD 1.

From x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, draw xS, cutting ef in p; and in the Vaniſhing Line xS, find a Point s, ſubtending with p an Angle equal to that ſubtended by S and p[f], and s will be the tranſpoſed Place of the Luminous Point; and the Projection $\alpha\beta\gamma\delta$ of the Figure $abcd$ on the Plane EFGH from the Point s, will be the Reflection ſought.

Dem. For xS being the Vaniſhing Line of a Plane perpendicular to the Reflecting Plane[g], and the Points S and s ſubtending equal Angles with p, all Lines whoſe Vaniſhing Point is s incline to the Plane $efgb$ in the ſame Angle, but the contrary way, with thoſe whoſe Vaniſhing Point is S[h]; s is therefore the Vaniſhing Point of the reflected Rays, and conſequently it is the tranſpoſed Place of the Luminous Point. *Q. E. I.*

METHOD 2.

Find a Point τ in the Vaniſhing Line EF, ſo that qt may be Harmonically divided in q, τ, y, and t; and τs drawn parallel to ef will cut Sx in s the tranſpoſed Place of the Luminous Point.

Dem. For in the Line Sx, the Radials of x and p being perpendicular[i], and the Angle ſubtended by S and s being biſected by the Radial of p, Sx is therefore Harmonically divided in x, s, p, and S[k]; and conſequently xq, $s\tau$, py, and St, being Harmonical

[a] Cor. 3. Lem. 8. B. III.

[b] Caſe 3. Prob. 6.

[c] Cor. 3. Lem. 8. B. III.

[d] Cor. 5. Theor. 12. and Theor. 14. B. I.

[e] Caſe 2. Prob. 1.

[f] Prop. 24. B. IV.

[g] Cor. 3. Prop. 20. B. IV.

[h] Prop. 24. B. IV.

[i] Cor. 4. Prop. 20. B. IV.

[k] Cor. Lem. 7. B. III.

monical Parallels, $q t$ is alſo Harmonically divided in q, τ, y, and t[a]; the Point s is therefore thus rightly determined. *Q. E. I.*

C O R.

The ſeveral Corollaries of the preceeding Caſes are alſo applicable to theſe.

Fig. 138. N°. 2, 3. Thus in Fig. N°. 2. S lies nearer to p than to x, s therefore falls beyond p[b]; and in Fig. N°. 3. S lies nearer to x than to p, s therefore falls beyond x[c]; but in either of theſe Caſes, s being a Vaniſhing Point, it may repreſent a Point at an infinite Diſtance either before or behind the Directing Plane, according as it happens to fall either above or below EF the Vaniſhing Line of the Original Plane.

Fig. 138. N°. 4. In Fig. N°. 4. the Interſection p of Sx with ef being out of reach, the Line qy is therefore uſed[d], which Line being biſected by t, it follows that S alſo biſects xp; its tranſpoſed Place is therefore at an infinite Diſtance in that Line, and anſwers to a Projecting Point at an infinite Diſtance in the Directing Plane, and the Direction si of the reflected Rays is therefore parallel to xS[e].

Fig. 138. N°. 5. In Fig. N°. 5. the Luminous Point is at an infinite Diſtance in the Directing Plane, $S\mathcal{J}$ being the Direction of the Rays of Light; the Parallel Seat τ of the tranſpoſed Place of the Luminous Point therefore biſects qy, and τs drawn parallel to ef cuts xs drawn parallel to $S\mathcal{J}$ in s, which alſo biſects xp[f].

In the two laſt Figures, the Vaniſhing Line xp being biſected by S or s, the fourth Point, which ſhould compleat the Harmonical Diviſion of that Line, is infinitely diſtant[g]; all Lines to which that Vaniſhing Point ſhould belong are therefore parallel to the Picture[h], and conſequently their Images are parallel to xp[i], to which their Direction si or $S\mathcal{J}$ is therefore alſo parallel.

Laſtly, if Sx be parallel to ef, the Point p being then infinitely diſtant, s will fall on the contrary Side of x from S, and at an equal Diſtance from it[k].

P R O B. XVI.

The Center and Diſtance of the Picture, and the Vaniſhing and Interſecting Lines of a Reflecting Plane which inclines to the Picture, being given, together with the Image of a Luminous Point, and its Seat on a given Original Plane; thence to find the Reflection of the Light on the Original Plane from any given determinate part of the Reflecting Plane, when the Vaniſhing Lines of thoſe Planes are either parallel or coincide.

C A S E 1. *and* 3.

When the Luminous Point is at a moderate Diſtance before, behind, or in the Directing Plane.

Fig. 139. N°. 1, 2. Let EFGH and $efgh$ be the Original and Reflecting Planes, MN their common Interſection, S the Luminous Point, and T and t its Oblique Seats on the Original and Reflecting Planes.

Through x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, draw xz the common Vertical Line of the given Planes[l], cutting ef and EF in their Centers y and z, and compleat the ſubſtituted Plane $zy\Delta D$, paſſing through the Support ST of the Luminous Point, and cutting the given Planes in yD and $z\Delta$[m]; then draw xS cutting yD in p, and make ps and pS repreſent equal Lines, and s will be the tranſpoſed Place of the Luminous Point, and $s\tau$ drawn parallel to xz will cut $z\Delta$ and yD in τ and t the Oblique Seats of s on the Original and Reflecting Planes, whence the Reflection $\alpha\beta\gamma\delta$ of the Part $abcd$ of the Reflecting Plane on the Original Plane, may be found as in the Figures[n].

Dem. For the ſubſtituted Plane $zy\Delta D$ being perpendicular to the Original and Reflecting Planes, p is the Perpendicular Seat of S on the Reflecting Plane[o], and ps and pS repreſenting equal Lines, s is therefore the tranſpoſed Place of the Luminous Point. *Q. E. I.*

Fig. 139. N°. 1. In Fig. N°. 1. the Original Plane is above the Eye, and the Reflecting Plane below it, and the Luminous Point S is at a moderate Diſtance before the Directing Plane, its Seat T on the Original Plane, which here may be called its *Point of Suſpenſion*, being in the Perſpective Part of that Plane; and in Fig. N°. 2. the Reflecting Plane is above

Fig. 139. N°. 2.

[a] Lem. 3. B. III.

[b] Cor. 2.

[c] Cor. 3.

[d] Meth. 2.

[e] Cor. 4.

[f] Cor. 4.

[g] Cor. 1. Lem. 1. B. III.

[h] Theor. 1. B I.

[i] Cor. 1. Theor. 15. B.I.

[k] Cor. 5.

[l] Cor. 2. Theor. 14. B.I. and Cor. 2. Prop. 20. B. IV.

[m] Caſe 2. Prop. 52. B. IV.

[n] Meth. 1. and 4. Caſe 1. Prob. 7.

[o] Caſe 2. Prop. 52. B. IV.

Fig. 138 N°. 3.

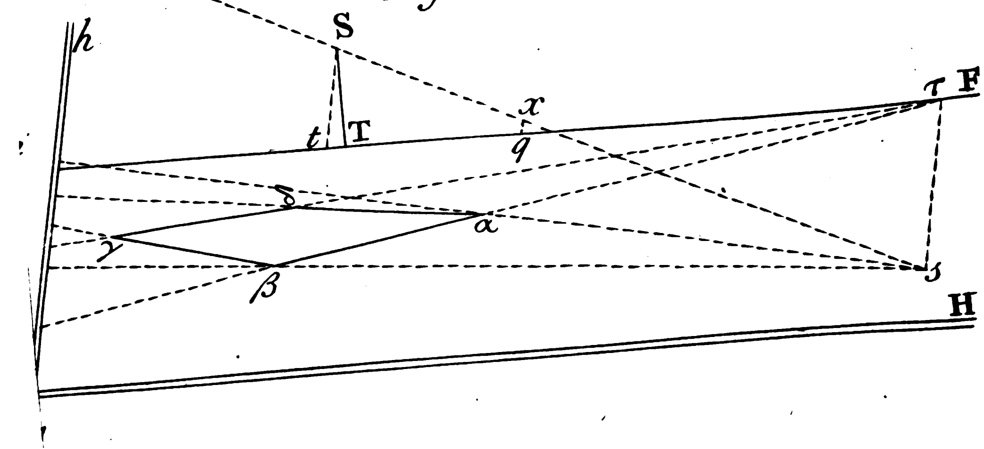
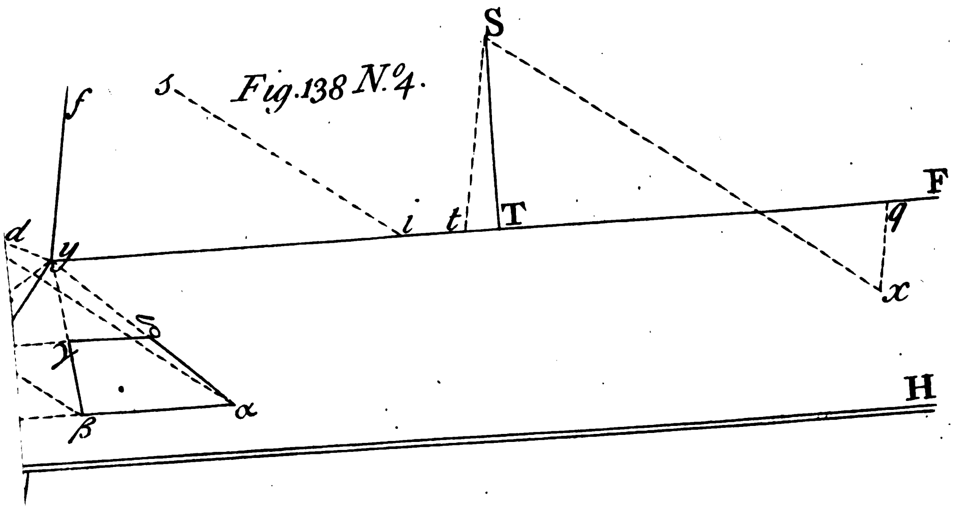
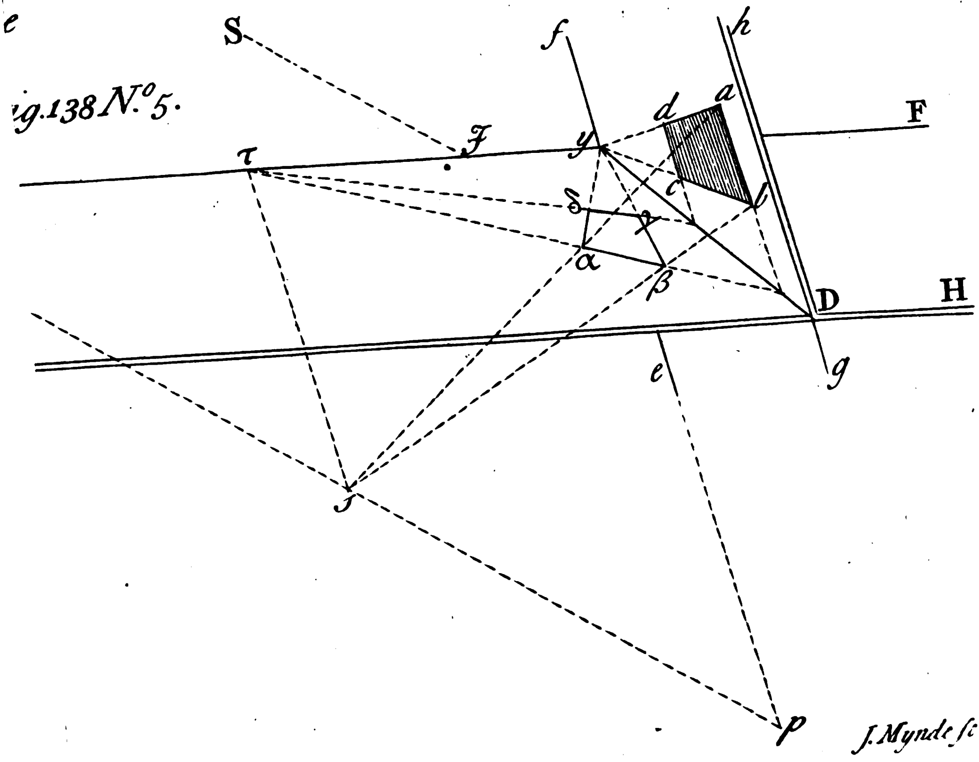


Fig. 138 N°. 4.



ig. 138 N°. 5.



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above the Eye and the Original Plane below it, and the Luminous Point S is at a moderate Diſtance behind the Directing Plane, its Seat T being in the Tranſprojective Part of the Original Plane.

Fig. 139. N°. 3. In Fig. N°. 3. the Original Plane is above the Eye, and the Reflecting Plane below it; the Luminous Point is at a moderate Diſtance in the Directing Plane, S*J* being the Direction of the Rays of Light, and T*J* the Direction of their Oblique Seats on the Original Plane, to which Δz the Interſection of the ſubſtituted Plane $zy\Delta D$ with the Original Plane is parallel[a]; xp is drawn parallel to S*J*, and cuts yD in p the Perpendicular Seat of the Luminous Point on the Reflecting Plane, and xp being biſected in s, s is the tranſpoſed Place of the Luminous Point[b].

COR.

Fig. 139. N°. 3. If s were the Luminous Point biſecting xp, the tranſpoſed Place of the Luminous Point would be in the Directing Plane; S*J* parallel to xp would be the Direction of the reflected Rays, and the Direction T*J* of their Seats on the Original Plane, would be parallel to $z\Delta$ the Interſection of that Plane with the ſubſtituted Plane $zy\Delta D$, which paſſes through $s\tau$ perpendicular to the given Planes[c].

CASE 2. *and* 4.

When the Luminous Point is at an infinite Diſtance before, behind, or in the Directing Plane.

Fig. 139. N°. 4, 5. Here, as in the correſponding Caſes of the laſt Problem, xp is a Vaniſhing Line, and either paſſes through S, or is drawn parallel to the Direction S*J*; s the tranſpoſed Place of the Luminous Point is found as before, according to the Situation of the Luminous Point with reſpect to x and p[d]; and the Point s or its Direction being found, the Reflection of $abcd$ is thence determined as in the Figures[e]. *Q. E. I.*

In Fig. N°. 4. the Luminous Point is at an infinite Diſtance behind the Directing Plane, and its tranſpoſed Place s is alſo to be conſidered as the Tranſprojected Image of a Point at an infinite Diſtance behind that Plane, by which means its Original being ſuppoſed under the Reflecting Plane, $\alpha\beta\gamma\delta$ becomes the Projection of $abcd$ on the Original Plane; whereas if s be taken as the direct Image of a Point at an infinite Diſtance before the Directing Plane, $\alpha\beta\gamma\delta$ becomes the Object, and $abcd$ its Projection on the Reflecting Plane[f].

The ſame Obſervation has Place in Fig. N°. 5. where the Luminous Point is at an infinite Diſtance in the Directing Plane, S*J* being the Direction of the Rays of Light, to which xp is drawn parallel, and is biſected by s; for here s muſt be conſidered as at an infinite Diſtance behind the Directing Plane, that ſo it being under the Reflecting Plane, $\alpha\beta\gamma\delta$ may be the Projection of $abcd$.

GENERAL COROLLARY 1.

When the Vaniſhing Lines of the Reflecting and Original Planes coincide, the Practice is in effect the ſame as before; the common Interſection MN of the given Planes in the Figures here referred to, becoming in that Caſe the ſame with their common Vaniſhing Line[g].

GENERAL COROLLARY 2.

In the Figures of the two preceeding Problems, the Center of the Picture is not marked, it being indifferent where it falls in the Vertical Line of the Reflecting Plane which paſſes through x[h]; and conſequently the Practice in all the Caſes of theſe Problems is the ſame, whether the Original Plane be perpendicular or inclining, either to the Picture or to the Reflecting Plane.

PROB. XVII.

The Center and Diſtance of the Picture, and the Vaniſhing and Interſecting Lines of a Reflecting Plane which inclines to the Picture, being given, together with the Image of a Luminous Point, and its Oblique Seat on that Plane; thence to find the Reflection of the Light from any given determinate part of the Reflecting Plane, on an Original Plane parallel to the Picture, whoſe Interſection with the Reflecting Plane is given.

[a] Caſe 3. Prob. 7.

[b] Cor. 4. Caſe 1. and 3. Prob. 15.

[c] Cor. 1. Caſe 1. and 3. Prob. 15.

[d] Cor. Caſe 2. and 4. Prob. 15.

[e] Caſe 2. and 4. Prob. 7.

[f] Gen. Cor. Prob. 10.

[g] Gen. Cor. Caſe 1. and 3. Prob. 7.

[h] Cor. Def. 13. B. I. and Cor. 2. Prop. 20. B. IV.

CASE 1. *and* 3.

When the Luminous Point is at a moderate Diſtance before, behind, or in the Directing Plane.

Fig. 140. N°. 1, 2. Let $efgh$ be the Reflecting Plane, and MN its Interſection with an Original Plane parallel to the Picture, and let S be the Luminous Point, and t its Oblique Seat on the Reflecting Plane.

Through x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, draw its Vertical Line xy, cutting ef in its Center y, and draw yt; then draw xS cutting yt in p, and make ps and pS repreſent equal Lines, and s will be the tranſpoſed Place of the Luminous Point; and if through a, the Interſection of yt with MN, a Line aτ be drawn parallel to xy, it will be cut by sy in τ the Oblique Seat of s on the Parallel Plane, whence the Reflection $\alpha\beta\gamma\delta$ of the part $abcd$ of the Reflecting Plane may be found as in the Figures[a].

Dem. For ty being the Interſection of the Reflecting Plane with a Plane perpendicular to it, paſſing through the Oblique Support St of the Luminous Point, p is therefore the Perpendicular Seat of S on the Reflecting Plane[b], and conſequently s is its tranſpoſed Place; and as aτ is the Interſection of this perpendicular ſubſtituted Plane with the Original Plane, $s\tau$ which paſſes through y, is the Oblique Support, and conſequently τ is the Oblique Seat of s on the Original Plane[c]. *Q. E. I.*

Fig. 140. N°. 1, 2. In Fig. N°. 1. the Luminous Point is at a moderate Diſtance before the Directing Plane; and in Fig. N°. 2. it is at a moderate Diſtance behind that Plane.

Fig. 140. N°. 3. In Fig. N°. 3. the Luminous Point is at a moderate Diſtance in the Directing Plane, S$\mathcal{J}$ is the Direction of the Rays of Light, and $t\mathcal{J}$ the Direction of their Oblique Seats on the Reflecting Plane, Dy is therefore drawn parallel to $t\mathcal{J}$, and xp to S$\mathcal{J}$, and xp is biſected by s[d].

CASE 2. *and* 4.

When the Luminous Point is at an infinite Diſtance before, behind, or in the Directing Plane.

Fig. 140. N°. 4, 5, 6. In theſe Caſes, the Method of finding the tranſpoſed Place of the Luminous Point differs in nothing from that propoſed at the correſponding Caſes of Prob. XV. and the Reflection $\alpha\beta\gamma\delta$ of the Figure $abcd$ is found by the Rules at Prob. IX. compared with Prob. VIII. *Q. E. I.*

Fig. 140. N°. 4. In Fig. N°. 4. the Luminous Point S is at an infinite Diſtance behind the Directing Plane, and its tranſpoſed Place s is alſo to be taken as at an infinite Diſtance behind that Plane, that the Reflected Rays flowing from a Vaniſhing Point beneath the Reflecting Plane, may project the Reflection $\alpha\beta\gamma\delta$ on the Original Plane[e]; and here, S and s fall one on each Side of p.

Fig. 140. N°. 5. In Fig. N°. 5. the Luminous Point is at an infinite Diſtance before the Directing Plane, but its tranſpoſed Place s muſt be taken as at an infinite Diſtance behind that Plane, for the Reaſon juſt mentioned; and here, S and s fall one on each Side of x.

Fig. 140. N°. 6. In Fig. N°. 6. the Luminous Point is at an infinite Diſtance in the Directing Plane, xp is therefore drawn parallel to S$\mathcal{J}$ the Direction of the Rays of Light, and is biſected by s[f], which Point muſt be taken as at an infinite Diſtance behind the Directing Plane.

COR.

If s were the Luminous Point biſecting xp, its tranſpoſed Place would be at an infinite Diſtance in the Directing Plane, and S$\mathcal{J}$ would be the Direction of the Reflected Rays[g]; but in that Caſe, no Reflection could be produced on the Original Plane, ſeeing the Reflected Rays being parallel to the Picture, would be alſo parallel to the Original Plane, and ſo could never meet it to determine the Reflection.

PROB. XVIII.

The Vaniſhing and Interſecting Lines of a Reflecting Plane perpendicular to the Picture, being given, together with the Image of a Luminous Point, and its Seat on a given Original Plane; thence to find the Reflection of the Light on the Original Plane, from any given determinate part of the Reflecting Plane, when the Vaniſhing Lines of the Original and Reflecting Planes interſect.

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[a] Prob. 9.

[b] Prop. 49. B. IV.

[c] Caſe 3. Prop. 52. B. IV.

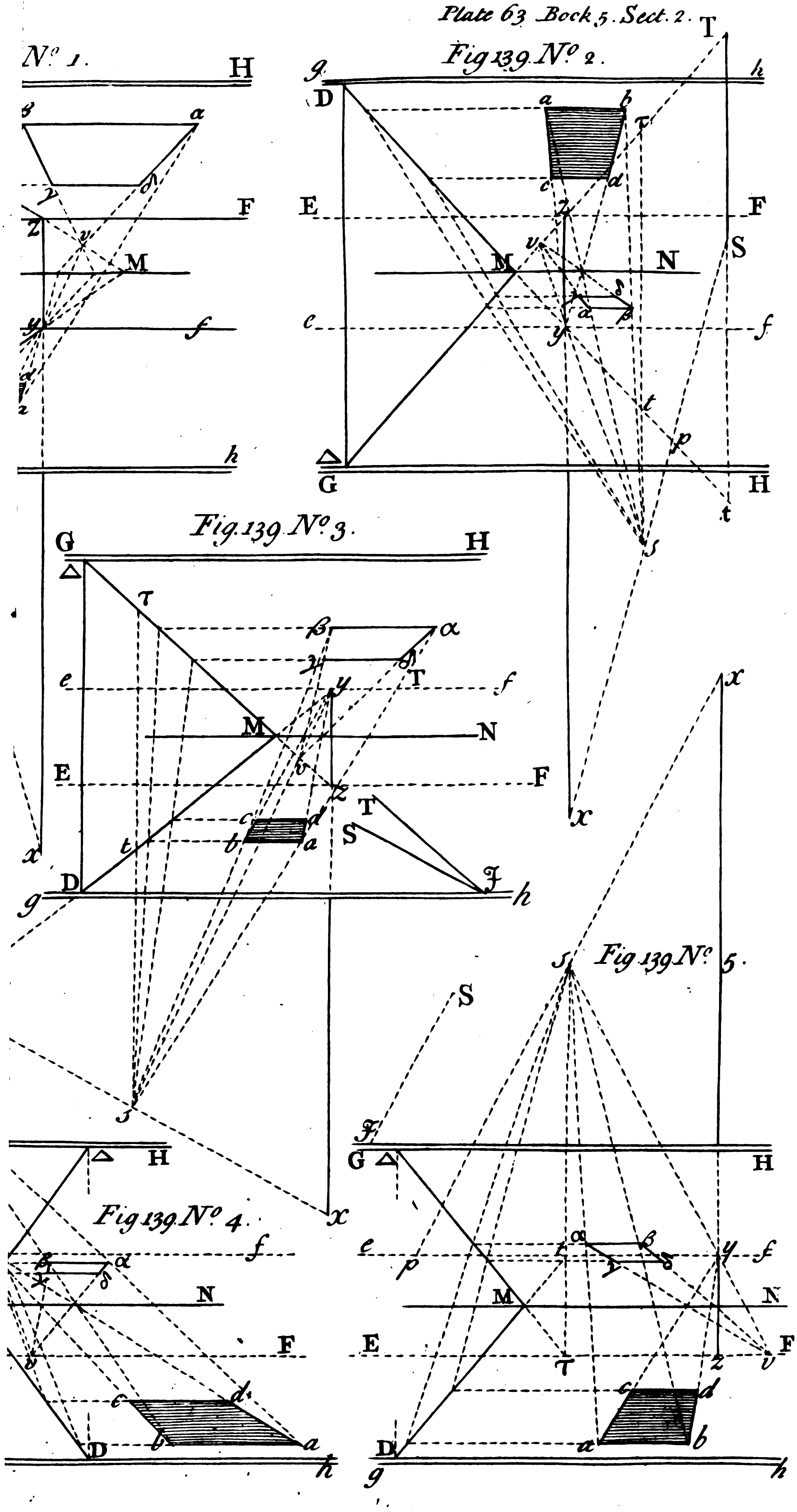
[d] Cor. 4. Caſe 1. and 3. Prob. 15.

[e] Caſe 2. and 4. Prob. 16.

[f] Cor. Caſe 2. and 4. Prob. 15.

[g] Cor. 4. Caſe 1. and 3. Prob. 15.

Plate 63 Book 5. Sect. 2.



No. 1.

Fig. 139 No. 2.

Fig. 139 No. 3.

Fig. 139 No. 4.

Fig. 139 No. 5.

J Mynde sc.

CASE 1. *and* 3.

When the Luminous Point is at a moderate Diſtance before, behind, or in the Directing Plane.

Let EFGH and $efgh$ be the Original and Reflecting Planes, S the Luminous Point, and T its Oblique Seat on the Original Plane.

Fig. 141. N°. 1, 2.

Having drawn Ss perpendicular to ef, and found its Interſection p with the Reflecting Plane[a], make ps equal to pS, and s will be the tranſpoſed Place of the Luminous Point, having τ for its Parallel Seat on the Original Plane with reſpect to the Reflecting Plane; whence the Reflection $\alpha\beta\gamma\delta$ of the Figure $abcd$ may be found as at Prob. XV.

Dem. For the Reflecting Plane being perpendicular to the Picture, all Lines perpendicular to that Plane are parallel to the Picture[b], and their Images are therefore perpendicular to ef[c]; wherefore Ss drawn perpendicular to ef is the Perpendicular Support, and p the Perpendicular Seat of S on the Reflecting Plane; and the Original of Ss being parallel to the Picture, sp and Sp, which are equal, repreſent equal Lines[d], and s is therefore the tranſpoſed Place of the Luminous Point. *Q. E. I.*

In Fig. N°. 1. the Luminous Point is before the Directing Plane; and in Fig. N°. 2. it is behind that Plane; in both, the Luminous Point and its tranſpoſed Place being in a Line parallel to the Picture, they are on the ſame Side of the Directing Plane.

Fig. 141. N°. 1, 2.

In Fig. N°. 3. the Luminous Point is at a moderate Diſtance in the Directing Plane, Sy being the Direction of the Rays of Light, and ty the Direction of their Parallel Seats on the Original Plane with reſpect to the Reflecting Plane.

Fig. 141. N°. 3.

And here, the tranſpoſed Place of the Luminous Point being alſo at a moderate Diſtance in the Directing Plane, the Directions of the Reflected Rays and their Parallel Seats are found in this manner;

The Directing Plane is ſuppoſed to be brought into the Picture, y being taken as the Place of the Eye; by which means S repreſents the Place of the Luminous Point in the Directing Plane[e], and Ss drawn perpendicular to gh (which in this View repreſents the Directing Line of the Reflecting Plane) cuts it in p the Perpendicular Seat of S on that Plane; and sp being made equal to Sp, s is the tranſpoſed Place of the Luminous Point in the Directing Plane; and the Parallel Seat τ of the Point s being found, sy and τy are the Directors of the Reflected Rays and their Parallel Seats, which may be uſed as the Directions of thoſe Lines; or any other Lines parallel to them, may be drawn from any convenient Point i in the Line EF.

CASE 2. *and* 4.

When the Luminous Point is at an infinite Diſtance before, behind, or in the Directing Plane.

When the Luminous Point is at an infinite Diſtance before or behind the Directing Plane, the Method of finding its tranſpoſed Place is the ſame as in the former Caſes, ſave that Ss becomes a Vaniſhing Line, and its Interſection with the Vaniſhing Line ef gives p; and in regard that Ss, which is perpendicular to ef, is then a Vaniſhing Line of Planes perpendicular to the Reflecting Plane, it is evident, that sp and Sp being made equal, s and S ſubtend equal Angles with p[f]; the Figures for this Caſe are not therefore drawn, they being eaſily ſupplied.

In Fig. N°. 4. the Luminous Point is at an infinite Diſtance in the Directing Plane, Sy being the Direction of the Rays of Light, made to cut EF in y; and the Point S being taken at pleaſure in yS, and Ss being drawn perpendicular to ef, cutting it in p, sp is made equal to Sp, which gives sy the Direction of the Reflected Rays; it being evident, that by this Conſtruction, the Angles pyS and syp are equal, and that therefore the Angle of Incidence is equal to the Angle of Reflection. *Q. E. I.*

Fig. 141. N°. 4.

GENERAL COROLLARY.

And here, as in the other Caſes of this Problem, the Luminous Point and its tranſpoſed Place are of the ſame kind, either both before, both behind, or both in the Directing Plane.

PROB. XIX.

The Vaniſhing and Interſecting Lines of a Reflecting Plane perpendicular to the Picture, being given, together with the Image of a Luminous

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[a] Cor. 2. Caſe 1. Prop. 52. B. IV.

[b] 38 and 6 El. 11.

[c] Cor. 2. Prop. 7. B. IV.

[d] Cor. 1. Theor. 23. B. I.

[e] Cor. 2. Caſe 3. Prob. 1. and Gen. Cor. Prob. 11.

[f] Caſe 1. Prop. 24 B. IV.

Luminous Point, and its Seat on a given Original Plane; thence to find the Reflection of the Light on the Original Plane from any given determinate part of the Reflecting Plane, when the Vaniſhing Lines of thoſe Planes are either parallel or coincide.

CASE 1. *and* 3.

When the Luminous Point is at a moderate Diſtance, before, behind, or in the Directing Plane.

Fig. 142. N°. 1. Let EFGH and $efgh$ be the Original and Reflecting Planes, S the Luminous Point, and T its Oblique Seat on the Original Plane.

Draw any ſubſtituted Plane $zy\Delta D$, paſſing through ST, and produce ST till it cut Dy in p; make ps equal to pS, and s will be the tranſpoſed Place of the Luminous Point; whence the Reflection $\alpha\beta\gamma\delta$ of the Figure $abcd$ is found as before[a].

Dem. For the Reflecting Plane being Perpendicular to the Picture, the ſubſtituted Plane $zy\Delta D$ is Perpendicular to the Reflecting Plane[b], and p is the Perpendicular as well as the Oblique Seat of S on that Plane, and conſequently s is the tranſpoſed Place of the Luminous Point. *Q. E. I.*

Fig. 142. N°. 1. In Fig. N°. 1. the Luminous Point S is at a moderate Diſtance before the Directing Plane, and T may be taken as its Point of Suſpenſion from the Original Plane; $v\varrho$ is the Line of the *Foci* of the Reflections on the Original Plane[c], and ϱ the *Focus* of the Reflections of ac and bd, whoſe Vaniſhing Point is O the Center of the Picture.

The Method is the ſame when the Luminous Point is behind the Directing Plane, the Figure for which Caſe is not therefore drawn.

Fig. 142. N°. 2. In Fig. N°. 2. the Luminous Point is at a moderate Diſtance in the Directing Plane, $S\mathcal{J}$ is the Direction of the Rays of Light, and Tz the Direction of their Oblique Seats on the Original Plane, and ſo placed, as to form a ſubſtituted Plane $\mathcal{J}zTp$, paſſing through the Oblique Support of the Luminous Point in the Directing Plane[d]; by which means p repreſents the Perpendicular Seat of S on the Directing Line of the Reflecting Plane, and ps made equal to pS, gives $s\mathcal{J}$ the Direction of the Reflected Rays; whence the Reflection is found as in the Figure[e].

CASE 2. *and* 4.

When the Luminous Point is at an infinite Diſtance, before, behind, or in the Directing Plane.

Fig. 142. N°. 3. In Fig. N°. 3. the Luminous Point is at an infinite Diſtance behind the Directing Plane, T is its Oblique Seat on the Vaniſhing Line of the Original Plane, and p is its Perpendicular Seat on the Vaniſhing Line of the Reflecting Plane; ps is made equal to pS, and the ſubſtituted Plane $Tp\Delta D$ hath Ss for its Vaniſhing Line; and the Reflection is found as in the Figure[f]. The ſame Method ſerves when the Luminous Point is before the Directing Plane, which therefore needs no Figure.

Fig. 142. N°. 4. In Fig. N°. 4. the Luminous Point is at an infinite Diſtance in the Directing Plane, $S\mathcal{J}$ being the Direction of the Rays of Light; and the ſubſtituted Plane $\mathcal{J}z\Delta D$ being formed, and its interſecting Line ΔD produced till it cut $S\mathcal{J}$ any where in S, ps is made equal to pS, whereby $s\mathcal{J}$ the Direction of the Reflected Rays is found, and thence the Reflection deſired[g].

For it is evident by this Conſtruction, that the ſubſtituted Plane $\mathcal{J}z\Delta D$ is Perpendicular to the Reflecting Plane, and that the Angles $S\mathcal{J}p$, $p\mathcal{J}s$ are equal. *Q. E. I.*

GENERAL COROLLARY.

When the given Planes are parallel, the Practice is the ſame in all the Caſes of this Problem; regard being had to the Coincidence of their Vaniſhing Lines.

PROB. XX.

The Vaniſhing and Interſecting Lines of a Reflecting Plane perpendicular to the Picture, being given, together with the Image of a Luminous Point, and its Seat on that Plane; thence to find the Reflection of the Light from any given determinate part of the Reflecting Plane, on an Original Plane parallel to the Picture.

CASE

[a] Meth. 1. and 4. Prob. 7.

[b] Cor. 3. Prop. 20. B. IV.

[c] Cor. Meth. 4. Prob. 7.

[d] Caſe 3. Prob. 7.

[e] Meth. 1. and 4. Caſe 3. Prob. 7.

[f] Caſe 2. Prob. 7.

[g] Caſe 4. Prob. 7.

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Fig. 140. No 5.

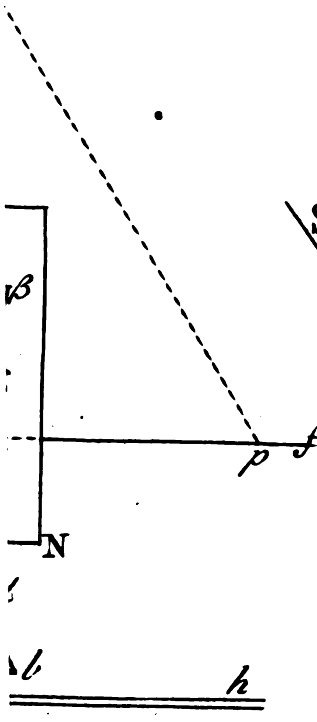
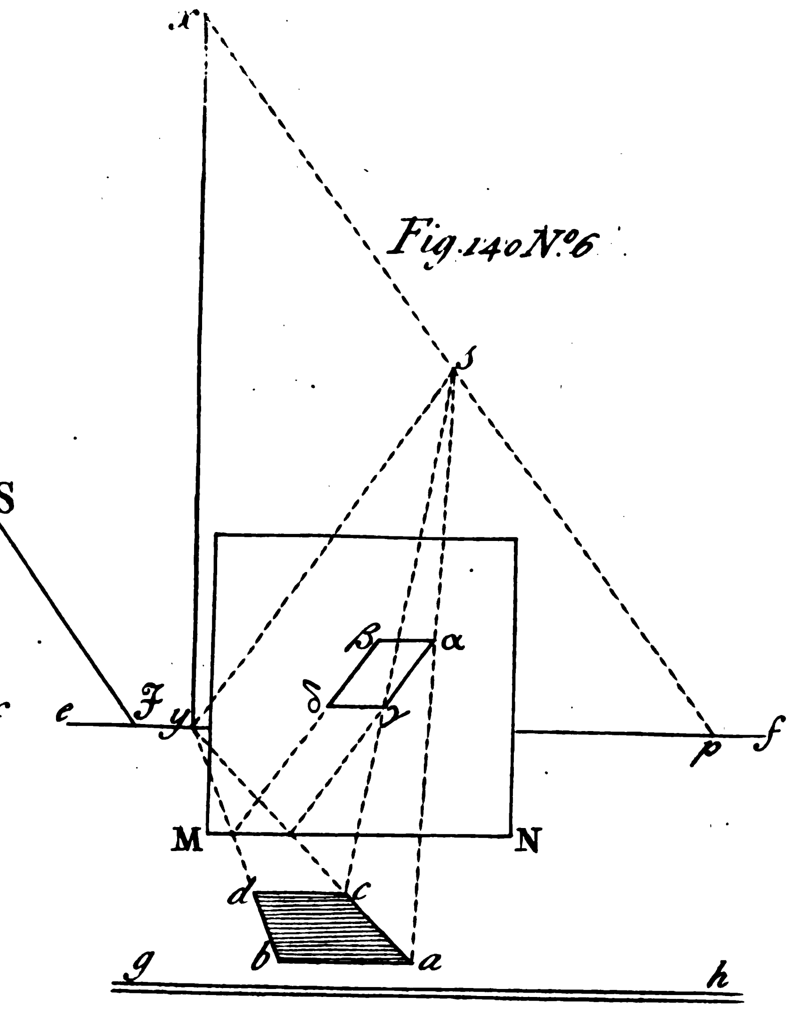


Fig. 140 No 6



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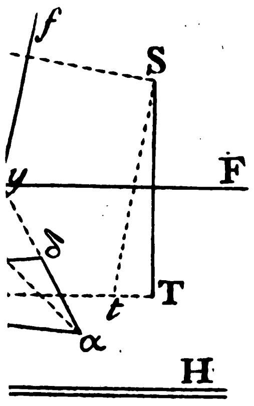


Fig. 141 No 2

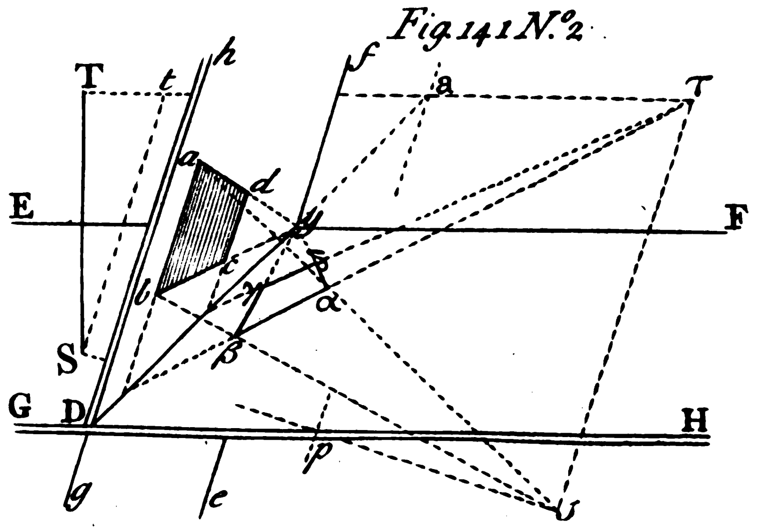


Fig. 141 No 3.

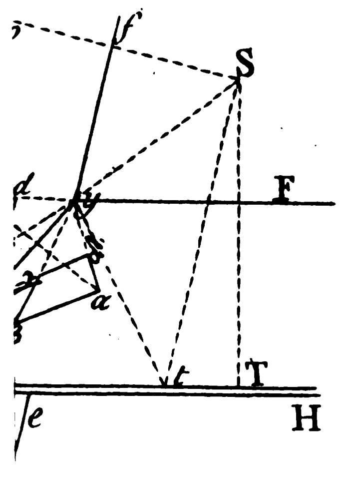
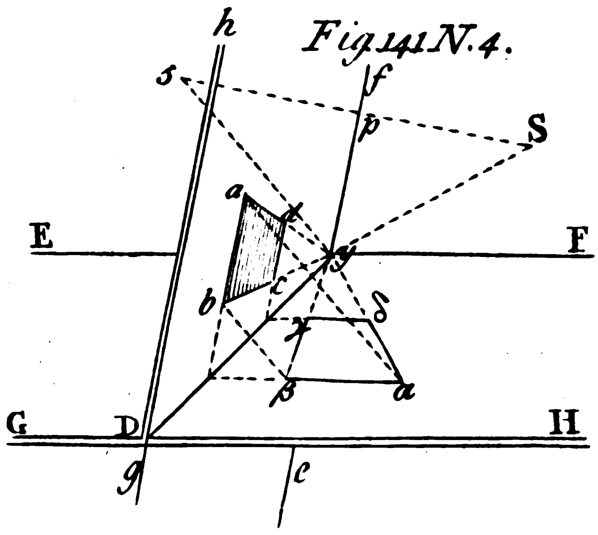


Fig. 141 N. 4.



CASE 1. *and* 3.

When the Luminous Point is at a moderate Diſtance, before, behind, or in the Directing Plane.

Let $efgh$ be the Reflecting Plane, and MN its Interſection with an Original Plane parallel to the Picture; and let S be a Luminous Point, and p its Perpendicular Seat on the Reflecting Plane. Fig. 143. N°. 1, 2.

Produce Sp to s, until sp and Sp be equal, and s will be the tranſpoſed Place of the Luminous Point, by which the Reflection $\alpha\beta\gamma\delta$ may be found[a]. *Q. E. I.*

In Fig. N°. 3. the Luminous Point is at a moderate Diſtance in the Directing Plane, SO being the Direction of the Rays of Light, and pO the Direction of their Perpendicular Seats on the Reflecting Plane; p therefore repreſents the Perpendicular Seat of S on the Directing Line of that Plane, and ps made equal to pS, gives s the tranſpoſed Place of the Luminous Point in the Directing Plane, whence sO the Direction of the Reflected Rays is found. Fig. 143. N°. 3.

CASE 2.

When the Luminous Point is at an infinite Diſtance behind the Directing Plane.

The Method in this Caſe is the ſame as before; ps being made equal to pS the Perpendicular Support of the Luminous Point S on the Vaniſhing Line of the Reflecting Plane, whereby its tranſpoſed Place s is determined. *Q. E. I.* Fig. 143. N°. 4.

COR.

If the Luminous Point be at an infinite Diſtance either before or in the Directing Plane, no viſible Reflection can be formed.

For if the Luminous Point be before the Directing Plane, and the given Part of the Reflecting Plane lye beyond the Original Plane, the Reflection muſt fall on the backſide of that Plane, and cannot therefore be ſeen; and if the given part of the Reflecting Plane be on the hither Side of the Original Plane, the Reflection being thrown towards the Eye, cannot therefore fall on the Original Plane: and laſtly, if the Luminous Point be at an infinite Diſtance in the Directing Plane, its tranſpoſed Place being alſo in that Plane, the Reflected Rays are parallel to the Picture, and conſequently to the Original Plane, on which therefore they can produce no Reflection.

PROB. XXI.

The Center and Diſtance of the Picture, and the Vaniſhing and Interſecting Lines of an Original Plane, being given, together with the Image of a Luminous Point, and its Seat on that Plane; thence to find the Reflection of the Light on that Plane, from any given determinate part of a Reflecting Plane parallel to the Picture, whoſe Interſection with the Original Plane is given.

CASE 1. *and* 3.

When the Luminous Point is at a moderate Diſtance before, behind, or in the Directing Plane.

Let EFGH be the Original Plane, and MN its Interſection with the Reflecting Plane parallel to the Picture; and let S be the Luminous Point, and T its Oblique Seat on the Original Plane. Fig. 144. N°. 1, 2.

From S to O the Center of the Picture, draw SO, and compleat the ſubſtituted Plane OoST, cutting the Reflecting Plane in ap; then in SO, make ps and pS repreſent equal Lines, and s will be the tranſpoſed Place of the Luminous Point; whereby $\alpha\beta\gamma\delta$ the Reflection of $abcd$ may be obtained[b].

Dem. For the ſubſtituted Plane OoST is perpendicular to the Reflecting Plane; and O being the Vaniſhing Point of Perpendiculars to that Plane[c], p is the Perpendicular Seat of S on it, and conſequently s is the tranſpoſed Place of the Luminous Point. *Q. E. I.*

In Fig. N°. 1. the Luminous Point is at a moderate Diſtance before the Directing Plane, and S falling on the outſide of p, s falls between p and O[d]. Fig. 144. N°. 1.

In Fig. N°. 2. the Luminous Point is at a moderate Diſtance behind the Directing Plane, and S lying on the contrary Side of O from p, s falls between O and p; in both, Sp is harmonically divided in S, O, s, and p[e]. Fig. 144. N°. 2.

[a] Prob. 17.

[b] Prob. 8.

[c] Cor. 3. Prop. 7. B. IV.

[d] Cor. 2. Prob. 15.

[e] Cor. 1. Lem. 8. B. III.

Here the Point *s* is found, either by making *s*O and *sp* repreſent equal Lines from S, taken as its Vaniſhing Point; or by making O *s* and OS repreſent equal Lines from *p*, taken as its Vaniſhing Point[a].

Fig. 144. N°. 3. In Fig. N°. 3. the Luminous Point is at a moderate Diſtance in the Directing Plane, S *J* being the Direction of the Rays of Light, and T *J* the Direction of their Seats on the Original Plane; the ſubſtituted Plane O*opa* is found by drawing *o*a parallel to T *J*, and O*p* parallel to S *J*[b], and the tranſpoſed Place of the Luminous Point is determined by biſecting O*p* in *s*[c].

CASE 2.

When the Luminous Point is at an infinite Diſtance, before or behind the Directing Plane.

Fig. 144. N°. 4, 5. Here, the Luminous Point having no Seat on the Reflecting Plane, the Point *p* hath no Place; SO and O*s* are therefore made equal[d], by which means it is evident that all Lines whoſe Vaniſhing Point is *s*, incline to the Picture, and conſequently to the Reflecting Plane, in the ſame Angle as thoſe whoſe Vaniſhing Point is S, S*s* being here a Vaniſhing Line of Planes perpendicular to the Picture and Reflecting Plane[e]. *Q.E.I.*

Fig. 144. N°. 4. In Fig. N°. 4. the Luminous Point is at an infinite Diſtance behind the Directing Plane, and its tranſpoſed Place *s* is at an infinite Diſtance before that Plane.

Fig. 144. N°. 5. In Fig. N°. 5. the Luminous Point is at an infinite Diſtance before the Directing Plane, and its tranſpoſed Place is behind that Plane; and here, the Reflecting Plane is ſeen on the backſide, its Reflecting Side being ſuppoſed to be turned towards the Luminous Point, without which no Reflection could be produced.

COR.

If the Luminous Point be at an infinite Diſtance in the Directing Plane, no Reflection can be produced by the Reflecting Plane on any Plane whatſoever.

For in that Caſe, the Rays of Light being ſuppoſed parallel to the Picture, they are alſo parallel to the Reflecting Plane, and cannot therefore be Reflected by it.

SCHOL.

It would be ſuperfluous to add a Problem for finding the Reflection of a determinate part of a Reflecting Plane on an Original Plane, when thoſe Planes are both parallel to the Picture; ſeeing, in that Caſe, the tranſpoſed Place of the Luminous Point is found by this Problem, and the Reflection by Prob. X.

Neither was it neceſſary, in the ſeveral Problems of this Section, to introduce any Variety in the Shape of the propoſed part of the Reflecting Plane, the chief thing there intended, being to determine the tranſpoſed Place of the Luminous Point, in all the Variety of Poſitions that it can have, with reſpect to the Picture and Reflecting Plane; which being found, the Reflection of any propoſed part of the Reflecting Plane, on any one or more Original Planes, may be had by the Methods already taught[f].

GENERAL COROLLARY.

The Luminous Body, whether at a moderate or infinite Diſtance, has in the two laſt Sections been conſidered ſtrictly as a Point, without regard to its Bulk; by which means, the Projection of the propoſed Figure, whether it be taken as the Shadow of that Figure, or as the Appearance of the Light tranſmitted through, or Reflected by it from the Luminous Point[g], is determined by the Interſection of the Plane of the Projection with one certain Pyramid of Rays, having the Luminous Point or its tranſpoſed Place for its Vertex, and the outline of the propoſed Figure for its Bounds, and is therefore exactly terminated at the Edges.

This ſerves well enough for finding the Projection of a given Object, whilſt the apparent Size of the Luminous Body bears but a ſmall Proportion to it, the Center of the Luminous Body being taken as the Projecting Point; for in this Manner, the ſtrong part of the Projection will be determined, and its fainter Edges may from thence with a little Judgment be deſcribed.

But when the Luminous Body is of a conſiderable Extent, ſuch as a large bright Cloud, a great Fire, or the like, the Projection of an Object thereby produced is much leſs defined; every Point of the Luminous Body producing a particular Projection, whether of Light or Shade, all which particular Projections blended together, compoſe the entire Projection of the Object from the whole Light; whence it neceſſarily follows, that the Compound Projection thus formed, cannot be ſo determinate or defined, as that which is produced by a ſingle Lucid Point; the Compound Projection being the

ſtrongeſt,

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[a] Cor. 6. Lem. 8. B. III.

[b] Caſe 3. Prob. 8.

[c] Cor. 4. Prob. 15.

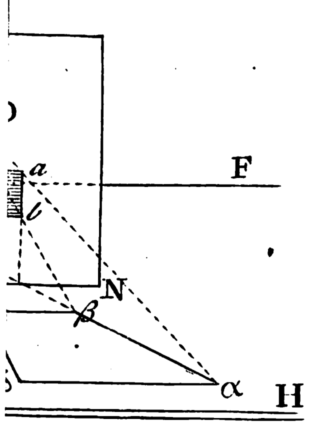
[d] Cor. 5. Prob. 15.

[e] Prop. 24. B. IV.

[f] Sect. 1.

[g] Gen. Cor. 3. Prob. 14.

No. 1.



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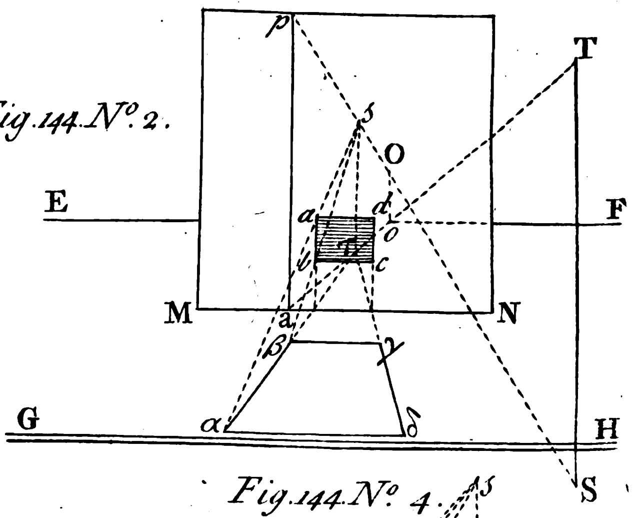

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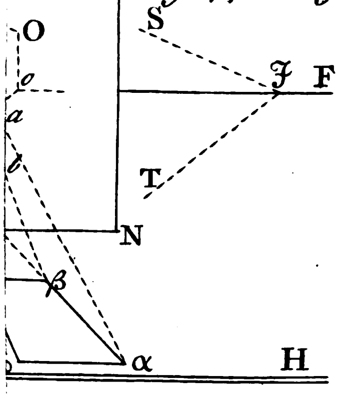

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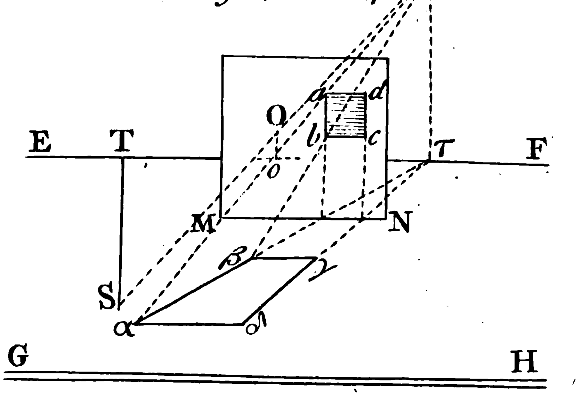


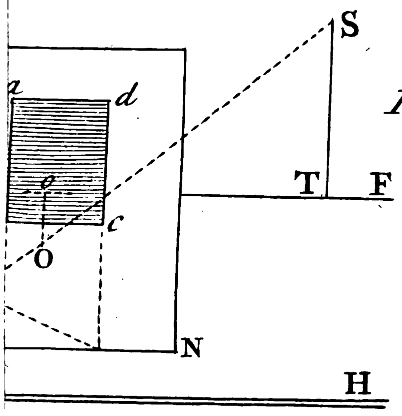
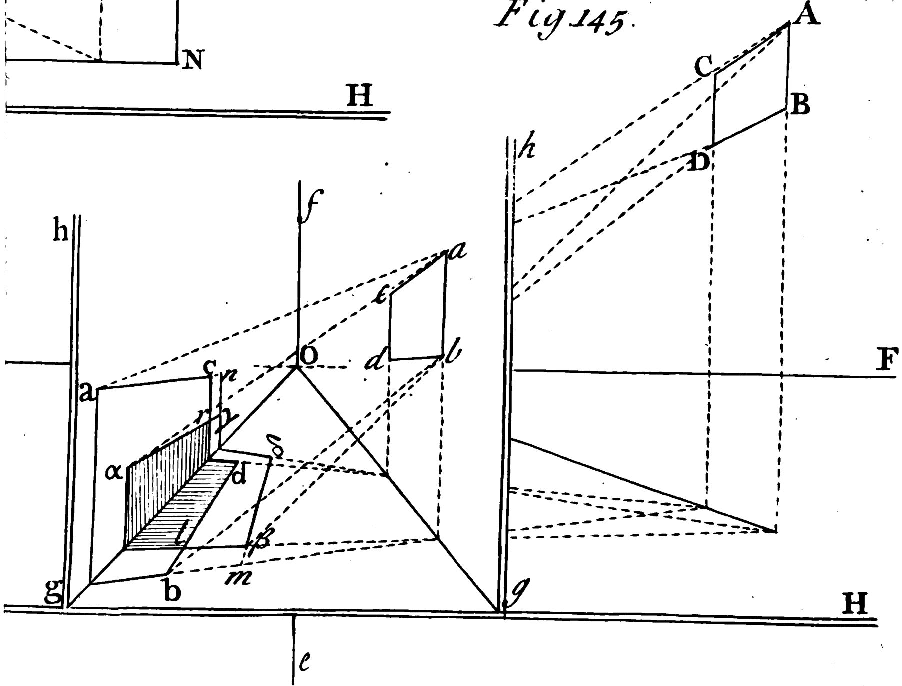


Fig. 144. No. 5.

Fig. 145.



J. Mynde sc.

ſtrongeſt, where moſt of the particular Projections unite to form it, and growing weaker towards the Edges, as the particular Projections ceaſe by degrees to cover each other.

This conſideration furniſhes a Method of deſcribing the Projections of Objects formed by ſuch a Light as we are now ſpeaking of; by chuſing a convenient Number of Points round the Borders of the Light, and finding the particular Projections of the Object from each of thoſe Points, by the help of which the compound Projection may be deſcribed; and altho' at firſt ſight, it may appear to be very laborious to trace out ſo many particular Projections, in order to find the Compound one, yet if the Points be properly choſen, two or three will for the moſt part be ſufficient, and the compound Projection may be thence compleated, which a little Obſervation and Experience will render eaſy; the rather for that the Projections of this Sort, whether of Light, or Shade, are generally ſo very imperfect and undefined at the Edges, that it is hardly perceivable preciſely where they begin or end, which therefore leaves a little Latitude for the Deſcription.

We ſhall illuſtrate this by an Example in the following Problem.

PROB. XXII.

A Luminous Body of a conſiderable Extent being given, together with a Figure in an Original Plane; thence to find the compound Projection of that Figure on one or more given Planes.

Let ABCD be the Luminous Body, and *abcd* a given Figure in the Plane *efgh*, the compound Projection of which is required on the Planes EFGH and *ef*gh. Fig. 145.

Having choſen any two extream Points A and D of the Luminous Body for projecting Points, find $\alpha\beta\gamma\delta$ and abcd the particular Projections of *abcd* on the given Planes from A and D[a], and compleat the Figure a$n\delta m$, and that will be the compound Projection of *abcd* from the entire Light ABCD; of which αrdl will be the Extent of the ſtrongeſt part, and a$n\delta m$ will be the utmoſt Verge of the weaker part; and the Space between theſe two, will be the Place wherein the Light gradually increaſes or decreaſes, according as the Projection required is the Shadow of *abcd* taken as an Opake Object, or as it is the Shape of the Light tranſmitted through *abcd* as an Aperture in the Plane *efgh*, or Reflected by it as part of a Reflecting Plane. In the firſt Caſe, αrdl will be the Place of the deepeſt Shadow, from whence its Strength will gradually decreaſe till it is loſt at the outward Verge in the full Light; in the other Caſe, αrdl will be the Place of the ſtrongeſt Light, which will decreaſe gradually till it is loſt at the outward Verge in the ſtrong Shade, to which no part of the Light from ABCD can reach.

Dem. For ac being the Projection of *ac* from D, and $\alpha\gamma$ being the Projection of the ſame Line from A, it is evident that the Projections of *ac* from all intermediate Points between A and D, muſt fall between $\alpha\gamma$ and ac, ſo that no Projection of *ac* from any Point between AC and BD, can fall below $\alpha\gamma$, or above ac; the ſame is to be underſtood of the Projections of the other Sides of the Figure *abcd*, which muſt all fall between their reſpective inward and outward Boundaries; conſequently αrdl is the Space wherein no part of the Projection of the Sides of the Figure *abcd* from any Point of ABCD can fall, and a$n\delta m$ is the Limit beyond which they can never reach. *Q.E.I.*

After this manner likewiſe may the *Penumbra*, or the faint part of the Shadows of Objects produced by the Sun, Moon, or a Candle, be found, where ſuch nicety is requiſite.

COR. 1.

A Window, a Door, or any other Aperture in a Building, has alſo the Effect of a large Light, when the Light without is uniform, ſuch as a clear ſerene Sky, without Clouds, or direct Sun Shine; in which Caſes, the Extent of the Light which the Aperture admits, is to be found by Lines drawn from convenient Points in the outward Boundary of the Aperture, and paſſing by its inner Boundary.

COR. 2.

Hence alſo appear the Effects of Shadows produced by two or more different Luminous Points; namely, that wherever any of the particular Shadows fall upon, or

croſs

[a] Prob. 6. and 14.

croſs any of the others, they there form a compound Shadow deeper than in the other Places, in proportion as more of the particular Shadows meet together.

COR. 3.

It is farther to be obſerved, that the Light which falls on a Plane from a near Radiant Point, as a Candle or a Lamp, is not equally bright; but that ſuch part of the Plane as lies perpendicularly expoſed to the Luminous Point, is the moſt enlightened by it, and the Light decreaſes by Degrees all round, as the Rays fall more obliquely on the Plane.

For the Rays proceeding from a Radiant Point diverge in their Progreſs, which Divergence of the Rays from a near Light is very ſenſible; and as a Perpendicular from the Luminous Point to the propoſed Plane, is the ſhorteſt Line that can from thence be drawn to it, the Rays of Light muſt be there the cloſeſt, and muſt therefore enlighten the Plane more ſtrongly in that part, than where they fall obliquely on it, and by reaſon of their greater Length or Diſtance from the Radiant Point, are ſo much the farther aſunder.

But when the Radiant Point is infinitely diſtant, the Rays of Light being then ſenſibly parallel, the Light received by a Plane expoſed to it, is to Appearance uniform and equal.

COR. 4.

A Shadow is ſaid to be ſtronger or weaker, in proportion to the Light the ſhadowed Part receives, compared with that which falls on its Borders, or according to the different Degrees of Light within and without the Shadow; the greater Difference there is between them, the ſtronger is the Shadow, and the leſs Difference there is, the Shadow is the weaker.

Hence if the Shadow be entirely deprived of Light, it may yet be ſaid to be ſtronger or weaker, as its Borders are more or leſs enlightened; and conſequently a Shadow produced by a total Interception of the Rays of the Sun, is more ſtrong than that occaſioned by the Interception of the Light of the Moon, or of a Candle; and on that account, the former would be more ſharp or defined than the latter, were not that Difference in ſome meaſure leſſened by the Refractive Power of the Air, which being greatly ſtored with Light from the Sunſhine, throws ſome ſhare of it upon the Shadow; making it thereby ſo much the weaker, which Refracted Light is not ſo plentiful by Moon or Candle Light, and the effect it produces is therefore the leſs.

SCHOL.

In this Section, Light has been conſidered as reflected by a Poliſhed Plain Surface; which Reflection is the moſt ſimple, every Ray of Light being regularly reflected according to its Angle of Incidence on the Plane; by which means, the Place of the Reflected Light is determined by the tranſpoſed Place of the Luminous Point, as has been ſhewn: but when the Reflection comes from a rough unpoliſhed Surface, the Rays not being then regularly reflected, the Place of the Reflected Light is but little governed by the Direction of the incident Rays; but in regard that the Light which falls on ſuch a Surface, is reflected with all kinds of Directions, but the Surface itſelf appears moſt enlightened when the Eye is perpendicularly oppoſed to it, as being then more compleatly ſeen; hence the Vaniſhing Point of Perpendiculars to that Surface may be uſed inſtead of the tranſpoſed Place of the Luminous Point, and may better ſerve the purpoſe; though from Bodies whoſe Surfaces are the leaſt rough, and which approach neareſt to a Poliſh, the Light may be more plentifully reflected towards its true Place than directly forward; ſo that the tranſpoſed Place of the Luminous Point may, in ſuch Caſes, ſerve to ſhew where the ſtrongeſt Reflection will fall: but this, with reſpect to the Quantity and Extent of the Reflected Light, muſt be managed with a due regard to the Smoothneſs, Shape, and Colour of the Surface which reflects it, as well as of that on which it falls; all which muſt be determined by Judgment and Obſervation.

For the different Degrees of Light and Shade, depending on ſo many different Circumſtances of the Situation, Colour, Shape, Materials, Opacity, Tranſparence, Smoothneſs, or Roughneſs of Objects, as well as on the different Brightneſs, Colour, Bulk, Poſition, and Direction of the primary Light, not to mention the various Effects of the Refraction as well as Reflection of Light, with the Differences of Colour and Strength thence ariſing, it would be impracticable to give Rules for ſo great and complicated

plicated a Variety; the ſureſt Method therefore of ſucceeding in the Deſcription of Light and Shade in this extenſive View, is a diligent Study and Imitation of Nature; which however may be greatly aſſiſted by the Rules here taught, whereby in general the Shapes and Extents of the Lights and Shades may be delineated, although their Strength or Colour is not determined.

SECTION III.

Of the Reflected Images of Points, Lines, and Plain Figures, as they appear in Reflecting Planes.

1. IN every Reflecting Plane, ſuch as a poliſhed Plain Mirror or Looking Glaſs, or the even ſmooth Surface of ſtanding Water, the Reflection of an Object is as far diſtant behind the Reflecting Surface, as the Object itſelf is before it; ſo that if from any Point of the Object, a Perpendicular be drawn to the Reflecting Plane, and produced to an equal Diſtance behind it, the Extremity of that Perpendicular will be the Reflection of the Point propoſed.

Let QR be a Reflecting Plane, a the propoſed Point, and A its Perpendicular Seat on that Plane; produce the Support aA of the given Point to α behind the Reflecting Plane, until Aα and Aa be equal, and α will be the Reflection of a. Fig. 146. N°. 1.

2. If an Original Line be expoſed to a Reflecting Plane, its Reflection will make the ſame Angle with that Plane behind it, as the Original Line doth before it, but with a contrary Direction; and the Original Line and its Reflection will be in a Plane paſſing through the Original Line perpendicular to the Reflecting Plane.

Let aD be the propoſed Line, AD its Perpendicular Seat on the Reflecting Plane QR, and D the Interſection of the given Line with its Seat: then D being a Point common to the given Line and its Reflection, Dα drawn from D to α the Reflection of any other Point a in aD, will be the Reflection of that Line; and the Triangles aAD, αAD, being ſimilar and equal, the Angles aDA, ADα are equal; and the Original Line aD and its Reflection αD are both in the Plane LNM, which paſſes through the given Line aD and its Perpendicular Seat AD on the Reflecting Plane, which laſt is therefore alſo the Perpendicular Seat of the Reflection αD on that Plane. Fig. 146. N°. 1.

Here, the Line αD is called the Reflection of aD, not that it is reflected back from the Plane QR, but as it is the Continuation behind that Plane of the Line Dd, which is the Reflection of aD taken as a Ray of Light falling on the Reflecting Plane[a]; ſo that the Difference between this kind of Reflection, and that treated of in the preceeding Section, is, that there, the real or effective part of the Reflection is the part Dd which lies before the Reflecting Plane, and here, it is αD ſuch part of it as lies behind that Plane; there, it is conſidered as the new Direction which a Ray of Light acquires by being reflected, and here, it is the Repreſentation of the propoſed Line, appearing as a real Line within and behind the Reflecting Plane.

3. If the propoſed Original Line be parallel to the Reflecting Plane, its Reflection will alſo be parallel to that Plane, and as far behind it as the Original Line is before it.

For all Points of the Original Line being equally diſtant from the Reflecting Plane, the Reflections of thoſe Points are alſo equally diſtant from that Plane, the Diſtances of every Original Point and its Reflection from the Reflecting Plane being equal[b].

4, If the Original Line be perpendicular to the Reflecting Plane, the Original Line and its Reflection will make one continued ſtraight Line[c].

Thus aA and its Reflection Aα make one continued ſtraight Line $a\alpha$.

5. The Reflection of any determinate part of an Original Line is equal to that part.

For the Triangles aAD, αAD being ſimilar and equal[d], aD and αD are equal; and for the ſame Reaſon, if β be the Reflection of b, bD and βD are equal, and conſequently $\alpha\beta$ is equal to ab.

6. The Reflections of any two Original Lines Da, Db, which interſect in D, make together the ſame Angle as the Originals do. Fig. 146. N°. 2.

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[a] Art. 1. Sect. 2.

[b] Art. 1.

[c] Art. 2.

[d] Art. 2.

For if a Triangle *ab*D be compleated on the two Original Lines, the Reflections of the Sides of that Triangle being equal to their reſpective Originals[a], their correſponding Angles will alſo be equal[b]; and conſequently the Angle *a*D*b* made by the propoſed Lines, will be equal to that made by their Reflections.

7. The Reflections of all parallel Original Lines are parallel.

For the Interſections of the Original Lines being infinitely diſtant, the Interſections of their Reflections are alſo infinitely diſtant.

8. The Reflection of any Plain Figure is ſimilar and equal to its Original.

For their correſponding Sides and Angles are equal[c].

9. The Reflection of an Original Plane inclines to the Reflecting Plane in the ſame Angle as the Original Plane doth, but the contrary way.

Fig. 146. N°. 3. Let QR be the Reflecting Plane, NL an Original Plane, and DN their common Interſection; and let *a*AD be another Plane perpendicular to DN, cutting the given Planes in AD and *a*D, and let αD be the Reflection of *a*D.

The Reflection of the Plane LN paſſing through the Reflections of DN and *a*D, and DN coinciding with its own Reflection, the Reflection of the Plane LN therefore paſſes through DN and αD; and becauſe *a*D and αD are both in the ſame Plane *a*AD[d], which is perpendicular to DN, the Plane *a*Dα is therefore perpendicular to all Planes which paſs through DN[e], and conſequently to the Reflecting Plane QR, the Original Plane LN, and its Reflection NM; and therefore the Angles *a*DA, ADα, made by the Interſections of the Plane *a*Dα with thoſe three Planes, are the Angles of Inclination of the Planes LN and MN to the Plane QR; but the Angles *a*DA and ADα are equal[f], therefore the Original Plane LN and its Reflection NM incline to the Reflecting Plane QR in equal Angles.

10. If the Original Plane be parallel to the Reflecting Plane, its Reflection will alſo be parallel to that Plane, and as far diſtant behind it, as the Original Plane is before it.

Becauſe the Reflection of every Line in the Original Plane is parallel to the Reflecting Plane, and at an equal Diſtance from it with its Original[g].

11. If the Original Plane be perpendicular to the Reflecting Plane, the Original Plane and its Reflection will make one continued Plane[h].

Fig. 146. N°. 1. Thus the Plane LN being perpendicular to the Reflecting Plane QR, its Reflection is AN the ſame Plane continued behind the Reflecting Plane.

12. If two Original Planes interſect, their Reflections will incline to each other in the ſame Angle as the Original Planes do.

Fig. 146. N°. 2. Let LN and NM be two Original Planes, and DN their common Interſection, and let *a*D and D*b* be their Interſections with a Plane perpendicular to them both, and conſequently *a*D*b* the Angle of their Inclination.

Then becauſe *a*D and *b*D are perpendicular to DN[i], their Reflections are alſo perpendicular to the Reflection of DN[k], and therefore meaſure the Angle of Inclination of the Reflections of the Planes LN and NM[l]; but the Reflection of the Angle *a*D*b* is equal to that Angle[m], therefore the Reflections of the Planes LN and NM incline to each other in the ſame Angle as the Original Planes do.

13. Hence if the Original Planes be perpendicular, their Reflections will be perpendicular; and if the Original Planes be parallel, their Reflections will be parallel[n].

SCHOL.

Thus far, the Reflections of Objects in a Reflecting Plane have been conſidered with regard to the true place where the Judgment conceives them to lie, as if they were real Objects exiſting behind the Reflecting Plane in thoſe particular Situations; but the Deſign of this Section being to ſhew how to deſcribe the Stereographical Appearance of thoſe Reflections, to an Eye in a given Poſition with reſpect to the Reflecting Plane, the Images of the Original Objects being given, we ſhall next conſider after what manner thoſe Reflections appear.

Fig. 146. N°. 1. 14. If an Eye be placed any where before the Reflecting Plane, the Reflected Image of any Line *a*D will appear the ſame as if its Reflection αD were the Original Line, ſeen through the Plane QR ſuppoſed to be tranſparent; or as the Image of αD on the Plane QR.

Thus if I be the Place of the Eye, and T its Perpendicular Seat on the Reflecting Plane QR, draw TA and Iα, cutting each other in a, and aD will be the imaginary Projection, or the Image of αD on the Plane QR[o], and conſequently it will be the Appearance of the Reflection of *a*D on the Plane QR, as ſeen by an Eye at I.

For

[a] Art. 5.

[b] 8 El. 1.

[c] Art. 5. and 6.

[d] Art. 2.

[e] 18 El. 11.

[f] Art. 2.

[g] Art. 3.

[h] Art. 9.

[i] 19 El. 11.

[k] Art. 6.

[l] Art. 9.

[m] Art. 6.

[n] Art. 6. and 7.

[o] Schol. 1. Caſe 4. Prob. 6.

Plate 68 Book 5 Sect. 3.

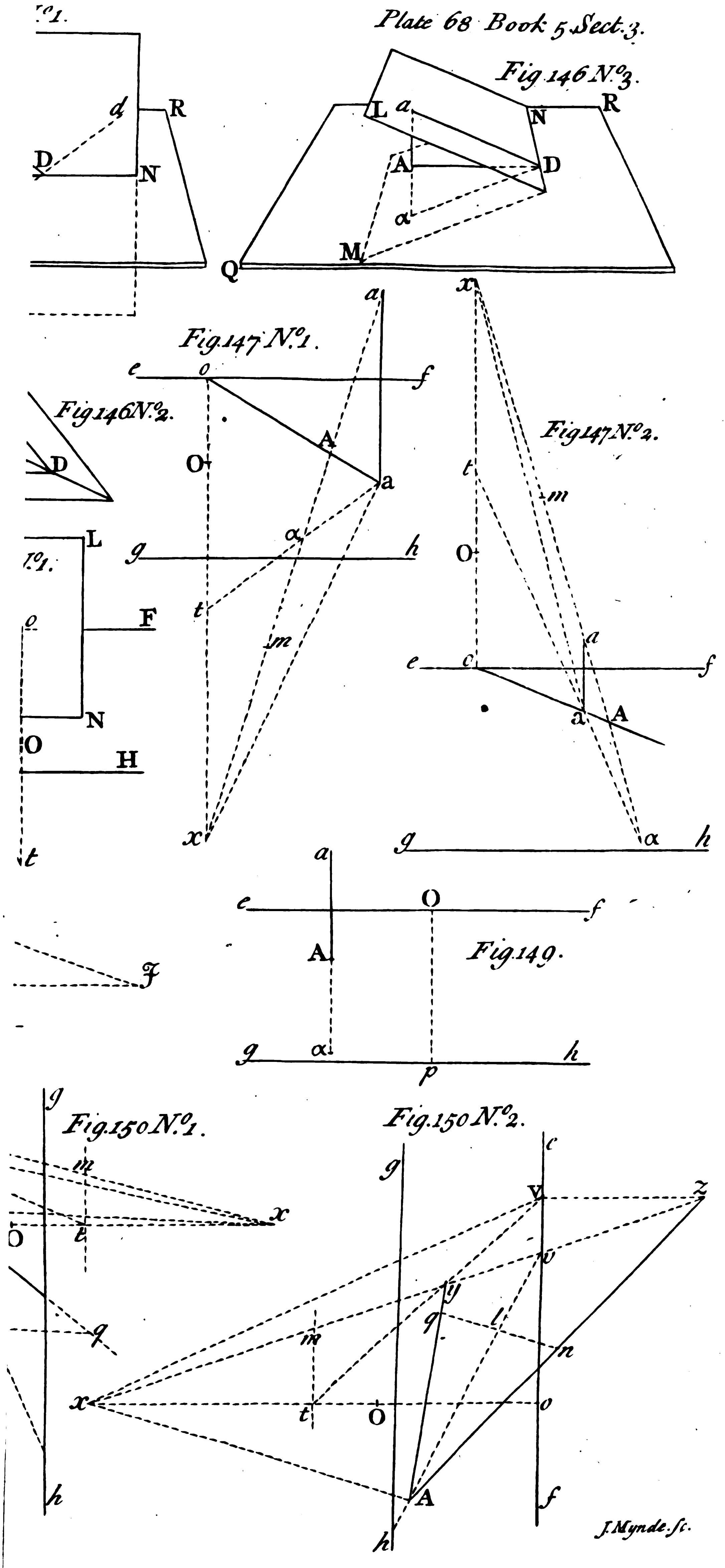

Fig. 146 N.o 3.

Fig. 147 N.o 1.

Fig. 146 N.o 2.

Fig. 147 N.o 2.

Fig. 149.

Fig. 150 N.o 1.

Fig. 150 N.o 2.

J. Mynde sc.

For a D is the Interſection of the Plane QR with a Plane paſſing through I and the Line α D.

15. If the Eye be removed to J in the Line IT produced to an equal Diſtance behind the Reflecting Plane, the ſame Line aD will be the Image of the Original Line aD on the Plane QR, as ſeen from J.

Draw Ja cutting TA in y, and yD will be the Image of aD on the Plane QR, as ſeen from J; it remains to be ſhewn that y coincides with a, and conſequently that aD continues the ſame whether the Eye be placed at I or J.

In the Similar Triangles ITa, aAα, $IT : A\alpha :: Ta : aA$
And by Compoſition $IT+A\alpha : A\alpha :: Ta+aA=TA : aA$
And in the Similar Triangles JTy, yAa, $JT : Aa :: Ty : yA$
And by Compoſition $JT+Aa : Aa :: Ty+yA=TA : yA$
But by Conſtruction $IT=JT$, and $Aa=A\alpha$,
Therefore $IT+A\alpha : A\alpha :: JT+Aa : Aa$
And conſequently $TA : aA :: TA : yA$
wherefore $aA=yA$, and the Points a and y therefore coincide.

16. Hence it is, that the Reflected Images of Objects exhibit the ſame Appearance to an Eye in a given Situation before the Reflecting Plane, as the Objects themſelves would do, were the Reflecting Plane tranſparent, and the Eye removed from its firſt Situation in a Line perpendicular to that Plane, to an equal Diſtance behind it; with this only Difference, that the Reflected Image of the Object ſeen from the Point I, will be the Reverſe of the direct Image of that Object ſeen from J; that is, there will be the ſame Difference between them, as there is between a Print viewed on the Side of the Impreſſion, and the Appearance of that Impreſſion ſeen on the backſide, when the Paper is oiled; or between the original Graving on a Copper Plate, and the Impreſſion taken off of it.

17. Hence if the Image of a Point be found, as far perpendicularly diſtant behind the Reflecting Plane, as the Eye is really before it, and that Point be conſidered as a Projecting Point repreſenting the Eye of a Spectator, and the Image of the Reflecting Plane be conſidered as a Picture expoſed to that Eye[a]; then the imaginary Projections on this Picture, of any Objects whoſe direct Images are given, will be the ſame with the Reflected Images of thoſe Objects, as they appear in the Reflecting Plane, to the Eye in its true Situation.

18. A reflexible Object may be either before or behind the Eye, ſo as the Object and the Eye lie both before the Reflecting Plane; that is, facing its Reflecting Side; but the Reflection muſt always appear behind the Reflecting Plane, and conſequently before the Eye; wherefore the Image of a reflexible Point in an Original Line, may be either in its Perſpective, Projective, or Tranſprojective Part; but the Image of its Reflection can only be in the Perſpective, or Projective Part of the Reflected Line, and can never fall out of the Bounds of the Reflecting Plane; and therefore ſuch part of the Indefinite Image of the Reflection of a Line as lies without thoſe Bounds, is not real but imaginary.

DEF. 13.

The Image of a Point, as far perpendicularly diſtant behind the Reflecting Plane, as the Eye is really before it, ſhall be called *the tranſpoſed Place of the Eye.*

COR.

The tranſpoſed Place of the Eye is the Image of its Reflection[b].

DEF. 14.

The Image of the Reflection of any Object, as of a Point, Line, or Plane, ſhall be called ſimply, *the Reflection of that Point, Line, or Plane*; or otherwiſe, *the Reflected Point, Line, or Plane*; and the direct Image of an Object ſhall be generally ſpoken of, as if it were the Original Object which it repreſents, for the Reaſon formerly mentioned[c].

PROB. XXIII.

The Center and Diſtance of the Picture, and the Vaniſhing and Interſecting Lines of a Reflecting Plane which inclines to the Picture, being given, together with a Point and its Seat on the Reflecting Plane; thence to find the Reflection of that Point.

Let

[a] Schol. 1. Caſe 4. Prob. 6.

[b] Art. 1.

[c] Schol. before Prob. 1.

Fig. 147. Nº. 1, 2. Let O be the Center of the Picture, and $efgh$ the Reflecting Plane, a the given Point, and a its Oblique Seat on that Plane.

METHOD 1.

From x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, draw its Vertical Line xo, and having drawn ao, draw xa cutting it in A; then make Aa and Aα repreſent equal Lines, and α will be the Reflection of the Point a.

Dem. For A being the Perpendicular Seat of a on the Reflecting Plane[a], and αA repreſenting a Line equal to the Original of Aa, α is the Image of a Point, as far perpendicularly diſtant behind the Reflecting Plane, as the Point a is before it, and is therefore its Reflection[b]. *Q. E. I.*

METHOD 2.

Biſect xo in t and draw ta, which will cut xa in α the Reflection ſought.

Dem. Becauſe x is the Vaniſhing Point of Perpendiculars to the Reflecting Plane, it is the Indefinite Image of a Line drawn from the Eye perpendicular to that Plane[c], and of every poſſible Point in that Line[d], the Point x therefore repreſents the tranſpoſed Place of the Eye[e]; and becauſe xo is biſected in t, t is the Oblique Seat of x on the Reflecting Plane[f]; the Point α where ta and xa interſect, is therefore the imaginary Projection, or the Image of a, on the Plane $efgh$, as ſeen from an Eye at x[g]; and conſequently it is the Reflection of the Point propoſed[h]. *Q. E. I.*

COR. 1.

The Point α found by both theſe Methods is the ſame.

Becauſe xo is biſected in t, and aa is parallel to it[i], ax, at, ao, and aa, are Harmonical Lines[k]; the Line xa which cuts them all four, is therefore Harmonically divided by them in x, α, A, and a[l]; but Aa and Aα repreſenting equal Parts of the Line xa, whoſe Vaniſhing Point is x[m], xa is Harmonically divided in x, α, A, and a[n]; and the Points x, a, and A, being the ſame in both Methods, the fourth Point α is likewiſe the ſame[o].

COR. 2.

In Fig. Nº. 1. x taken as the tranſpoſed Place of the Eye, repreſents a Projecting Point at a moderate Diſtance before the Directing Plane; and in Fig. Nº. 2. it repreſents a Point at a moderate Diſtance behind that Plane; in both, it is under or behind the Reflecting Plane[p], x falling below its Seat t in Fig. Nº. 1. and above it in Fig. Nº. 2. its Support xt being there inverted by Tranſprojection.

PROB. XXIV.

The Center and Diſtance of the Picture, and a Reflecting Plane parallel to the Picture, being given, together with a Point and its Seat on that Plane; thence to find the Reflection of that Point.

Fig. 148. Nº. 1. Let O be the Center of the Picture, LMN the Reflecting Plane, a the given Point, and A its Perpendicular Seat on that Plane.

METHOD 1.

Produce the Perpendicular Support Aa of the given Point to its Vaniſhing Point O, and make Aα and Aa repreſent equal Lines, and α will be the Reflection of a.

Dem. For the Reflecting Plane being parallel to the Picture, the Vaniſhing Point of Perpendiculars to that Plane is at O the Center of the Picture; wherefore α repreſents a Point as far perpendicularly behind the Reflecting Plane, as the Point a is before it, and is therefore its Reflection. *Q. E. I.*

METHOD 2.

Let a be the given Point, and a its Oblique Seat on the Reflecting Plane with reſpect to any Original Plane EFGH, not perpendicular to the Picture.

Having drawn oO the Vertical Line of the Plane EFGH, take Ot in that Line equal to Oo, and draw ta, which will cut aO in α the Reflection of a.

Fig. 148. Nº. 2. *Dem.* Let I$k\tau p$ repreſent the Vertical Plane of the Original Plane EFGH, and τp its Interſection with the Reflecting Plane, and let I repreſent the Eye, and kp the Line of Station; then Iτ parallel to kp, will give τ the Oblique Seat of the Eye on the

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[a] Prop. 49. B. IV.

[b] Art. 1.

[c] Prop. 7. B. IV.

[d] Cor. Theor. 8. B. I.

[e] Def. 13.

[f] Caſe 1. Prob. 43. B. IV.

[g] Prob. 1.

[h] Art. 17.

[i] Meth. 2.

[k] Lem. 4. B. III.

[l] Lem. 8. B. III.

[m] Meth. 1.

[n] Cor. 6. Lem. 8. B. III.

[o] Lem. 2. B. III.

[p] Art. 15.

the Reflecting Plane; IT perpendicular to *τp*, will give T the Perpendicular Seat of the Eye on that Plane; *J*T being made equal to IT, *J* will be the Original of the tranſpoſed Place of the Eye; and *Jt* parallel to *kp*, gives *t* the Oblique Seat of *J* on the Reflecting Plane: now, becauſe of the Similar Triangles I*τ*T, T*Jt*, *J*T and TI being equal, T*τ* and T*t* are alſo equal; and *τt* being a Line parallel to the Picture, the Images of T*τ* and T*t* are equal: but O is the Image of T, as well as of Fig. 148. the tranſpoſed Place of the Eye, and *o* is the Image of *τ*[a]; wherefore O*t* being made N°. 1. equal to O*o*, gives *t* the Image of the Oblique Seat of the tranſpoſed Place of the Eye on the Reflecting Plane; conſequently *t*a cuts *a*O in *α*, the imaginary Projection of *a* on that Plane, and *α* is therefore the Reflection of *a*. *Q. E. I.*

COR. 1.

The Point *α* found by both theſe Methods, is the ſame.

Draw *a*A parallel to O*o*, which will cut *a*O in A the Perpendicular Seat of *a* on the Reflecting Plane, *a*A being the Interſection of that Plane with a Plane perpendicular to it, paſſing through *a*; now, a*o*, aO, a*t*, and aA, being Harmonical Lines, *a*O is Harmonically divided by them in *a*, A, *α* and O, as it is by the firſt Method, A*a* and A*α* repreſenting equal Lines[b]; the Point *α* is therefore the ſame in both Methods.

COR. 2.

The Point O, which is here the Reflection of the Spectator's Eye, repreſents a Projecting Point at a moderate Diſtance beyond the Reflecting Plane, having *t* for its Oblique Seat on that Plane with reſpect to the Plane EFGH.

PROB. XXV.

A Reflecting Plane perpendicular to the Picture, and a Point with its Seat on that Plane, being given; thence to find its Reflection.

Let O be the Center of the Picture, and *efgb* the Reflecting Plane, *a* the given Fig. 149. Point, and A its Perpendicular Seat on that Plane.

Make A*α* equal to A*a*, and *α* will be the Reflection deſired.

Dem. For the Reflecting Plane being Perpendicular to the Picture, the Vaniſhing Point of Perpendiculars to that Plane is infinitely diſtant, the Perpendicular Support A*a* of the Point *a* on that Plane, is therefore parallel to the Picture, and conſequently A*a* and A*α* which are equal, repreſent equal Lines. *Q. E. I.*

COR. 1.

Here, the tranſpoſed Place of the Eye repreſents a Projecting Point at a moderate Diſtance in the Directing Plane; and the Directions of the Projecting Lines and of their Seats on the Reflecting Plane, are both Perpendicular to the Vaniſhing Line of that Plane.

For the Reflecting Plane being Perpendicular to the Picture, the Eyes Director of that Plane is Perpendicular to it[c], in which Line the tranſpoſed Place of the Eye, as well as its Seat on the Reflecting Plane, muſt therefore lye[d]; and the Directions of all Lines proceeding from either of theſe Points, are therefore parallel to the Eyes Director, and conſequently perpendicular to the Vaniſhing Line of the Reflecting Plane.

COR. 2.

Hence, the ſecond Method propoſed in the two foregoing Problems cannot be here applied.

For the Images of the Projecting Lines which proceed from the tranſpoſed Place of the Eye and its Seat on the Reflecting Plane, and paſs through the propoſed Point *a* and its Seat A on that Plane, being both perpendicular to *ef*[e], they muſt coincide in the ſame ſtraight Line *aα*, and cannot therefore, by their Interſection, determine the Projection required.

However, the Analogy between this, and the preceeding Caſes, appears from hence, that as there, the Line *aα* was Harmonically divided in *a*, A, *α*, and the Vaniſhing Point of Perpendiculars to the Reflecting Plane[f]; here, this fourth Point being infinitely diſtant, the Line *aα* is biſected in A.

PROB. XXVI.

The Center and Diſtance of the Picture, and the Vaniſhing and In-

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[a] Cor. Theor. 8. B. I.

[b] Cor. 1. Prob. 23.

[c] Cor. 1. Theor. 9. B. I.

[d] Def. 13.

[e] Cor. 1.

[f] Cor. 1. Lem. 1. B. III.

terſecting Lines of a Reflecting Plane inclining to the Picture, and the indefinite Image of a Line out of that Plane being given; thence to find its Reflection.

Fig. 150. N°. 1, 2, 3, 4. Let $efgh$ be the Reflecting Plane, Az the indefinite Image of a Line, z its Vaniſhing Point, and A its Interſection with the Reflecting Plane.

METHOD 1.

From x, the Vaniſhing Point of Perpendiculars to the Reflecting Plane, draw xz, cutting ef in v, and in the Vaniſhing Line xz, find a Point y ſubtending with v, an Angle equal to that ſubtended by v and z[a]; and Ay will be the Reflection of Az, and y its Vaniſhing Point.

Dem. For xz being the Vaniſhing Line of a Plane perpendicular to the Reflecting Plane, paſſing through Az[b], it is alſo the Vaniſhing Line of the Plane in which the Reflection of Az lyes[c]; and the Vaniſhing Points y and z ſubtending equal Angles with v, the Line Ay inclines to the Reflecting Plane in the ſame Angle, but the contrary way, that the given Line Az doth[d]; and A being a Point common to that Line and its Reflection, Ay is therefore the Reflection of Az[e], and y is its Vaniſhing Point. *Q. E. I.*

METHOD 2.

Biſect the Vertical Line xo in t, and having drawn zv the Oblique Support of z on ef, draw tv, which will cut xz in the ſame Point y, whence Ay is determined as before.

Dem. For t being the Oblique Seat of x the tranſpoſed Place of the Eye on the Reflecting Plane[f], tv cuts xz in y the *Focus* of the Projection of Az from the Point x[g]; and vo, vt, vx, and vz being Harmonical Lines, xz is Harmonically divided by them in x, y, v, and z[h]; but xz is alſo Harmonically divided by the former Method[i], and the Points x, v and z being the ſame in both, the Point y is alſo the ſame. *Q. E. I.*

COR.

Hence, the *Focus* of the Projection of any Line on any given Plane, from the tranſpoſed Place of the Eye, taken as a Projecting Point, is the ſame with the Vaniſhing Point of the Reflection of that Line by the ſame Plane, taken as a Reflecting Plane.

METHOD 3.

Having found Av the Perpendicular Seat of Az on the Reflecting Plane, draw Ax, and from any Point n in Az, draw nl parallel to Ax, cutting Av in l, and produce nl to q, till nl and lq be equal; then a Line Ay drawn through A and q will be the indefinite Reflection of Az.

Dem. For the given Line Az, its Seat Av, its Reflection Ay, and the Line Ax, being always Harmonical Lines[k], the Line nq drawn parallel to Ax, one of theſe Harmonicals, is biſected by the other three[l]. *Q. E. I.*

COR. 1.

If xv be biſected in m, then, if either of the Points z or y fall between m and v, the other will fall beyond v; and *vice verſa*, if either of them fall beyond v, the other will fall between m and v[m].

Fig. 150. N°. 1, 2. Thus, in Fig. N°. 1. z falls between m and v, y therefore falls beyond v; and in Fig. N°. 2. z falls beyond v, y therefore falls between m and v.

COR. 2.

If either of the Points z or y fall between m and x, the other will fall beyond x; and *vice verſa*, if either of them fall beyond x, the other will fall between m and x[n].

Fig. 150. N°. 3, 4. Thus, in Fig. N°. 3. z falls between m and x, y therefore falls beyond x; and in Fig. N°. 4. z falls beyond x, y therefore falls between m and x.

COR. 3.

If either of the Points z or y biſect xv, the other will be infinitely diſtant[o], and the Line which ought to tend to that Point, will therefore be parallel to xv, and conſequently repreſent a Line parallel to the Picture.

Fig. 150. N°. 5, 6. Thus, in Fig. N°. 5. z biſects xv, the Point y is therefore infinitely diſtant, and the Re-

[a] Prop. 24. B. IV.

[b] Cor. 3. Prop. 20. B. IV.

[c] Art. 2.

[d] Prop. 24. B. IV.

[e] Art. 2.

[f] Meth. 2. Prob. 23.

[g] Cor. 1. Caſe 1. Prob. 3.

[h] Cor. 1. Meth. 2. Prob. 23.

[i] Meth. 2. Caſe 2. and 4. Prob. 15.

[k] Def. 2. B III.

[l] Lem. 7. B. III.

[m] Cor. 2. Prob. 15.

[n] Cor. 3. Prob. 15.

[o] Cor. Caſe 2. and 4. Prob. 15.

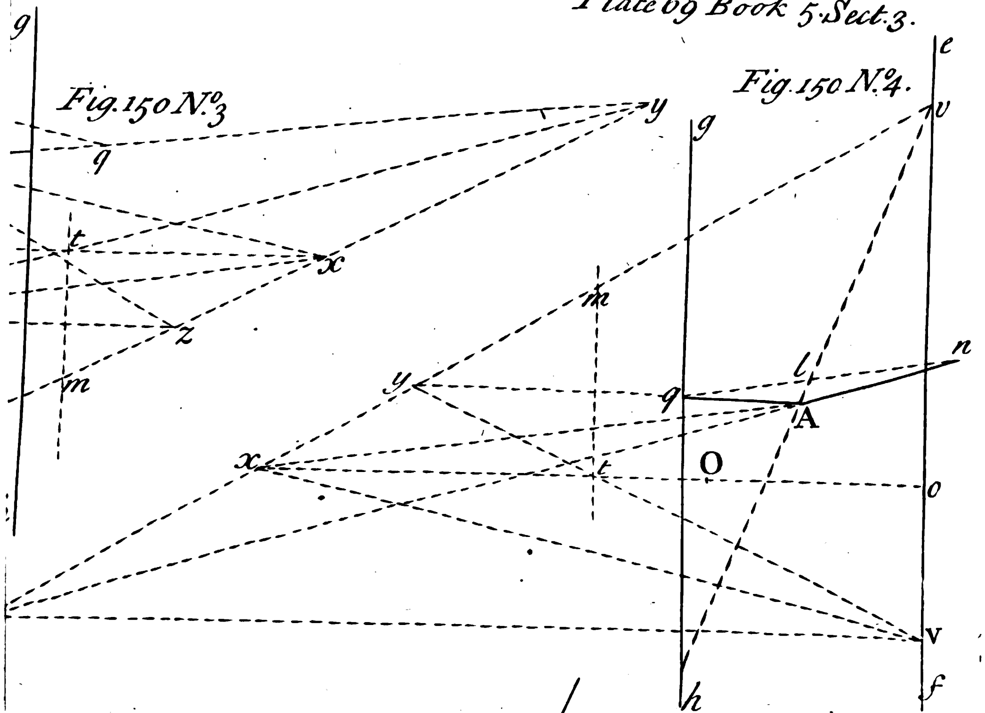
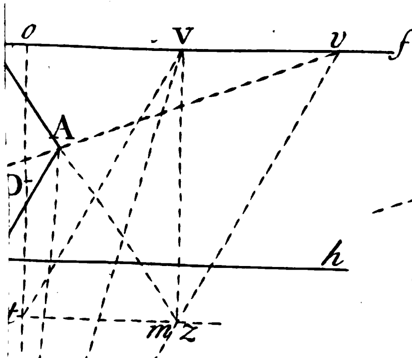


Fig. 150 N°. 3.

Fig. 150 N°. 4.



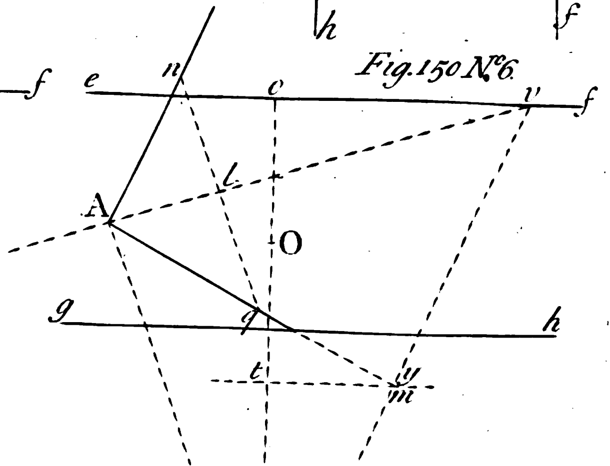
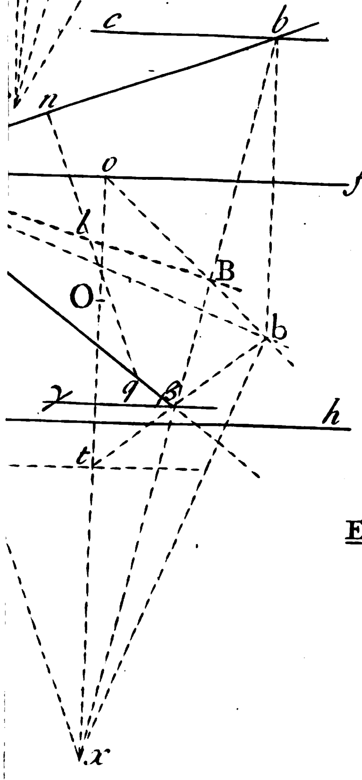


Fig. 150 N°. 6.



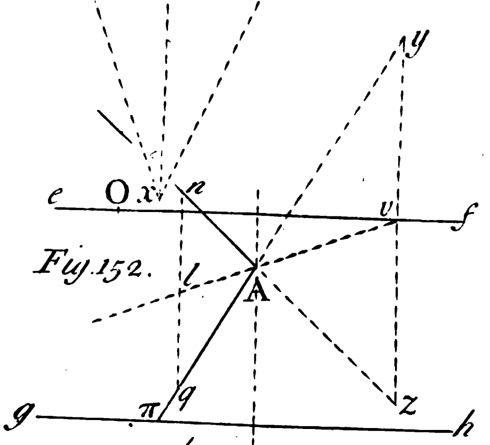


Fig. 152.

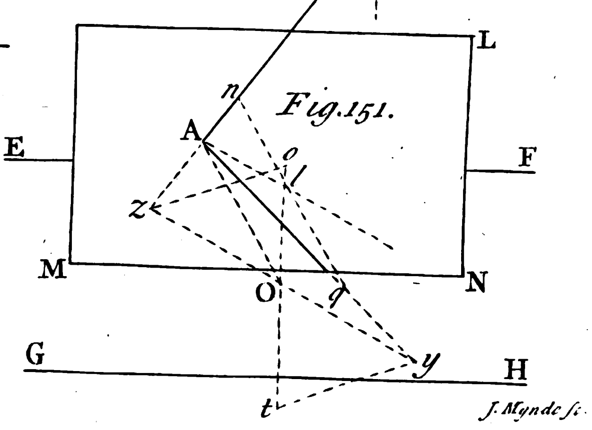


Fig. 151.

J. Mynde sc.

Reflection Aq is parallel to xv the Vaniſhing Line of the Plane in which it lies, and therefore repreſents a Line parallel to the Picture[a]; and here, tv and xz being parallel, the Projection Aq is parallel to them[b].

In Fig. N°. 6. the given Line An is ſuppoſed parallel to the Picture, therefore xv drawn parallel to An is the Vaniſhing Line of the Plane of its Perpendicular Seat on the Reflecting Plane[c], in which Plane its Reflection alſo lies; and the Vaniſhing Point of An being infinitely diſtant, the Vaniſhing Point y of its Reflection therefore biſects xv.

And here, y is the Parallel Seat of x the tranſpoſed Place of the Eye on the Reflecting Plane with reſpect to the given Line An, the ſame with the *Focus* of the Projection Aq[d].

COR. 4.

Fig. 150. N°. 6.

If the propoſed Line An be parallel to the Picture, through t draw tm parallel to ef, and from x draw xy parallel to An, which will cut tm in y the Vaniſhing Point of Aq the Reflection of An.

For xo being biſected in t[e], xv is biſected by tm in y.

COR. 5.

The Line tm is the Place of the Vaniſhing Points of the Reflections of all Original Lines whatſoever which are parallel to the Picture.

For a Line drawn from x parallel to any ſuch Original Line, will cut tm in the Vaniſhing Point of its Reflection[f].

Fig. 150. N°. 5.

The Line tm is alſo the Place of the Vaniſhing Points of all Original Lines whoſe Reflections are parallel to the Picture[g].

COR. 6.

If the propoſed Line be parallel to the Reflecting Plane, the Seat of ſome one Point of that Line on the Reflecting Plane muſt be given; and the Reflection of that Point being thence found[h], a Line drawn through the Reflected Point, ſo as to repreſent a Parallel to the propoſed Line, will be its Reflection[i].

Fig. 150. N°. 7.

Thus if zb be the given Line, and β the Reflection of b, $y\beta$ will be the Reflection of zb.

And here, the Points A, z, and y, all coincide in ef the Vaniſhing Line of the Reflecting Plane; and as bx, bt, bo, and bb are Harmonical Lines, ſo Ab, AB, $A\beta$, and Ax are Harmonical Lines, and conſequently nq parallel to Ax, is biſected in l[k].

Or if cb be the propoſed Line parallel both to the Picture and to the Reflecting Plane, $\beta\gamma$ drawn through β parallel to cb will be its Reflection; and cb, $\gamma\beta$, and two Lines drawn through B and x parallel to them, will be Harmonical Parallels[l].

And here, a Line from x parallel to cb is parallel to tm, ſo that the Vaniſhing Point of the Reflection $\beta\gamma$ is infinitely diſtant[m].

COR. 7.

If the propoſed Line be perpendicular to the Reflecting Plane, its Reflection is the ſame Line continued behind the Reflecting Plane[n]; and the Reflection of any Point of the propoſed Line, is found by the Methods at Prob. XXIII. ſeeing the Interſection of the propoſed Line with the Reflecting Plane, is the Perpendicular Seat of every Point of that Line on that Plane.

Fig. 147. N°. 1, 2.

Thus if aA be the propoſed Line, $A\alpha$ is its Reflection, and α the Reflection of the Point a of that Line.

COR. 8.

Fig. 147. N°. 1, 2.

If xA be biſected in m, then, in Fig. N°. 1. mA will be the Reflection of ſo much of the Original Line as lies between the Original of A and its Directing Point, that is, m will be the Reflection of the Directing Point of Aa; and mx will be the Reflection of the whole Tranſprojective Part of the Original Line; but in Fig. N°. 2. the Original of mA is the whole of that Line which can be reflected, were the Reflected Line $A\alpha$ continued on to its Directing Point, that is, the imaginary Reflection of m, is at the Directing Point of $A\alpha$.

For xa being always Harmonically divided in x, α, A, and a[o]; if the Point a (Fig. N°. 1.) fall beyond A, the Point α will be between m and A; if a fall beyond x, α will fall between m and x; and if a be infinitely diſtant, α will coincide with m which biſects xA[p]: On the contrary, if α (Fig. N°. 2.) fall beyond A, the Point a muſt

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[a] Cor. 2. Theor. 15. B. I.

[b] Cor. 2. Caſe 1. Prob. 3.

[c] Cor. 2. Caſe 2. Prop. 40. B. IV.

[d] Gen. Cor. 2. Prob. 3.

[e] Meth. 2.

[f] Cor. 4.

[g] Cor. 3.

[h] Prob. 23.

[i] Art. 3.

[k] Meth. 3.

[l] Def. 3. B. III.

[m] Cor. 4.

[n] Art. 4.

[o] Cor. 1. Meth. 2. Prob. 23.

[p] Cor. 1, 2, 3.

muſt fall between A and m, and if α be infinitely diſtant, a will coincide with m; but α cannot fall beyond x[a], therefore a can never fall between m and x.

PROB. XXVII.

The Center and Diſtance of the Picture, and a Reflecting Plane parallel to the Picture, being given, together with the Indefinite Image of a Line out of that Plane; thence to find its Reflection.

Fig. 151. Let O be the Center of the Picture, zn the given Line, z its Vaniſhing Point, and A its Interſection with the Reflecting Plane LMN.

METHOD 1.

Draw zO and produce it to y till Oy and Oz be equal, and yA will be the Reflection of zn, and y its Vaniſhing Point.

Dem. For zy being the Vaniſhing Line of a Plane perpendicular to the Reflecting Plane paſſing through zn, in which Plane its Reflection alſo lies[b], and the Vaniſhing Points z and y ſubtending equal Angles with O the Vaniſhing Point of Perpendiculars to the Reflecting Plane, the Lines zn and yA incline to the Reflecting Plane in equal Angles, and yA is therefore the Reflection deſired. Q. *E. I.*

METHOD 2.

If any Original Plane EFGH not perpendicular to the Picture be given, cutting the Reflecting Plane in MN; the Reflection of zn may be alſo found in this manner.

Draw the Vertical Line Oo of the Plane EFGH, and produce it to t till Ot and Oo be equal, and having drawn zo, draw ty parallel to it, and a Line zO will cut ty in the ſame Point y, whence Ay is found as before.

Dem. For t being the Oblique Seat of O the tranſpoſed Place of the Eye on the Reflecting Plane with reſpect to the Plane EFGH[c], and zo being the Vaniſhing Line of the Plane of the Oblique Seat of zn on the Reflecting Plane[d], ty drawn parallel to zo, is the Interſection of the Reflecting Plane with a Plane paſſing through the Oblique Support of the Tranſpoſed Place of the Eye, parallel to ozn the Plane of the Oblique Seat of zn[e], ty is therefore the Line of the *Foci* of the Projections on the Plane LMN of all Lines in the Plane ozn[f]; wherefore zO cuts ty in y, the *Focus* of the Projection of zn on that Plane from the Point O, and y is therefore the Vaniſhing Point of the Reflection of zn[g]. Q. *E. I.*

COR.

The Point y found by both theſe Methods is the ſame.

For the Triangles zOo, tOy being ſimilar, and tO being equal to Oo, zO and Oy are equal.

METHOD 3.

Draw Al parallel to zy; and having drawn OA, from any Point n in the given Line, draw nl parallel to OA, cutting Al in l; produce nl to q till lq and ln be equal, and Aq will be the Reflection deſired.

Dem. For Al parallel to zy is the Perpendicular Seat of zn on the Reflecting Plane, and zy being biſected in O, Az, AO, Ay, and Al, are Harmonical Lines; wherefore nq parallel to AO, one of theſe Harmonicals, is biſected by the other three[h]. Q. *E. I.*

COR. 1.

If the propoſed Line be parallel to the Picture, it will alſo be parallel to the Reflecting Plane, and conſequently to its Reflection; wherefore the Reflection of any Point of the propoſed Line being found[i], the Reflection of the Line itſelf is thence determined[k].

COR. 2.

If the propoſed Line be perpendicular to the Picture, it will alſo be perpendicular to the Reflecting Plane; its Reflection is therefore the ſame Line continued behind that Plane, and the Reflections of any Points of the propoſed Line are found as before[l].

COR. 3.

Here, zy being always biſected by O the tranſpoſed Place of the Eye, the fourth Point which ſhould compleat the Harmonical Diviſion of that Line, namely the Vaniſhing Point of the Perpendicular Seat of the propoſed Line on the Reflecting Plane[m], is in-

finitely

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[a] Art. 18.

[b] Art. 2.

[c] Meth. 2. Prob. 24.

[d] Cor. 1. Prop. 50. B. IV.

[e] Cor. Theor. 3. B. I.

[f] Cor. 1. Meth. 4. Prob. 6.

[g] Cor. Meth. 2. Prob. 26.

[h] Meth. 3. Prob. 26.

[i] Prob. 24.

[k] Cor. 6. Prob. 26.

[l] Cor. 7. Prob. 26. and Prob. 24.

[m] Prob. 26.

finitely diſtant[a], and conſequently that Seat is parallel to the Picture; as it muſt be from another Conſideration, the Reflecting Plane being here parallel to the Picture.

PROB. XXVIII.

A Reflecting Plane perpendicular to the Picture, and the Indefinite Image of a Line out of that Plane, being given; thence to find its Reflection.

Let O be the Center of the Picture, $efgh$ the Reflecting Plane, zn the given Line, z its Vaniſhing Point, and A its Interſection with the Reflecting Plane. Fig. 152.

Draw zv Perpendicular to ef, cutting it in v, and produce zv to y, till yv and vz be equal; then yq drawn through A, will be the Reflection of zn, and y its Vaniſhing Point.

Dem. For zy is the Vaniſhing Line of a Plane perpendicular to the Reflecting Plane, paſſing through the given Line zn, and the Points z and y ſubtend equal Angles with the Vaniſhing Point v[b]; wherefore yq is the Reflection of zn. *Q. E. I.*

COR. 1.

The ſecond Method of the two laſt Problems is not here applicable, for the reaſon already mentioned[c]; and the third Method becomes in Effect the ſame as the firſt.

For zy being biſected in v, Az, Av, Ay, and a Line drawn through A, parallel to zy, are Harmonical Lines; and therefore nq drawn parallel to this laſt, is biſected by the other three. And here q is alſo the Reflection of n[d].

COR. 2.

If the propoſed Line be parallel or perpendicular to the Reflecting Plane, its Reflection is found as before[e].

COR. 3.

Here, zy being always biſected by v, the Point x is infinitely diſtant; all perpendiculars to the Reflecting Plane are therefore parallel to the Picture[f], and to zy.

GENERAL COROLLARY 1.

In all Situations of the propoſed Line and its Reflection, except when they are in the ſame ſtraight Line[g], a Line drawn from the tranſpoſed Place of the Eye to any Point of the propoſed Line, will cut its Reflection in the Reflection of that Point; and, *vice verſa*, a Line drawn from the tranſpoſed Place of the Eye to any Point of the indefinite Reflection, will cut the propoſed Line in the Point whoſe Reflection it is[h].

Thus, xb and xc drawn from x to any Points b and c of the propoſed Line zc, cut its Reflection $y\gamma$ in β and γ, the Reflections of b and c. Fig. 153. N°. 1, 2.

GENERAL COROLLARY 2.

Hence, a Line xv, drawn from x through z the Vaniſhing Point of the propoſed Line zc, cuts its Reflection $y\gamma$ in y its Vaniſhing Point; a Line $x\delta$, drawn from x parallel to zc, cuts $y\gamma$ in δ the Reflection of the Directing Point of the propoſed Line; a Line from x, through π, where $y\gamma$ croſſes gh, will cut zc in p, the Image of the fartheſt Point of the Original Line which can be Reflected within the Bounds of the Reflecting Plane; and a Line xd, drawn from x parallel to $y\gamma$, cuts zc in a Point d, the Reflection of which is infinitely diſtant, or the directing Point of the Reflected Line $y\gamma$.

For zc and $x\delta$ being parallel, and in the ſame Plane, they have the ſame Directing Point, and $y\gamma$ and xd being parallel and in the ſame Plane, they have the ſame Directing Point[i]; wherefore δ is the Reflection of the Directing Point of zc, and the Reflection of d is at the Directing Point of $y\gamma$; the reſt is evident.

SCHOL.

In Fig. N°. 1. y the Vaniſhing Point of the Reflected Line $y\gamma$, falling beyond ef, the part yA is above or before the Reflecting Plane, and the part Aγ Indefinitely produced beyond γ, is under or behind that Plane; yA is therefore the Imaginary Part, and Aγ indefinitely produced beyond γ, is the Real Part of the Reflected Line, or that part of it, in which all poſſible real Reflections muſt lye, were the Reflecting Plane continued on to its Directing Line[k]. Fig. 153. N°. 1.

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[a] Cor. 1. Lem. 1. B. III.

[b] Caſe 1. Prop. 24. B. IV.

[c] Cor. 1. and 2. Prob. 25.

[d] Prob. 25.

[e] Cor. 6. and 7. Prob. 26. and Prob. 25.

[f] Cor. 3. Prob. 27.

[g] Cor. 7. Prob. 26.

[h] Gen. Cor. 2. Prob. 4.

[i] Cor. 5. Theor. 12. B. I.

[k] Art. 18.

Fig. 153. N°. 2. In Fig. N°. 2. the Vaniſhing Point y falling below ef, the Part Ay of the Reflected Line is under the Reflecting Plane, and is therefore the whole poſſible Real part of the Reflection; the Remainder of that Line indefinitely produced beyond A, being above the Reflecting Plane, and therefore only Imaginary.

In either Caſe, all ſuch part of the propoſed Line zc is Reflexible, from whence Lines may be drawn through x, to any Point in the Real part of its Reflection; whether that part of the propoſed Line lye in the Perſpective, Projective, or Tranſprojective Parts of its indefinite Image.

By theſe Rules, it will be eaſy to diſtinguiſh what part of a Reflected Line is Real, and what of it is Imaginary, and what part of an Original Line is Reflexible, and what not, in all poſſible Situations of the propoſed Line and its Reflection, with reſpect to the Reflecting Plane.

GENERAL COROLLARY 3.

If an Original Line be divided into any Number of Parts in a known Proportion, and the indefinite Reflection of that Line, with the determinate Reflection of any one of thoſe Parts be given; the Reflections of all the other Parts of the propoſed Line may be thence found, by uſing the Reflected Line, as if it were the indefinite Image of the propoſed Line; and finding therein the Images of the Parts required by the common Rules already taught[a].

For the Reflection of any determinate Part of an Original Line, being equal to that Part[b], their Images repreſent equal Lines; and therefore, if in the Reflected Line, any Parts be found repreſenting Diviſions, reſpectively equal to the Diviſions of the Original Line, the Parts thus found will be the Reflections of the correſponding Parts of the Original Line.

SCHOL.

After this manner, by the help of the indefinite Reflection alone, the Reflections of ſuch Parts of the Original Line as lye near its Directing Point, before or behind the Directing Plane, may be found without their Images (which may be out of reach, or impoſſible to be had) the true Meaſures of the propoſed Parts of the Original Line being known, and regard being had to the contrary Order in which they lye, with reſpect to A the common Interſection of the Reflecting Plane with the Original and Reflected Lines.

Fig. 153. N°. 1. Thus, if bc and its Reflection $\beta\gamma$ be given, and the Reflection of another Part of zc on the oppoſite Side of c from A were required, of double the Length of the Original of bc; find a Point α in the indefinite Reflection $y\gamma$, likewiſe on the oppoſite Side of γ from A, ſo that $\gamma\alpha$ may repreſent a Line double the Length of the Original of $\gamma\beta$, and $\gamma\alpha$ will be the Reflection of the required Part of the Original Line; and which is thus found without its Image, which here cannot be had; in regard that the Original of the propoſed Part lies partly before and partly behind the Directing Plane, and that the Image of one of its Extremities falls at an inacceſſible Diſtance beyond p in the Tranſprojective Part of zc.

PROB. XXIX.

The Center and Diſtance of the Picture, and a Reflecting Plane inclining to the Picture, together with an Original Plane, being given; thence to find the Reflected Plane, and the Reflections of any propoſed Lines in the Original Plane, when the Original and Reflecting Planes interſect in a Line not Parallel to the Picture.

CASE 1.

When the Original Plane inclines to the Reflecting Plane.

Fig. 154. N°. 1, 2. Let O be the Center of the Picture, $efgh$ and EFGH the Reflecting and Original Planes, Dy their Interſection, and x the Vaniſhing Point of Perpendiculars to the Reflecting Plane.

METHOD 1.

From x, to any Vaniſhing Point z in EF, draw xz cutting ef in v, and find the Reflection y of the Vaniſhing Point z[c]; then $\varepsilon\phi$ drawn through y and y, will be the Reflection of the Vaniſhing Line EF, or the Vaniſhing Line of the Reflected Plane; and

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[a] Sect. 2. B. II.

[b] Art. 5.

[c] Prob. 26.

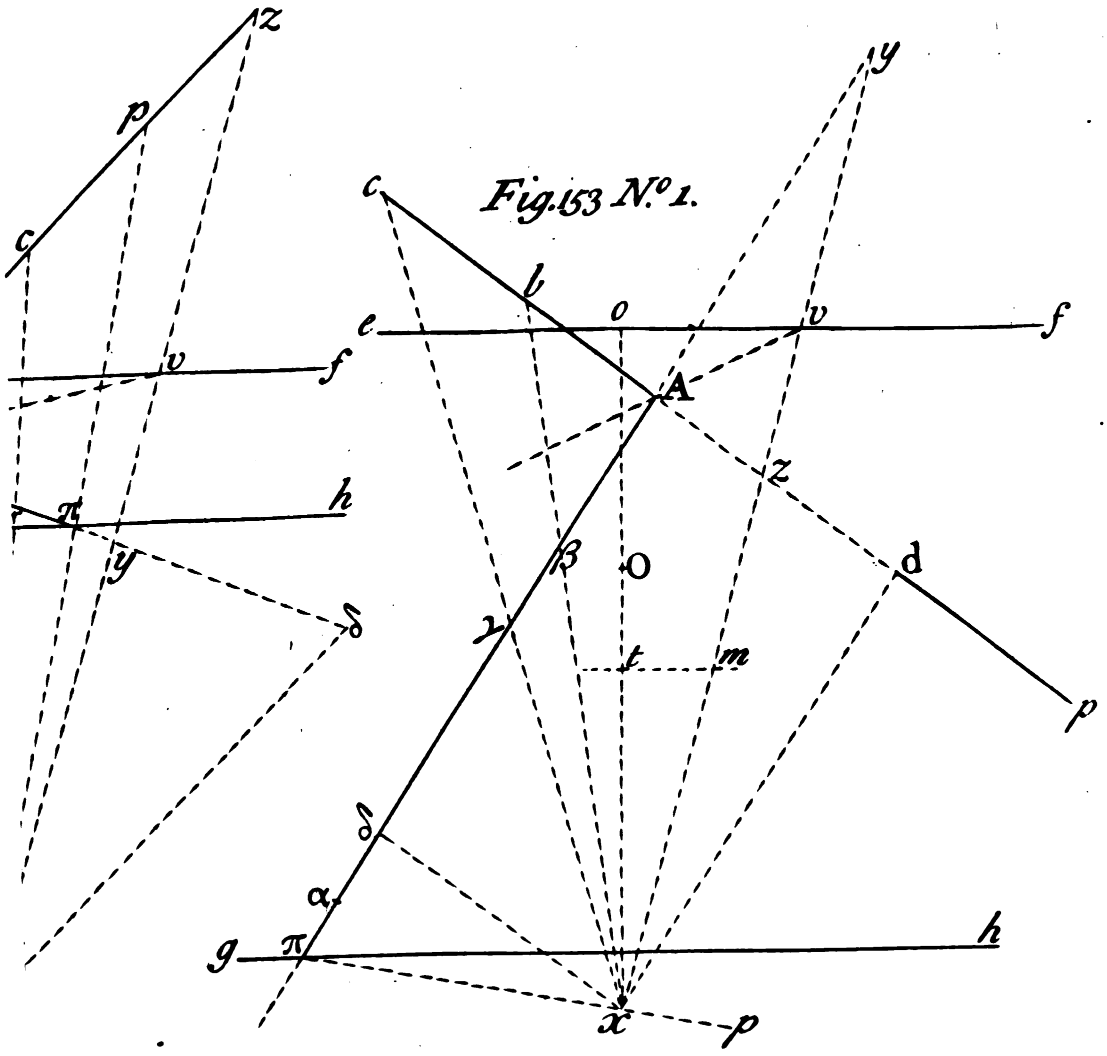

Fig. 153 N.° 1.

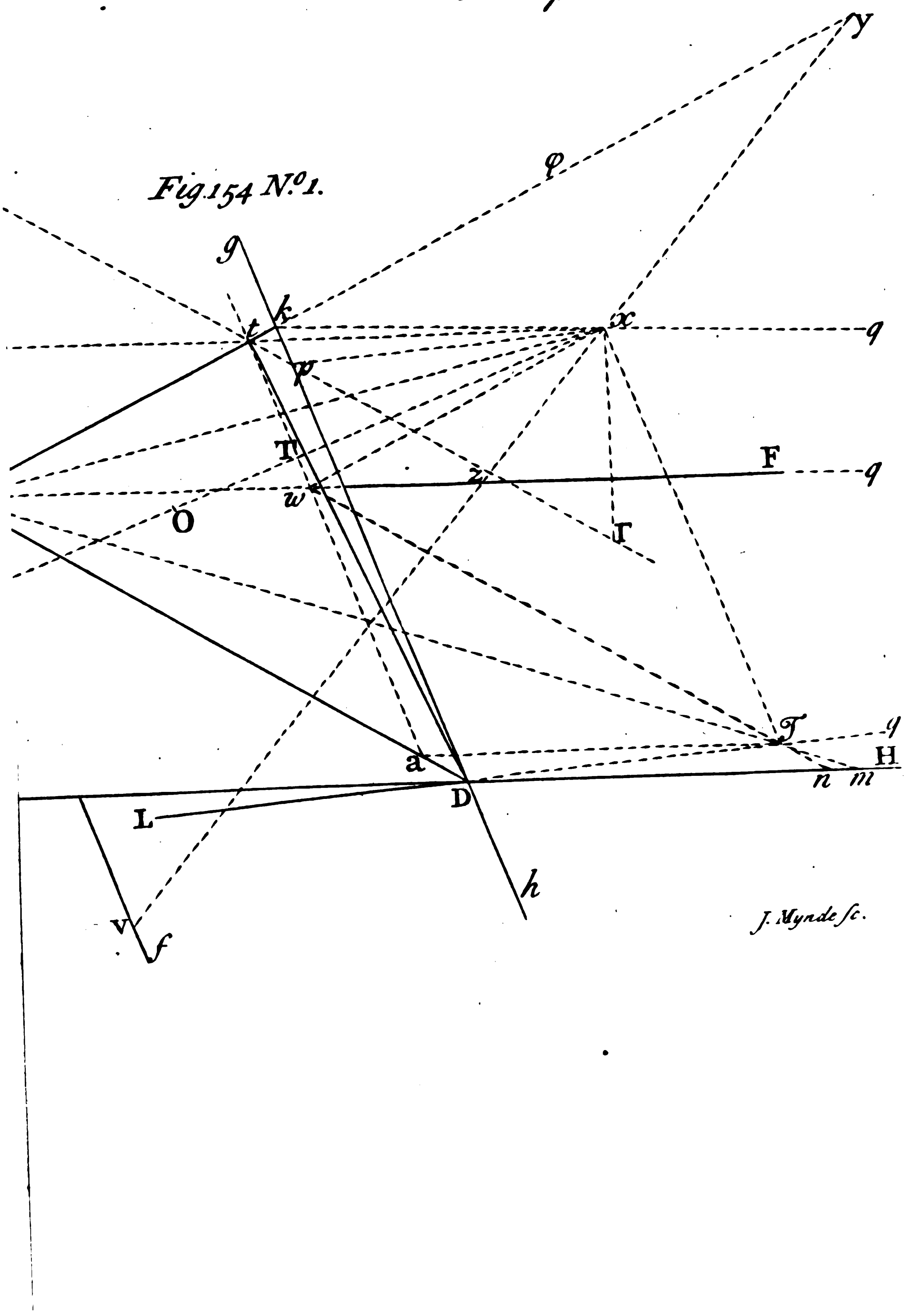

Fig. 154 N.° 1.

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and if from x, through any Vaniſhing Point in EF, a Line be drawn, it will cut $\varepsilon\phi$ in the Reflection of that Vaniſhing Point.

Dem. For y being the Vaniſhing Point of the Reflections of all Lines in the Plane EFGH, whoſe Vaniſhing Point is z, y is therefore a Point in the Vaniſhing Line of the Reflected Plane; and y being the Vaniſhing Point of the Interſection of that Plane with the Original and Reflecting Planes, $\varepsilon\phi$ drawn through y and y is therefore the Vaniſhing Line of the Reflected Plane; the reſt is evident. *Q. E. I.*

COR.

Here zx being Harmonically divided in v, z, x, and y[a], the Place of the Point y, and conſequently the Poſition of $\varepsilon\phi$ depends on the Situation of z with reſpect to x and v, according as it falls nearer to the one or the other of them, or in the middle between both[b]; it is evident alſo, that yy, yx, yv, and yz, being Harmonical Lines, the Plane $\varepsilon\phi$D inclines the contrary way to the Reflecting Plane $efgh$, in the ſame Angle as the Original Plane EFGH doth[c], and is therefore the Reflected Plane[d]; and a Line through D parallel to $\varepsilon\phi$, is its Interſecting Line.

METHOD 2.

Biſect the Vertical Line xo of the Reflecting Plane in T the Oblique Seat of the tranſpoſed Place of the Eye on that Plane[e], and draw Tt parallel to ef; then draw xl parallel to EF, cutting Tt in t; and yt drawn through y and t, will be the reflected Vaniſhing Line, the ſame with $\varepsilon\phi$ before found.

Dem. For if x be conſidered as the tranſpoſed Place of the Eye, and $efgh$ be taken as the Plane of the Projection, then t is the Parallel Seat of x on the Plane $efgh$ with reſpect to the Plane EFGH, and conſequently ty is the Line of the *Foci* of the Projections on the Plane $efgh$ of all Lines in the Plane EFGH from the Point x[f]; and as the *Focus* of the Projection of any Line from the Point x is the ſame with the Vaniſhing Point of its Reflection[g], the Line ty is therefore the Place of the Vaniſhing Points of the Reflections of all Lines in the Plane EFGH, and conſequently it is the Vaniſhing Line of the Reflected Plane. *Q. E. I.*

METHOD 3.

Through x draw xl parallel to EF, cutting ef in l, and biſect xl, which will give the ſame Point t, by which yt or $\varepsilon\phi$ is found as before.

Dem. For ef, EF, $\varepsilon\phi$, and xy which meet in y, being Harmonical Lines[h], the Line xl drawn parallel to EF, one of theſe Harmonicals, is biſected by the other three[i]; and in the ſimilar Triangles xol, xTt, xo being biſected in T[k], xl is alſo biſected in t, wherefore t found by either of theſe two laſt Methods is the ſame. *Q. E. I.*

SCHOL.

When x is conſidered as the tranſpoſed Place of the Eye[l], it is then taken as a Projecting Point at a moderate Diſtance before or behind the Directing Plane[m], and the Reflection of any Line in the Plane EFGH thereby found, repreſents the Projection of that Line on the Plane $efgh$ from the Point x, and conſequently may be taken as a Line lying in that Plane; but when x is conſidered as the Vaniſhing Point of Perpendiculars to the Reflecting Plane[n], then the Reflection thereby found, is the Image of the Reflected Line, and conſequently repreſents a Line in the Reflected Plane whoſe Vaniſhing Line is $\varepsilon\phi$, on which laſt Conſideration alone it becomes truly manageable[o], although the Indefinite Reflection found either way be in the ſame ſtraight Line[p]; and that Point, which on the firſt Conſideration is the *Focus* of the Projection, becomes the real Vaniſhing Point of that Line conſidered as the Reflected Line[q].

COR. 1.

Through t draw pv parallel to Dy, and pv will be the Indefinite Reflection of the Directing Line of the Plane EFGH; and conſequently if from x, a Line be drawn parallel to any propoſed Line in the Plane EFGH, it will cut pv in the Reflection of the Directing Point of that Line[r], which is therefore a Point in its Indefinite Reflection.

SCHOL.

If x be conſidered as the tranſpoſed Place of the Eye, then xt repreſents a Line parallel to the Picture, in a Plane whoſe Vaniſhing Line is vw parallel to EF, and which

[a] Meth. 2. Prob. 26.

[b] Cor. 1, 2, 3. Prob. 26.

[c] Cor. 3. Caſe 2. Prop. 25. B. IV.

[d] Art. 9.

[e] Meth. 2. Prob. 23.

[f] Meth. 4. Prob. 6. and Schol. 1. after that Prob.

[g] Cor. Meth. 2. Prob. 26.

[h] Cor. 3. Caſe 2. Prop. 25. B. IV.

[i] Lem. 7. B. III.

[k] Meth. 2.

[l] Meth. 2.

[m] Cor. 2. Meth. 2. Prob. 23.

[n] Meth. 1.

[o] Gen. Cor. 3. Prob. 28.

[p] Art. 14, 15.

[q] Cor. 3. Meth. 4. Prob. 6.

[r] Meth. 6. and Cor. 1. Prob. 6.

which cuts the Plane $efgh$ in vp parallel to yD; but if x be taken as the Vaniſhing Point of Perpendiculars to the Reflecting Plane, then xt parallel to EF is the Vaniſhing Line of a Plane perpendicular to the Reflecting Plane, and cutting the Reflected Plane in tp parallel to Dy; ſo that whether vw or xt be the Vaniſhing Line of the ſuppoſed Plane which paſſes through x, ſtill that Plane will have the ſame Directing Line with the Plane EFGH[a], on which laſt Circumſtance the Demonſtration of the preceeding Corollary is founded.

COR. 2.

By the help of t, find T the Parallel Seat of the tranſpoſed Place of the Eye on the Plane EFGH with reſpect to the Reflecting Plane[b]; then if through T and y a Line my be drawn, the imaginary Reflection of that Line will coincide with ef the Vaniſhing Line of the Reflecting Plane; and conſequently, if from the Interſection of my with any propoſed Line in the Plane EFGH, a Line be drawn through x, it will cut ef in a Point of the Indefinite Reflection of that Line[c].

SCHOL.

Here, if the Reflection were conſidered as the Projection of the propoſed Line on the Plane $efgh$ from x, the Point found by this Corollary would be the Vaniſhing Point of that Projection; but as the real Reflection doth not lie in the Reflecting, but in the Reflected Plane, the Point thus found is not the Vaniſhing Point of the Reflected Line, but only a Point through which it paſſes; the true Vaniſhing Point of the Reflection being its Interſection with $\varepsilon\varphi$ the Vaniſhing Line of the Reflected Plane.

COR. 3.

If through T a Line nw be drawn parallel to Dy, the imaginary Reflection of nw will coincide with the Directing Line of the Reflected Plane; and if from the Interſection of nw with any propoſed Line in the Plane EFGH, a Line be drawn to x, it will be parallel to the Reflection of the propoſed Line.

And here Tt and nw cut EF in the ſame Point w, and xw is parallel to $\varepsilon\varphi$.

For Dy and nw being parallel, yw and aT are equal; but $aT = tx = tl$, wherefore xw is parallel to $\varepsilon\varphi$, and tw is parallel to ef, and therefore coincides with Tt.

SCHOL.

If x be taken as the tranſpoſed Place of the Eye, and tw be taken as the Vaniſhing Line of the Plane which paſſes through xT and nw, then the Planes $efgh$ and $twxT$ having the ſame Directing Line, the imaginary Projection of nw would coincide with the Directing Line of the Reflecting Plane[d]; but if x be conſidered as the Vaniſhing Point of Perpendiculars to the Reflecting Plane, and xw be taken as the Vaniſhing Line of a Plane perpendicular to the Reflecting Plane, paſſing through nw, and which is the real Plane in which the Reflection of nw lies, that Reflection will then coincide with the Directing Line of the Reflected Plane, ſeeing thoſe two Planes have alſo the ſame Directing Line[e]; ſo that whether the Reflection of the propoſed Line be conſidered as lying in the Reflecting or in the Reflected Plane, ſtill its Image will be parallel to a Line drawn from x to the Interſection of nw with the Line propoſed.

COR. 4.

The Point t is the Vaniſhing Point of the Reflections of all Lines in the Plane EFGH, which are parallel to the Picture[f]; all ſuch Lines being parallel to EF, and conſequently to xt[g].

And hence Dt (Fig. N°. 1.) is the Reflection of GH the Interſecting Line of the Original Plane.

COR. 5.

The Reflections of all Lines in the Plane EFGH which have w for their Vaniſhing Point, are parallel to the Picture, and conſequently to $\varepsilon\varphi$[h]; xw being parallel to $\varepsilon\varphi$[i].

COR. 6.

The Reflections of all Lines in the Plane EFGH which paſs through T, are parallel to ef, wherefore the Reflection of TD indefinitely produced beyond L, coincides with gh the Interſecting Line of the Reflecting Plane, and DL is the Boundary of the Reflexible Part of the Plane EFGH within the Compaſs of the Interſecting Line of the Reflecting Plane[k].

For

[a] Cor. Caſe 2. Prop. 46. B. IV.

[b] Prop. 52. B. IV.

[c] Meth. 5. and Cor. 1. Prob. 6.

[d] Meth. 7. and Cor. 1. Prob. 6.

[e] Cor. Caſe 2. Prop. 46. B. IV.

[f] Cor. 5. Prob. 26

[g] Cor. 1.

[h] Cor. 5. Prob. 26.

[i] Cor. 3.

[k] Cor. 3. Meth. 5. Prob. 6.

For DL cutting nw in T, xT is parallel to the Reflection of DL[a], and xT being parallel to ef, and conſequently to gb, Dg is therefore the Reflection of DL.

SCHOL.

The Part EyDL is the whole Reflexible Part of the Plane EFGH, and the Part kyD is the whole of the Reflecting Plane within the Compaſs of its Interſecting Line, wherein any Part of the Reflection can appear.

If the tranſpoſed Place of the Eye repreſent a Point at a moderate Diſtance before the Directing Plane, and its Parallel Seat T fall between the Vaniſhing and Interſecting Lines of the Plane EFGH; then the Space EyDL will be Indefinite, and Part of it will extend behind the Directing Plane, and the Reflection of that Tranſprojective Part will lie in the Part tpk of the Reflecting Plane. Fig. 154. N°. 1.

But if the Parallel Seat T of the Point x on the Plane EFGH, fall below its Interſecting Line GH[1], or if the tranſpoſed Place of the Eye repreſent a Point at a moderate Diſtance behind the Directing Plane[2]; in either Caſe, the Space EyDL which is the Reflexible Part of the Plane EFGH will be confined to the Triangle yDq.

Laſtly, if the tranſpoſed Place of the Eye repreſent a Point either in the Plane EFGH or behind it, no part of that Plane can be reflected.

For if the tranſpoſed Place of the Eye be in the Plane EFGH, x then coinciding with T, the whole Reflection of that Plane muſt lie in the Line Dy; and if it be under the Plane EFGH, any poſſible Reflection of that Plane muſt alſo fall under Dy and cannot therefore be ſeen.

COR. 7.

If any Line in the Plane EFGH be parallel to Dy, its Reflection will alſo be parallel to Dy.

For all ſuch Lines having the ſame Directing Point with Dy, and that Directing Point being common to them and their Reflections, thoſe Reflections are therefore parallel to Dy[b].

COR. 8.

If from x a Perpendicular to EF be drawn till it cut vp in Γ, the Point Γ will be the imaginary Reflection of the Foot of the Eye's Director with reſpect to the Plane EFGH. Fig. 154. N°. 1, 2, 3,

For the Reflection of the Eye's Director is in a Plane paſſing through that Line perpendicular to the Reflecting Plane, the intire Image of which Plane is only a ſtraight Line parallel to the Eye's Director[c], and conſequently perpendicular to EF; and x being the Reflection of the Eye, $x\Gamma$ perpendicular to EF is the Indefinite Reflection of the Eye's Director, and conſequently Γ, where it cuts vp the Reflection of the Directing Line of the Plane EFGH[d], is the Reflection of the Foot of the Eye's Director or Point of Station.

COR. 9.

If from x through k, where $\epsilon\phi$ croſſes gb, a Line xk be drawn, it will cut EF in q the Vaniſhing Point of DL; and if from x through p, where vp croſſes gb, a Line xp be drawn, it will be parallel to DL. Fig. 154. N°. 1, 2.

For Dk being the Reflection of DL[e], k is the Vaniſhing Point of that Reflection[f], and p is the Reflection of the Directing Point of DL[g].

CASE 2.

When the Original Plane is perpendicular to the Reflecting Plane.

In this Caſe, the Original Plane and its Reflection make one continued Plane[h], and conſequently EF and $\epsilon\phi$ coincide, which all the Methods of this Problem alſo prove. Fig. 154. N°. 3.

For x being here a Point in EF[i], a Line from x to any Vaniſhing Point in EF neceſſarily coincides with it, and the Point v (Fig. N°. 1, 2.) coincides with y. Now the Reflection of any Vaniſhing Point z in EF being had, by finding a Point y in that Line, ſo that xy may be Harmonically divided in x, z, y, and y[k], the Line yy which coincides with EF is therefore the Reflected Vaniſhing Line[l]; again xo being biſected in T, Tt drawn parallel to ef cuts xt (which here coincides with EF) in t, through which and y the Reflected Vaniſhing Line paſſes[m]; and laſtly, as xo is biſected in T, ſo xl or xy is biſected in t[n]. *Q. E. I.*

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[a] Cor. 3.

[1] Fig. 154. N°. 3.

[2] Fig. 154. N°. 2.

[b] Cor. 4, 5. Theor. 12. B. I.

[c] Cor. 1. Theor. 17. B. I.

[d] Cor. 3.

[e] Cor. 6.

[f] Meth. 1.

[g] Cor. 1.

[h] Art. 11.

[i] Cor. 3. Prop. 20. B. IV.

[k] Prob. 26.

[l] Meth. 1.

[m] Meth. 2.

[n] Meth. 3.

COR.

The Points w and t coincide, and pv and nw are in the ſame ſtraight Line, and the Lines my and DL, and alſo the Point Γ, are found as in the firſt Caſe of this Problem; and with theſe Obſervations all the Corollaries of that Caſe are applicable to this.

For w is the Interſection of Tt with EF[a], the ſame with the Point t; wherefore pv and wn, which are both parallel to Dy[b], and paſs through the ſame Point t or w, make one continued ſtraight Line. The reſt of this Corollary is ſufficiently evident.

SCHOL.

If the Original Plane were perpendicular to the Picture, as well as to the Reflecting Plane, EF and ef would be perpendicular[c], and the Points T and t would coincide; and in regard that $x\mathcal{T}$ would then be perpendicular to EF, the Points $\mathcal{T}$ and Γ would be the ſame[d]; but thiswould make no Difference, either in the Practice or Demonſtration.

PROB. XXX.

The Center and Diſtance of the Picture, and a Reflecting Plane inclining to the Picture, together with an Original Plane, being given; thence to find the Reflected Plane, and the Reflections of any propoſed Lines in the Original Plane, when the Original and Reflecting Planes interſect in a Line parallel to the Picture.

CASE I.

When the Original Plane inclines to the Reflecting Plane.

Fig. 155. N°. 1. Let O be the Center of the Picture, EFGH and $efgh$ the Original and Reflecting Planes, MN their common Interſection, and x the Vaniſhing Point of Perpendiculars to the Reflecting Plane.

METHOD I.

Draw xo the Common Vertical Line of the given Planes, cutting ef and EF in o and z, and find v the Reflection of the Vaniſhing Point z[e]; then $\varepsilon\phi$ drawn through v parallel to ef, will be the Vaniſhing Line of the Reflected Plane; and if from x through any Vaniſhing Point in EF, a Line be drawn, it will cut $\varepsilon\phi$ in the Reflection of that Vaniſhing Point.

Dem. For the Reflected Plane paſſing through MN a Line parallel to the Picture, its Vaniſhing Line muſt be parallel to MN[f], and v being a Point in that Vaniſhing Line, $\varepsilon\phi$ drawn through v, parallel to ef or MN, is therefore the Reflected Vaniſhing Line. Q. *E. I.*

COR.

Here, xv being Harmonically divided in x, z, o, and v, the Vaniſhing Lines EF, ef, $\varepsilon\phi$, and a Line through x parallel to them, are Harmonical Parallels; wherefore the Place of v, and conſequently the Poſition of $\varepsilon\phi$, depends on the Situation of z with reſpect to x and o[g]; and hence if the Point z biſect xo, the Point v will be infinitely diſtant, and conſequently the Reflected Plane will be parallel to the Picture[h]; and if z be infinitely diſtant, that is, if the Original Plane be parallel to the Picture, the Point v, and conſequently $\varepsilon\phi$ the Vaniſhing Line of the Reflected Plane will biſect xo.

METHOD 2.

Biſect xo in T the Oblique Seat of the tranſpoſed Place of the Eye on the Reflecting Plane, and through T draw any Line Ty, cutting ef in y, and compleat the ſubſtituted Plane $y\mathrm{y}\Delta D$; then draw $x\mathrm{y}$ cutting Dy in r, and $\varepsilon\phi$ drawn through r parallel to ef, will be the Reflected Vaniſhing Line.

Dem. For r is the *Focus* of the Projection of $\Delta\mathrm{y}$, a Line in the Plane EFGH, on the Plane $efgh$ from the Point x[i], and is therefore the Vaniſhing Point of the Reflection of $\Delta\mathrm{y}$[k], conſequently $\varepsilon\phi$ drawn through r parallel to ef is the Reflected Vaniſhing Line. Q. *E. I.*

SCHOL.

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[a] Cor. 3.

[b] Cor. 1. 3.

[c] Cor. 3. Theor. 16. B. I.

[d] Cor. 8. Caſe 1. and Cor. of this.

[e] Prob. 26.

[f] Cor. 1. Theor. 15. B. I.

[g] Cor. Meth. 1. Prob. 29.

[h] Cor. 2. Caſe 2. Prop. 25. B. IV.

[i] Meth. 4. Prob. 7.

[k] Cor. Meth. 2. Prob. 26.

Fig 154 No. 2.

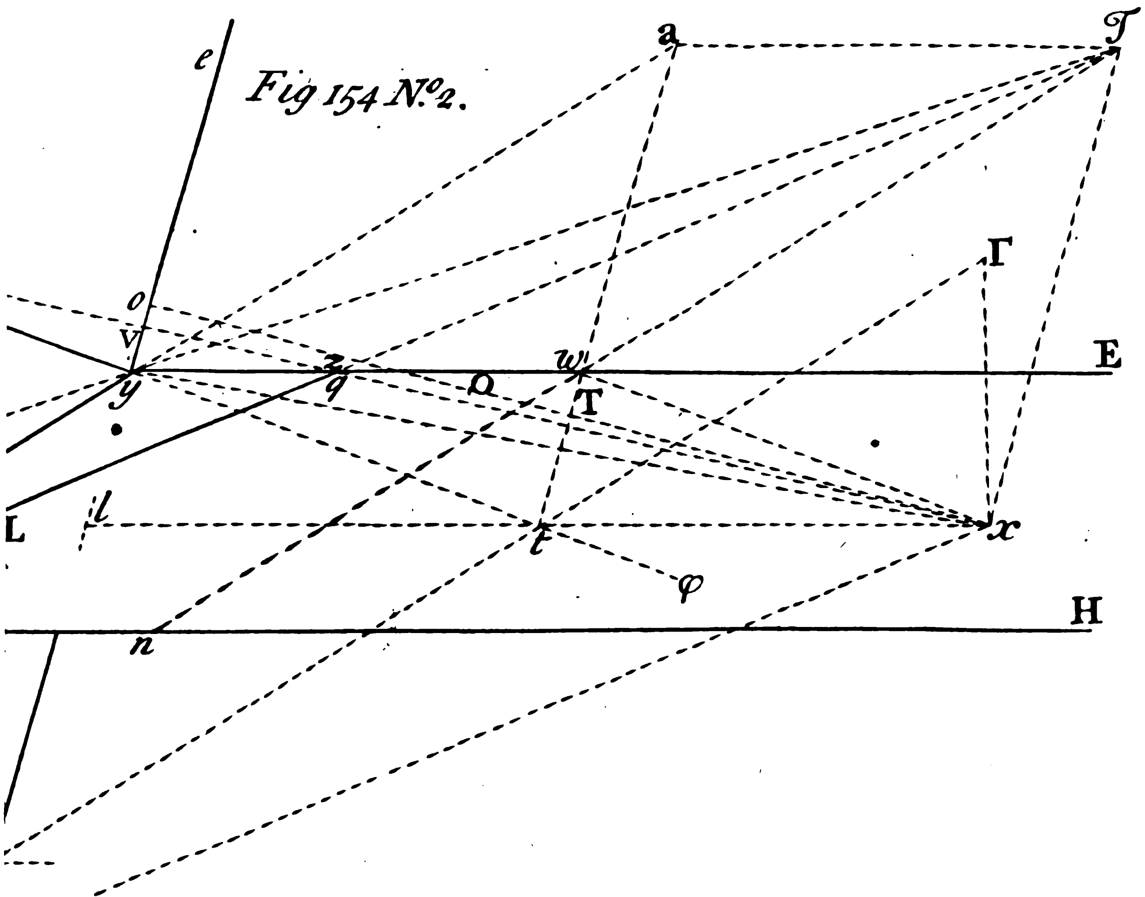
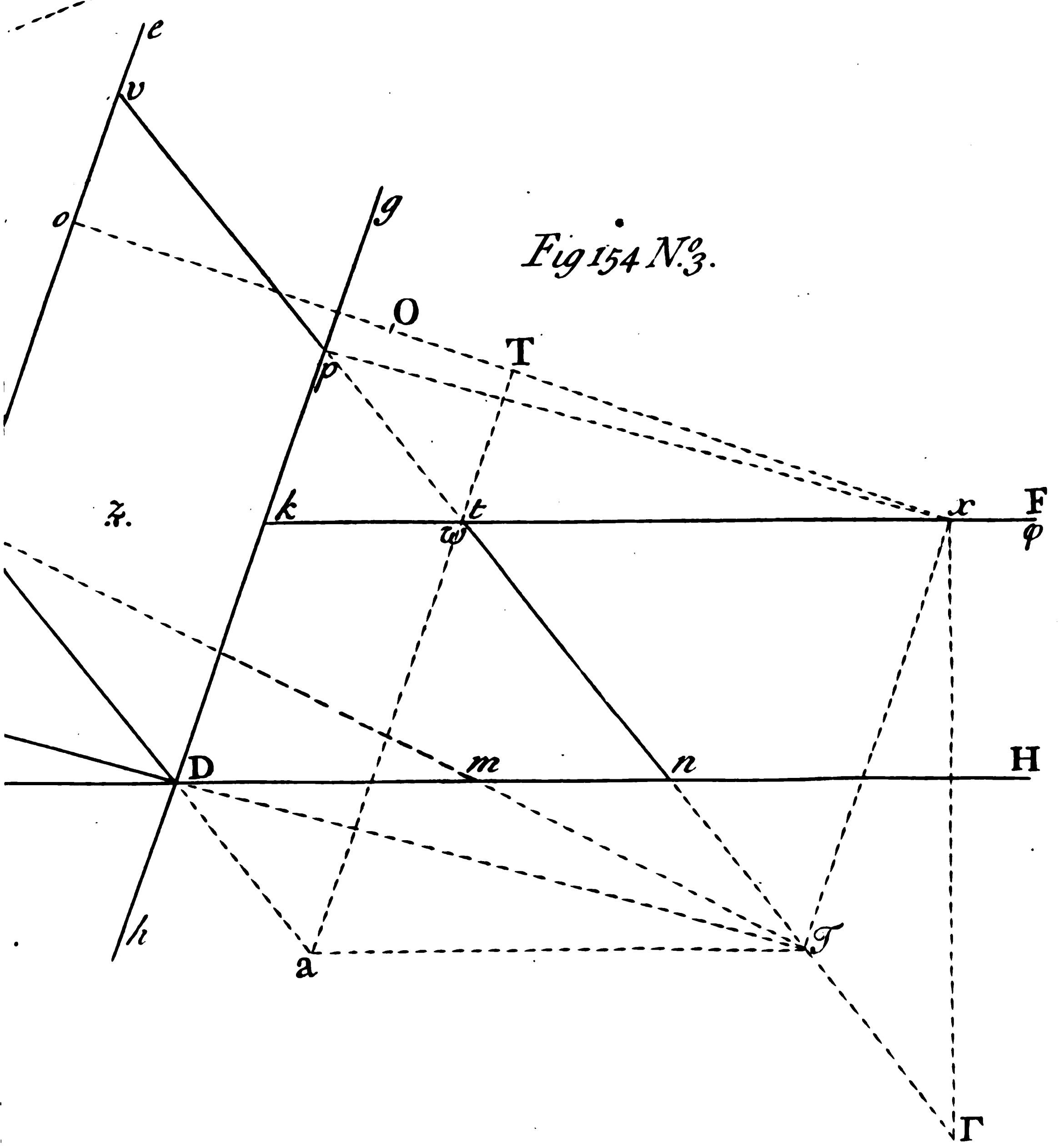


Fig 154 No. 3.



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SCHOL.

The Vaniſhing Line $\epsilon\varphi$ found by either of theſe Methods, is the ſame.

For xo being biſected in T, yo, yT, yx, and yy are Harmonical Lines; wherefore xr, which cuts them all four, is Harmonically divided by them in x, y, q and r; and conſequently $\epsilon\varphi$, ef, EF, and a Line through x, parallel to them, are Harmonical Parallels, as they were ſhewn to be in the firſt Method[a].

COR. 1.

From x, draw xp parallel to yΔ, cutting yD in p; and $p\pi$ drawn parallel to MN, will be the Reflection of the Directing Line of the Plane EFGH[b]; and conſequently, if from x, a Line be drawn parallel to any propoſed Line in the Plane EFGH, it will cut $p\pi$ in the Reflection of the Directing Point of that Line[c].

And here, Γ, the Interſection of $p\pi$ with xo, is the Reflection of the foot of the Eye's Director with reſpect to the Plane EFGH[d].

COR. 2.

From x, draw xy cutting yΔ in m, and draw $m\mu$ parallel to EF; then the Reflection of $m\mu$ will coincide with ef the Vaniſhing Line of the Reflecting Plane[e]; and conſequently, if from the Interſection of $m\mu$ with any propoſed Line in the Plane EFGH, a Line be drawn through x, it will cut ef in a Point of the indefinite Reflection of that Line, which Point however is not the Vaniſhing Point of the Reflection, but only a Point through which the Reflection paſſes[f].

COR. 3.

From x, draw xn parallel to yD, cutting yΔ in n, and draw $n\nu$ parallel to MN; then the imaginary Reflection of $n\nu$ will coincide with the Directing Line of the Reflected Plane[g]; and conſequently, if from the Interſection of $n\nu$ with any propoſed Line in the Plane EFGH, a Line be drawn through x, it will be parallel to the Reflection of the propoſed Line[h].

COR. 4.

From x, through D, draw xD cutting yΔ in L, and draw Ll parallel to EF; then the Reflection of Ll will coincide with gh the Interſecting Line of the Reflecting Plane, and Ll will be the Boundary of the Reflexible Part of the Plane EFGH within the Compaſs of the Interſecting Line of the Reflecting Plane[i]; ſo that the whole Reflexible Part of the Plane EFGH lies between Ll and MN.

COR. 5.

From x, through Δ, draw $x\Delta$ cutting yD in d, and draw dδ parallel to ef; then dδ will be the Reflection of GH the Interſecting Line of the Original Plane; and conſequently the Part MNdδ is the whole of the Reflecting Plane, within which the Reflection of any Point in the Perſpective Part of the Original Plane can appear[k].

COR. 6.

The Reflection of any Line in the Plane EFGH, parallel to the Picture, is had by the ſame Method as the Reflections of GH, and Ll are found, in the two laſt Corollaries[l].

SCHOL.

The ſeveral Methods in this Problem, and its Corollaries, are alſo applicable, when the Vaniſhing Lines of the given Planes are not parallel, but incline ſo Obliquely, that the Methods in Prob. XXIX. become inconvenient.

Fig. 155. N°. 2.

Let the ſame Letters mark the ſame things as in the laſt Figure; here, the Points r, p, m, n, L and d, are found by the help of the ſubſtituted Plane yyΔD, paſſing through xT, as in this Problem; but $\epsilon\varphi$ and $m\mu$ muſt be drawn tending to N the Interſection of EF with ef, and $p\pi$ and $n\nu$ muſt be drawn parallel to MN the common Interſection of the given Planes; Ll muſt be drawn tending to the Interſection of $n\nu$, with a Line from x parallel to gh, which Interſection is the ſame with the Point marked $\mathcal{T}$, in Fig. 154. and dδ muſt be drawn tending to the Interſection of $p\pi$, with a Line from x parallel to GH, which Interſection is the ſame with the Point marked t, in that Figure; the Lines marked $m\mu$, $p\pi$, $n\nu$, Ll, and dδ, in the Figures of this Problem, correſponding reſpectively to my, pv, nw, DL, and Dt, in the Figures of Prob. XXIX.

Here,

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[a] Cor. Meth. 1.

[b] Meth. 6. and Cor. Caſe 1. Prob. 7.

[c] Cor. 1. Prob. 29.

[d] Cor. 8. Prob. 29.

[e] Meth 5. and Cor. Caſe 1. Prob. 7.

[f] Cor. 2. and Schol. Prob. 29.

[g] Meth. 7. and Cor. Caſe 1. Prob. 7.

[h] Cor. 3. and Schol. Prob. 29.

[i] Meth. 1. and Cor. 2 Caſe 1. Prob. 7.

[k] Meth. 1. and Cor. 2. Caſe 1. Prob. 7.

[l] Cor. 1. Meth. 1. Caſe 1. Prob. 7.

Here alſo, GH, gh, Ll, and dδ being produced, will all meet in the Interſecting Point of MN, the ſame with that marked D, in Fig. 154.

All this will appear ſufficiently evident, if compared with the Scholium at the End of Problem VII. and with Prob. XXIX. and its Corollaries.

GENERAL COROLLARY 1.

Fig. 155. N°. 3. If the Vaniſhing Line EF paſs through the Point T, which biſects the Vertical Line xo of the Reflecting Plane; then the Reflected Plane being parallel to the Picture[a], its Vaniſhing Line $\epsilon\phi$ will be infinitely diſtant, and the Line $n\nu$ will coincide with EF; but every thing elſe is found as before.

For if Ty be drawn, and the ſubſtituted Plane yyΔD paſſing through xT be compleated; it is evident that xo being biſected in T, and To and yy being equal, xy will be parallel to Dy, and their Interſection r, through which $\epsilon\phi$ ought to paſs[b], is therefore infinitely diſtant; and xy parallel to Dy, cutting Δy in y, the Points y and n coincide[c], wherefore $n\nu$ coincides with EF; and conſequently a Line from x, to the Interſection of EF with any Line in the Plane EFGH (which Interſection is the Vaniſhing Point of that Line) will be parallel to its Reflection; which it ought to be from another Conſideration, for if xy be produced to q its Interſection with ef, it will be biſected in y; wherefore the Reflection AD of the Line yΔ muſt be parallel to xy[d].

The Lines $p\pi$, $m\mu$, Ll and dδ are found, as in the firſt, ſecond, fourth and fifth Corollaries; MNRR is the viſible Part of the Reflected Plane parallel to the Picture, and MNLl is the whole Reflexible Part of the Original Plane.

And here, EF is the Vaniſhing Line of all Planes whatſoever, whoſe Reflections are parallel to the Picture[e].

GENERAL COROLLARY 2.

Fig. 155. N°. 4. If the Original Plane MNRR be parallel to the Picture, the Vaniſhing Line $\epsilon\phi$ of the Reflected Plane will paſs through T which biſects xo[f], and the Lines $p\pi$ and dδ will coincide with it; but every thing elſe is found as before.

For if Ty be drawn, and a ſubſtituted Plane paſſing through Ty and xT be compleated, the Interſection of that Plane with the Original Plane will be AL parallel to xT[g]; and the Vaniſhing and Interſecting Points y and Δ of the Line AL being therefore infinitely diſtant, Lines from x tending to thoſe Points will be parallel to AL, and coincide with xo, and T being the Interſection of xo with Dy, the Points r, p and d, will all coincide with T[h]; wherefore $\epsilon\phi$, $p\pi$ and dδ will alſo coincide, the Vaniſhing, Directing, and Interſecting Lines of the Original Plane, which ſhould produce thoſe Reflected Lines, being all infinitely diſtant.

The Lines $m\mu$, $n\nu$, and Ll are found, as in the ſecond, third, and fourth Corollaries; MNLl is the Reflexible Part of the Original Plane, and MNgh is the Place of its Reflection.

And here, as all Planes parallel to the Picture, are parallel to each other, ſo $\epsilon\phi$ the Vaniſhing Line of the Reflection of the Plane MNRR, is alſo the Vaniſhing Line of the Reflections of all Planes whatſoever which are parallel to the Picture[i].

SCHOL.

Here, the Point Γ, which coincides with T[k], is not the Reflection of the Point of Station on the Original Plane, there being no ſuch Point; but Γ is the Vaniſhing Point of the Reflection of the Eye's Director with reſpect to the Reflecting Plane $efgh$, which Director being parallel to the Original Plane, their Interſection is infinitely diſtant.

CASE 2.

When the Original Plane is perpendicular to the Reflecting Plane.

Fig. 155. N°. 5. Here, EF which paſſes through x, coincides with $\epsilon\phi$, and the Lines $p\pi$ and $n\nu$ alſo Coincide[l]; the Reflection of any Vaniſhing Point y is found at r, by taking xr equal to xy, and every thing elſe is found as before.

Dem. For xo being biſected in T, and the ſubſtituted Plane yyΔD being drawn, it is evident that xy cuts Dy in r, a Point in EF, wherefore EF and $\epsilon\phi$ coincide[m]; and becauſe xo is biſected in T, the Triangles rTx, oTy, are ſimilar and equal, rx is therefore equal to oy, which is equal to xy; and becauſe xp is parallel to Δy[n], and xn to Dy[o], the Triangles rpx, xny, are alſo ſimilar and equal, wherefore xp

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[a] Cor. Meth 1.

[b] Meth. 2.

[c] Cor. 3.

[d] Cor. 3. Meth. 3. Prob. 26.

[e] Art. 13.

[f] Cor. Meth. 1.

[g] Cor. Theor. 3. B. I.

[h] Meth. 2. and Cor. 1. and 5.

[i] Art. 13.

[k] Cor. 1. and Cor. 8. Prob. 29.

[l] Caſe 2. and Cor. Prob. 29.

[m] Meth. 2.

[n] Cor. 1.

[o] Cor. 3.

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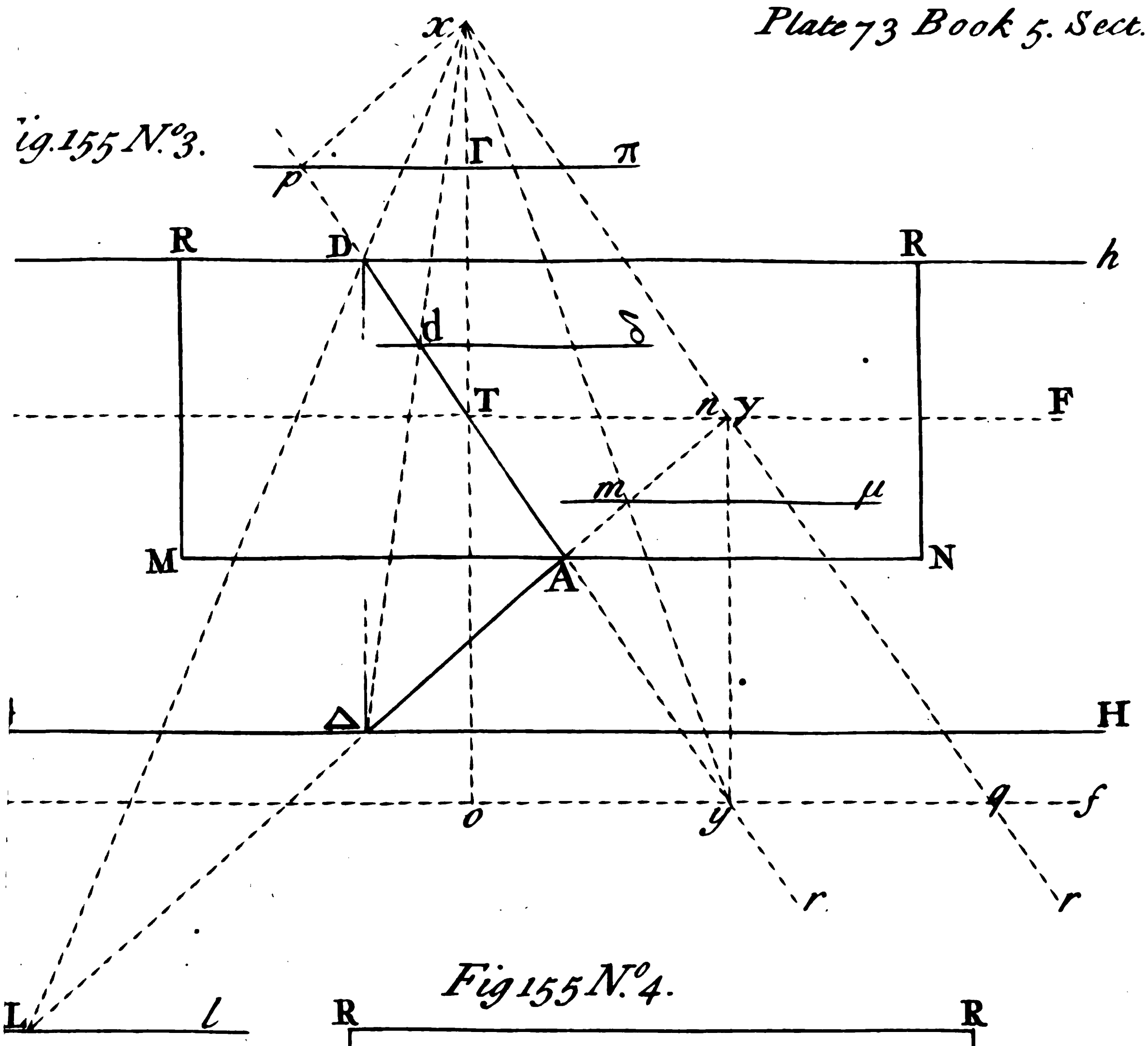
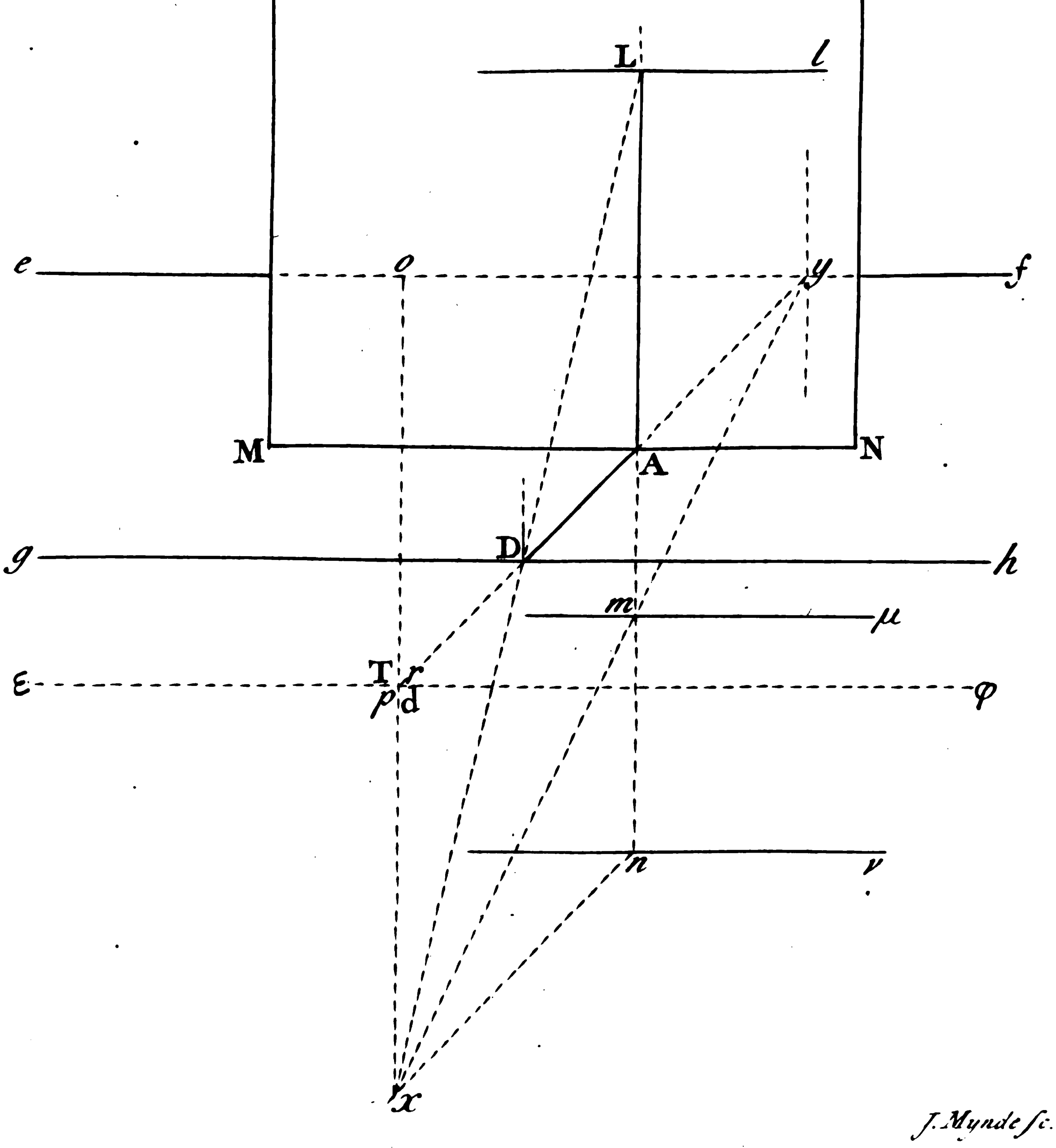


Fig 155 N°4.



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and yn are equal, and pn is therefore parallel to EF, and conſequently $p\pi$ and $n\nu$ coincide: laſtly, becauſe the Interſection of EF with ef is infinitely diſtant, the Point x biſects the Diſtance between the Vaniſhing Point y and its Reflection r; which is alſo apparent, for that the Vaniſhing Points y and r ſubtend equal Angles with x. *Q. E. I.*

CASE 3.

When the Original Plane is parallel to the Reflecting Plane.

In this Caſe, the Reflected Plane being alſo parallel to the Original and Reflecting Planes[a], the Vaniſhing Lines EF, ef, and $\varepsilon\varphi$ coincide, and all Vaniſhing Points in EF coincide with their Reflections[b]; the Lines $p\pi$, $n\nu$, Ll, and dδ, are found by the help of the ſubſtituted Plane yyΔD paſſing through xT as before; and if the Interſecting Line of the Reflected Plane be wanted, it is found by taking Dd equal to DΔ, and through d, drawing gh parallel to EF; in regard that the Original and Reflected Planes being equally diſtant from the Reflecting Plane, their Interſecting Lines muſt be equally diſtant from gh the Interſecting Line of the Reflecting Plane. *Q. E. I.*

Fig. 155. N°. 6.

[a] Art. 10.

[b] Art. 7.

Here alſo, the Interſection of the given Planes being infinitely diſtant, MN coincides with EF, as does alſo the Line $m\mu$; ſeeing xy can cut Δy only in y[c], the Points y and y being the ſame.

[c] Cor. 2. Caſe 1.

PROB. XXXI.

A Reflecting Plane perpendicular to the Picture, together with an Original Plane, being given; thence to find the Reflected Plane, and the Reflections of any propoſed Lines in the Original Plane, when the Original and Reflecting Planes interſect in a Line not parallel to the Picture.

CASE 1.

When the Original Plane inclines to the Reflecting Plane.

Let O be the Center of the Picture, $efgh$ and EFGH the Reflecting and Original Planes, and Dy their Interſection.

Fig. 156. N°. 1.

From any Vaniſhing Point z in EF, draw zy perpendicular to ef, cutting it in v, and take vy equal to vz; then $\varepsilon\varphi$ drawn through y and y will be the Vaniſhing Line of the Reflected Plane; and if from any Vaniſhing Point in EF, a Line be drawn perpendicular to ef, it will cut $\varepsilon\varphi$ in the Reflection of that Vaniſhing Point.

Dem. For y being the Reflection of z[d], it is a Point in the Reflected Vaniſhing Line, and y being another Point of that Line, $\varepsilon\varphi$ is therefore the Reflected Vaniſhing Line; and it is evident that all Lines perpendicular to ef, and terminated by EF and $\varepsilon\varphi$, will be biſected by ef. *Q. E. I.*

[d] Prob. 28.

SCHOL.

Here, x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, being infinitely diſtant, the Line zy is biſected in v[e], the Vaniſhing Lines EF and $\varepsilon\varphi$ incline in equal Angles to ef, and yz the Diſtance between y and any Vaniſhing Point z, is always equal to yy the Diſtance between y and the Reflection y of that Vaniſhing Point.

[e] Cor. Meth. 1. Prob. 29.

Alſo, the tranſpoſed Place of the Eye being a Point at a moderate Diſtance in the Directing Plane, and the Direction of all Lines proceeding from thence being parallel to zy[f], all Lines which ſhould tend to the tranſpoſed Place of the Eye, muſt be drawn perpendicular to ef.

[f] Cor. 1. Prob. 25.

COR. 1.

If any Line in the Plane EFGH be parallel to the Picture, and conſequently to EF, its Reflection will be parallel to $\varepsilon\varphi$; and the Reflection of any propoſed Part of ſuch a Line will be equal to that Part.

For if a Plane be imagined to paſs through the propoſed Line parallel to the Picture, that Plane being perpendicular to the Reflecting Plane, its Reflection will make one continued Plane with it[g]; wherefore the Reflections of all Lines in this Parallel Plane, are alſo in the ſame Plane, and conſequently parallel to the Picture; but the Reflection of the propoſed Line, is a Line in the Reflected Plane, and being parallel to the Picture, its Image is therefore parallel to $\varepsilon\varphi$; and the propoſed Line and its Reflection being

[g] Art. 11.

both in the ſame Plane parallel to the Picture, their equal Parts repreſent equal Lines[a].

COR. 2.

Hence, gh drawn through D, parallel to $\epsilon\phi$, and which is the Interſecting Line of the Reflected Plane, is alſo the Imaginary Reflection of GH the Interſecting Line of the Original Plane.

COR. 3.

All Lines in the Plane EFGH, which have Parallel Images, have Parallel Reflections.

For the Directing Plane being perpendicular to the Reflecting Plane, it coincides with its own Reflection; wherefore the Imaginary Reflection of every Point or Line in the Directing Plane, is alſo in the Directing Plane; but all Lines in the Plane EFGH which have Parallel Images, have the ſame Directing Point, the Reflection of which is a Point common to the Reflections of thoſe Lines; and this Reflected Point being alſo a Directing Point, the Images of thoſe Reflections are therefore parallel.

COR. 4.

Hence, the Reflection of the Directing Line of the Original Plane, is the Directing Line of the Reflected Plane, neither of which can be repreſented; ſo that the Lines marked pv and nw, in Fig. 154. have here no Place[b].

COR. 5.

If from D, a Line Dz be drawn parallel to any propoſed Line in the Plane EFGH, cutting EF any where in z; find y the Reflection of z, and draw Dy, then Dy will be parallel to the Reflection of the propoſed Line.

For Dy being the Reflection of Dz, it is parallel to the Reflections of all Lines in the Original Plane which are parallel to Dz[c].

And hence, if any Line in the Plane EFGH be parallel to Dy, its Reflection will be parallel to it[d].

COR. 6.

From k, where $\epsilon\phi$ croſſes gh, draw kq perpendicular to ef, cutting EF in q; then qD will be the Boundary of the Reflexible Part of the Original Plane[e], and yDq will therefore be the whole Reflexible Part of that Plane, and yDk the whole of its Reflection.

For k being the Reflection of the Vaniſhing Point q, Dk which coincides with gh, is the Reflection of Dq.

And here, as qk is biſected in T, ſo ql is biſected in y, and yk and yl are therefore equal.

COR. 7.

From y, draw ym parallel to qD, and the Imaginary Reflection of ym will coincide with ef[f].

For the Reflections of qD and ym being parallel[g], and y being a Point in the Reflection of ym, ef parallel to Dk the Reflection of qD[h], is therefore the Imaginary Reflection of ym.

SCHOL.

If EFGH be taken to repreſent the Directing Plane, and q or $\mathcal{Y}$ the Place of the Eye[i], then qy and yl being equal[k], ef will repreſent the Directing Line of the Reflecting Plane[l]; and qT and Tk being equal, k will repreſent the tranſpoſed Place of the Eye in the Directing Plane, and T its Perpendicular Seat on the Directing Line of the Reflecting Plane; and if kt be drawn parallel to EF, cutting ef in t, t will be the Parallel Seat of k on the Directing Line ef with reſpect to the Original Plane, and qt the Direction of all Lines proceeding from that Seat; wherefore yk or $\epsilon\phi$ drawn from y parallel to qt, will be the Line of the *Foci* of the Projections on the Plane $efgh$, of all Lines in the Plane EFGH from the tranſpoſed Place of the Eye[m], and is therefore the Reflected Vaniſhing Line[n].

Likewiſe, kD which is parallel to ef, cutting GH in D, D repreſents the Parallel Seat of k on the Directing Line GH of the Original Plane with reſpect to the Reflecting Plane, and qD is its Direction; wherefore ym drawn parallel to qD, is the Interſection of the Plane EFGH with a Plane paſſing through the tranſpoſed Place of the Eye, parallel to the Reflecting Plane[o]; and conſequently the Imaginary Reflection of ym coincides with ef[p].

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[a] Art. 5. and Cor. 2. Theor. 23. B. I:

[b] Cor. 1. and 3. Prob. 29.

[c] Cor. 3.

[d] Cor. 7. Prob. 29.

[e] Cor. 9. Prob. 29.

[f] Cor. 2. Prob. 29.

[g] Cor. 3.

[h] Cor. 6.

[i] Cor. 2. Caſe 3. Prob. 1.

[k] Cor. 6.

[l] Cor. 3. Def. 18. B. I.

[m] Meth. 4. Caſe 3. Prob. 6.

[n] Meth. 2. Prob. 29.

[o] Meth. 5. Caſe 3. Prob. 6.

[p] Cor. 2. Prob. 29.

It is evident alſo, that in this View, kq, which is perpendicular to ef, is the Direction of all Lines proceeding from k the tranſpoſed Place of the Eye in the Directing Plane.

CASE 2.

When the Original Plane is perpendicular to the Reflecting Plane.

Fig. 156. N°. 2.

Here, the Original and Reflected Planes making one continued Plane[a], their Vaniſhing Lines EF and $\varepsilon\varphi$ coincide, and the Direction of the Reflected Rays or Projecting Lines is parallel to EF, as being perpendicular to ef.

The Reflection of any Vaniſhing Point z in EF, is at y in the ſame Line, yy being taken equal to yz[b]; and all Lines in the Original Plane which are parallel to the Picture, being perpendicular to the Reflecting Plane, make one continued ſtraight Line with their Reflections[c].

The Boundary qD is found by making yq equal to yk or yl[d], which here coincide; and the Line ym is had by drawing it parallel to qD[e]. Q. *E. I.*

COR.

The other Corollaries of the laſt Caſe are likewiſe applicable here; and in either Caſe, the Practice is the ſame, whether the Original Plane be perpendicular or inclining to the Picture; it making no Difference, in whatever Point of ef the Center of the Picture O falls.

PROB. XXXII.

A Reflecting Plane perpendicular to the Picture, together with an Original Plane, being given; thence to find the Reflected Plane, and the Reflections of any propoſed Lines in the Original Plane, when the Reflecting and Original Planes interſect in a Line parallel to the Picture.

CASE 1.

When the Original Plane inclines to the Reflecting Plane.

Fig. 157. N°. 1.

Let $efgh$ and EFGH be the Reflecting and Original Planes, MM their Interſection, and O the Center of the Picture.

Draw any Line zy perpendicular to ef, cutting EF and ef in z and v, and in it take vy equal to vz; then $\varepsilon\varphi$ drawn through y parallel to ef, will be the Reflected Vaniſhing Line; and if from any Vaniſhing Point in EF, a Line be drawn perpendicular to ef, it will cut $\varepsilon\varphi$ in the Reflection of that Vaniſhing Point.

Dem. For y being the Reflection of z[f], $\varepsilon\varphi$ drawn through y parallel to ef, is the Reflected Vaniſhing Line[g]. Q. *E. I.*

COR. 1.

Draw any Line Δd parallel to zy, cutting gh and GH in D and Δ; and in it take Dd equal to DΔ; then gh drawn through d parallel to ef, will be the Reflection of GH, and alſo the Interſecting Line of the Reflected Plane.

For d being the Reflection of Δ[h], gh is the Reflection of GH, and conſequently the Interſecting Line of the Reflected Plane[i].

COR. 2.

Draw $z\Delta$ and yd, and from b the Interſection of yd with gh, draw bL perpendicular to ef, cutting $z\Delta$ in L; and Ll, drawn through L parallel to MM, will be the Boundary of the Reflexible Part of the Original Plane; ſo that the whole Reflexible Part of that Plane will be between Ll and MM, and the whole of its Reflection between MM and gb[k].

For yd being the Reflection of $z\Delta$, b is the Reflection of L, and conſequently the Reflection of Ll coincides with gb.

COR. 3.

From c the Interſection of yd with ef, draw cm perpendicular to ef, cutting $z\Delta$ in m, and through m draw $m\mu$ parallel to ef; and the imaginary Reflection of $m\mu$ will coincide with ef[l].

For the imaginary Reflection of m is at c, and conſequently ef is the imaginary Reflection of $m\mu$.

COR.

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[a] Art. 11.

[b] Schol. Caſe 1.

[c] Art. 4.

[d] Cor. 6. Caſe 1.

[e] Cor. 7.

[f] Prob. 28.

[g] Meth. 1. Prob. 30.

[h] Prob. 25.

[i] Cor. 2. Prob. 31.

[k] Cor. 6. Prob. 31.

[l] Cor. 7. Prob. 31.

COR. 4.

Here, the Lines marked $p\pi$ and $n\nu$ (Fig. 155.[a]) have no Place, for the Reaſon already mentioned[b].

CASE 2.

When the Original Plane is perpendicular to the Reflecting Plane, and conſequently parallel to the Picture.

Fig. 157. N°. 2. Here, the Original and Reflected Planes making one continued Plane RRMM*rr* parallel to the Picture, the Lines $\epsilon\phi$ and gh are infinitely diſtant; and the Reflection $A\alpha$ of any Line Aa in the Original Plane, makes the ſame Angle with MM the common Interſection of the given Planes, as the Original Line doth, but the contrary way; if therefore from any Point a in Aa, a Line $a\alpha$ be drawn perpendicular to MM, cutting it in a, take aα equal to aa, and α will be the Reflection of a, and conſequently $A\alpha$ the Reflection of Aa. *Q. E. I.*

COR. 1.

If from β, where $A\alpha$ croſſes gb, a Line bβ be drawn perpendicular to MM, cutting it in b; take bL equal to bβ, and through L draw Ll parallel to MM, and Ll will be the Boundary of the Reflexible Part of the Original Plane.

For β being the Reflection of L, the Reflection of Ll will coincide with gb the Interſecting Line of the Reflecting Plane.

COR. 2.

If from c, where $A\alpha$ cuts ef, a Line cm be drawn perpendicular to MM, cutting Aa in m; through m draw $m\mu$ parallel to MM, and the imaginary Reflection of $m\mu$ will coincide with ef; ſeeing the imaginary Reflection of m is at c.

It is evident alſo, that if from γ, where Aa croſſes ef, a Perpendicular to MM be drawn, it will cut $A\alpha$ in μ, a Point of $m\mu$, and that ef and $m\mu$ are equally diſtant from MM.

CASE 3.

When the Original Plane is parallel to the Reflecting Plane.

Fig. 157. N°. 3. Here, the Vaniſhing Lines EF, ef, and $\epsilon\phi$ coincide, and all Vaniſhing Points in EF coincide with their Reflections[c].

The Interſecting Line gh of the Reflected Plane, which is alſo the Reflection of GH, is found by drawing any Line ΔD perpendicular to EF, and making Dd equal to DΔ[d]; and if from any Point z in EF, $z\Delta$ and zd be drawn, a Line bL drawn from b the Interſection of zd with gb, perpendicular to EF, will cut $z\Delta$ in L, through which the Boundary Ll paſſes[e]. *Q. E. I.*

And here, $m\mu$ and MM coincide with EF[f].

PROB. XXXIII.

The Center and Diſtance of the Picture, and a Reflecting Plane parallel to the Picture, together with an Original Plane, being given; thence to find the Reflected Plane, and the Reflections of any propoſed Lines in the Original Plane.

CASE 1.

When the Original Plane inclines to the Reflecting Plane.

Fig. 158. N°. 1. Let O be the Center of the Picture, EFGH the Original Plane, RRMM the Reflecting Plane parallel to the Picture, and MM their common Interſection.

Draw the Vertical Line Oo of the Plane EFGH, and produce it to t, till Ot and Oo be equal; then $\epsilon\phi$ drawn through t parallel to MM, will be the Vaniſhing Line of the Reflected Plane; and if from any Vaniſhing Point z in EF, a Line be drawn through O, it will cut $\epsilon\phi$ in y the Reflection of z, and zy will be biſected in O.

Dem. For t being the Reflection of the Vaniſhing Point o[g], it is a Point in the Reflected Vaniſhing Line, and MM the Interſection of the Reflected Plane with the Original Plane, being parallel to EF, the Reflected Vaniſhing Line is alſo parallel to EF,

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[a] Cor. 1. and 3. Prob. 30.

[b] Cor. 4. Prob. 31.

[c] Caſe 3. Prob. 30.

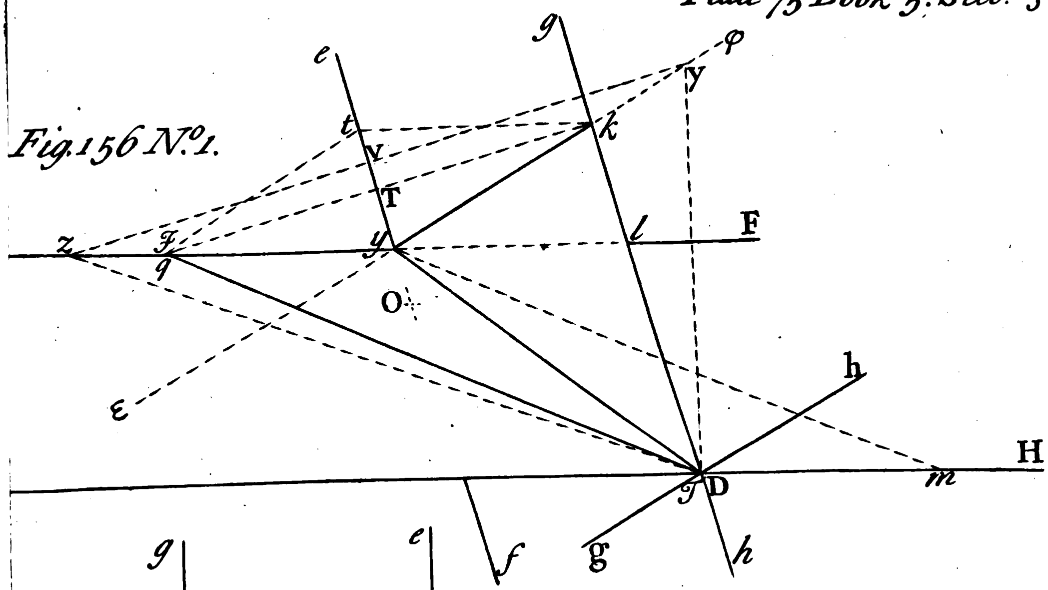
[d] Cor. 1. Caſe 1.

[e] Cor. 2. Caſe 1.

[f] Caſe 3. Prob. 30.

[g] Meth. 1. Prob. 27.

Fig. 156 No. 1.



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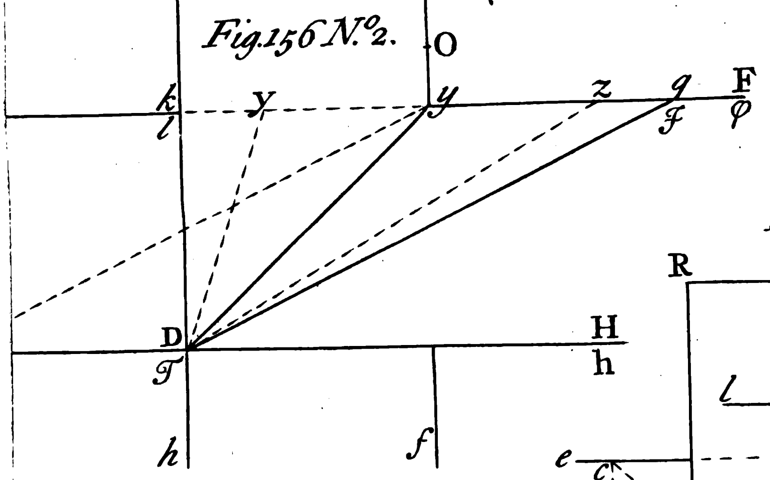

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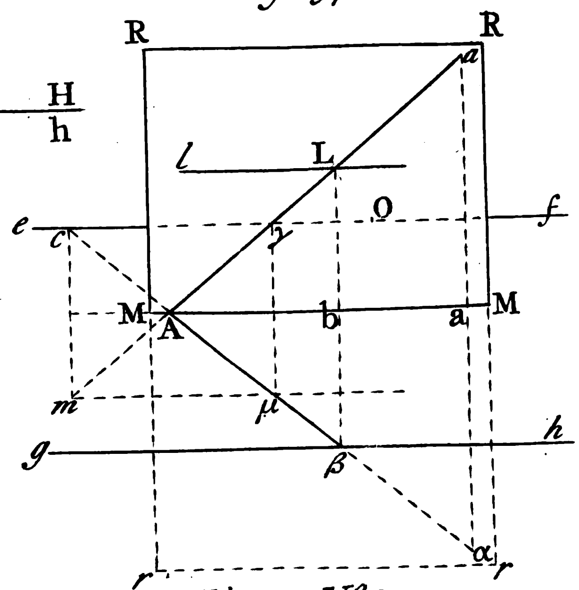


Fig. 157 No. 1.

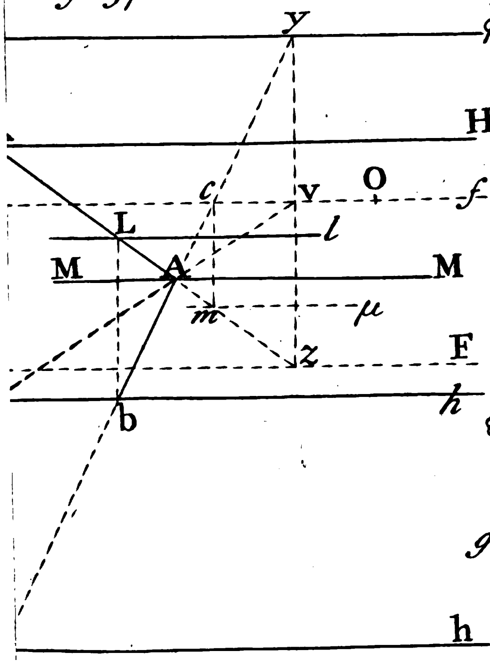
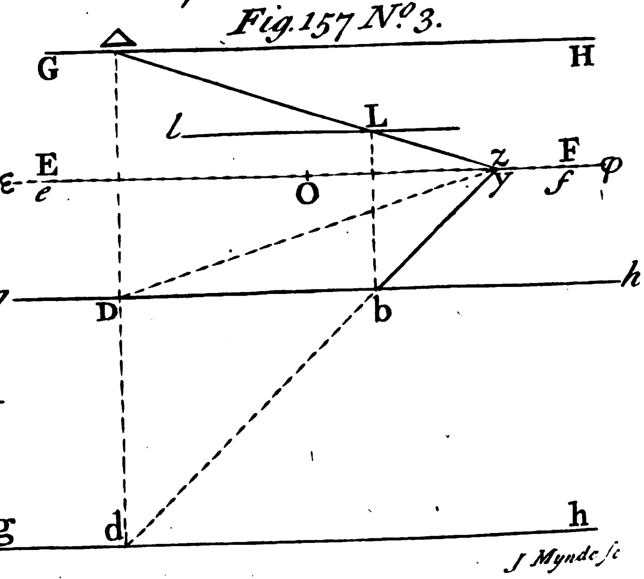


Fig. 157 No. 3.



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EF[a], and conſequently to MM; wherefore $\epsilon\phi$ drawn through t parallel to MM is the Reflected Vaniſhing Line; in the next place it is evident, that Oo and Ot being equal, zO and Oy are alſo equal, and that therefore y is the Reflection of z[b].

Or thus, t being the Oblique Seat of the tranſpoſed Place of the Eye on the Reflecting Plane with reſpect to the Plane EFGH[c], $\epsilon\phi$ is the Line of the *Foci* of the Projections on the Plane RRMM of all Lines in the Plane EFGH from the Point O[d], and is therefore the Vaniſhing Line of the Reflected Plane[e]. *Q. E. I.*

COR. 1.

Through o draw any Line $o\Delta$ in the Plane EFGH, cutting MM and GH in A and Δ, and draw At; then from O draw Op parallel to $o\Delta$, cutting At in p; and $p\pi$ drawn parallel to EF, will be the Reflection of the Directing Line of the Plane EFGH; the Point Γ, where $p\pi$ cuts Oo, will be the Reflection of the Foot of the Eye's Director with reſpect to the Plane EFGH[f], and the Line $p\pi$ will biſect the Diſtance between MM and $\epsilon\phi$.

For At being the Reflection of ΔA[g], p is the Reflection of the Directing Point of ΔA, and conſequently a Point in the Reflection of the Directing Line of the Plane EFGH; which Directing Line being parallel to EF, its Reflection is alſo parallel to EF; and as ot is biſected in O, At is alſo biſected in p, and conſequently Nt in Γ.

COR. 2.

From O draw On parallel to At, cutting Δo in n, and draw $n\nu$ parallel to MM; then the imaginary Reflection of $n\nu$ will coincide with the Directing Line of the Reflected Plane[h], and $n\nu$ will biſect the Diſtance between MM and EF.

For At and On being parallel, and in the ſame Plane OoA, they have the ſame Directing Point; and as to is biſected in O, Ao is biſected in n, and conſequently No in ν.

COR. 3.

The Line $n\nu$ alſo ſupplies the Places of $m\mu$ and Ll, the Reflections of which ſhould coincide reſpectively with the Vaniſhing and Interſecting Lines of the Reflecting Plane[i].

For the Point ν which biſects No is the Oblique Seat of O, taken as the tranſpoſed Place of the Eye, on the Original Plane[k]; and a Plane paſſing through O and the Line $n\nu$, is therefore parallel to the Picture and to the Reflecting Plane, and conſequently may be ſuppoſed to meet this laſt, either in its Vaniſhing, Interſecting, or Directing Lines, all which are infinitely diſtant.

COR. 4.

Here, the whole of the Original Plane, from MM indefinitely produced behind the Directing Plane, is Reflexible, and its intire Reflection lies between MM and $\epsilon\phi$.

COR. 5.

From O draw OΔ cutting At in d, and draw dδ parallel to MM; and dδ will be the Reflection of GH[l].

And thus the Reflection of any Line parallel to the Picture in the Plane EFGH may be had[m].

COR. 6.

If the Point O fall below EF, but nearer to it than to MM, ſome Part of the Reflection of the Original Plane will be viſible; but if O biſect the Diſtance between EF and MM, or if it be nearer to MM than to EF, no Part of the Original Plane can be Reflected.

For if O fall below o, but above ν which biſects oN[n], then oO being leſs than $o\nu$, Ot which is equal to oO[o], will alſo be leſs than $o\nu$ or νN, and therefore t, and conſequently $\epsilon\phi$ will not reach ſo low as MM, and ſome Part of the Reflection of the Original Plane will therefore be viſible[p]; but if O fall either at or below ν, t and conſequently $\epsilon\phi$ will fall at or below MM, in either of which Caſes no viſible Reflection can be produced[q].

SCHOL.

When the Reflecting Plane is parallel to the Picture, Ot the Oblique Support of the tranſpoſed Place of the Eye, being a Line in the Vertical Plane[r], there can no ſubſti-

[a] Theor. 14. B. I.

[b] Meth. 1. Prob. 27.

[c] Cor. 2. Prob. 24.

[d] Meth. 2. Prob. 27.

[e] Meth. 2. Prob. 29.

[f] Cor. 1. Prob. 30.

[g] Meth. 1. Prob. 27.

[h] Cor. 3. Prob. 30.

[i] Cor. 2. and 4. Prob. 30.

[k] Caſe 1. Prop. 43. B. IV.

[l] Cor. 5. Prob. 30.

[m] Cor. 6. Prob. 30.

[n] Cor. 2.

[o] Prob.

[p] Cor. 4.

[q] Schol. Cor. 6. Prob. 29.

[r] Prop. 3 B. IV.

ſubſtituted Plane paſs through that Line perpendicular to the Reflecting Plane, ſave only the Vertical Plane itſelf, the intire Image of which is the Vertical Line O*o*; which being therefore unfit to be uſed as the ſubſtituted Plane $yz\Delta D$[a] in the laſt Problem, it becomes neceſſary to have recourſe to the Method here propoſed.

[a] Fig. 157.

CASE 2.

When the Original Plane is perpendicular to the Reflecting Plane, and conſequently to the Picture.

Fig. 158. N°. 2.

Let O be the Center of the Picture, EFGH the Original Plane, and MM its Interſection with the Reflecting Plane RRMM.

Here, O coinciding with *o*, the Point *t* coincides with them, $\varepsilon\varphi$ therefore coincides with EF, and the Reflection of any Vaniſhing Point *z* in EF is found at *y* in the ſame Line, by taking O*y* equal to O*z*[a]; alſo the Lines $p\pi$ and $n\nu$ coincide, and biſect the Diſtance between MM and EF, and the Line $d\delta$ is found as before[b].

[a] Caſe 2. Prob. 30.
[b] Cor. 5. Caſe 1.

Dem. Draw any Line Δz in the Plane EFGH, cutting MM in A, and having taken O*y* equal to O*z*, draw A*y* the Reflection of ΔA; then draw O*p* parallel to Δz, cutting A*y* in *p*, and O*n* parallel to A*y*, cutting Δz in *n*, and draw *pn*; laſtly, having drawn OΔ cutting A*y* in d, draw δd parallel to MM: then becauſe *zy* is biſected in O, A*y* is biſected in *p*, and A*z* in *n*; wherefore *pn* is parallel to EF and biſects NO, and the Lines $p\pi$ and $n\nu$, which muſt paſs through *p* and *n* parallel to EF or MM, are therefore the ſame; it is evident alſo, that d being the Reflection of Δ, δd parallel to MM is the Reflection of GH. *Q. E. I.*

COR.

Here, all Lines in the Plane EFGH, which have O for their Vaniſhing Point, being perpendicular to the Reflecting Plane, they make one continued ſtraight Line with their Reflections; and the Reflections of any Points in ſuch Lines are found by the Methods before ſhewn[c].

[c] Cor. 7. and 8. Prob. 26.

CASE 3.

When the Original Plane is parallel to the Reflecting Plane, and conſequently to the Picture.

Fig. 158. N°. 3.

Let O be the Center of the Picture, RRMM the Reflecting Plane, and LLK the Original Plane, both parallel to the Picture.

Having drawn any ſubſtituted Plane EFGH perpendicular to the Picture, cutting the Reflecting and Original Planes in MM and LL[d], from O to any Point L in LL, draw OL cutting MM in A, and make A*l* and AL repreſent equal Lines; then draw *ll* parallel to MM, and on it deſcribe a Plane *llk* parallel to the Picture, and that will be the Reflected Plane.

[d] Meth. 2. Prob. 10.

Dem. For *l* being the Reflection of L[e], *ll* is the Reflection of LL, and is therefore a Line in the Reflected Plane; *llk* is therefore the Reflected Plane, it repreſenting a Plane parallel to the Reflecting Plane, and as far beyond it as the Original Plane is before it[f]. *Q. E. I.*

[e] Cor. Caſe 2.
[f] Art. 10.

COR. 1.

The Reflections of all Lines in the Plane LLK are parallel to their Originals, and the Reflection $\alpha\beta\gamma$ of any Figure *abc* in the Original Plane, is ſimilar to it[g], and their Sides are in the ſame Proportion to each other as *ll* to LL[h].

[g] Art. 7. and 8.
[h] Prop. 38. B. IV.

SCHOL.

Here, the Reflection $\alpha\beta\gamma$ is ſimilar to, and alike poſited with the Triangle *abc*, although it be really in a contrary Poſition to that whoſe Reflection it is; for *abc* is not the real Figure which is reflected, but the Appearance of it as ſeen through the Original Plane taken as tranſparent, the Original of the Reflection $\alpha\beta\gamma$ being on that Side of the Plane LLK which fronts the Reflecting Plane, and ſo cannot be otherwiſe ſeen than by its Reverſe.

COR. 2.

Fig. 158. N°. 4.

If the Original Plane coincide with the Directing Plane, then AO will be biſected in *l*, and therefore *ll* will biſect the Diſtance between MM and EF.

For in this Caſe *L* repreſenting the Directing Point of AO, its Reflection *l* muſt biſect AO[i].

[i] Cor. 8. Prob. 26.

Plate 76 Book 5. Sect. 3.

158 N°1.

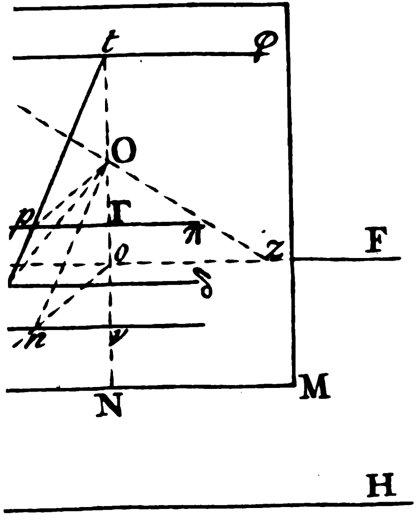
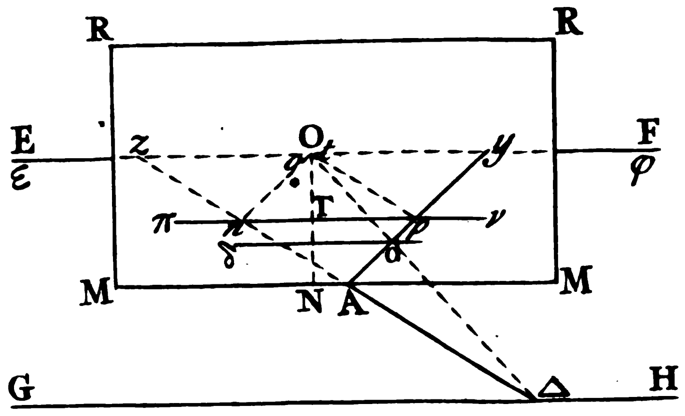


Fig. 158 N°2.



158 N°3.

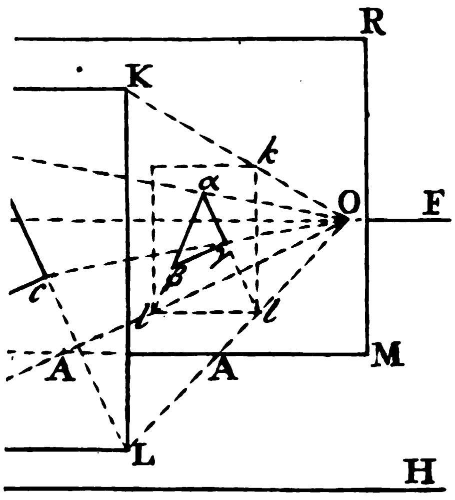


Fig. 158 N°4.

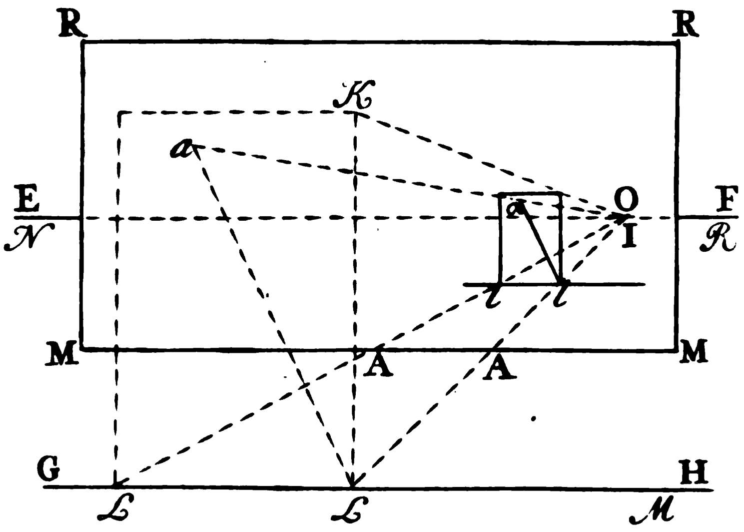
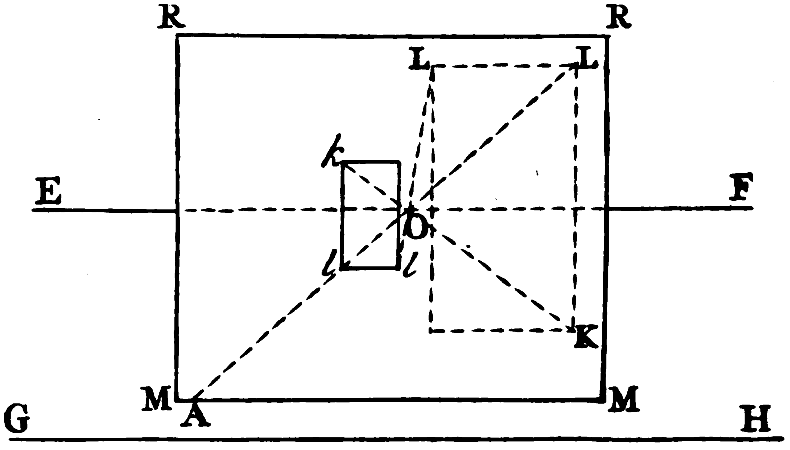


Fig. 158 N°5.



J. Mynde sc.

SCHOL.

Here, as the Original Figure to be Reflected can have no real Image, the Directing Plane muſt be drawn, for which purpoſe the Plane EFGH may be made to ſerve[a]; in which, the Original Plane *LLK* being placed, in the ſame Poſition with reſpect to O, as it really has with reſpect to the Eye in the Directing Plane, *LM* will then repreſent the Interſection of the Original Plane with the Plane EFGH, the ſame with the Directing Line of this Plane; and if from O, any Line O*L* be drawn, cutting *LM* in *L*, O*L* will be the Direction of the Line in which the Reflection of *L* lies; and in regard that O and I are here the ſame, OA is alſo the Image of that Line, which being biſected in *l*, gives the Reflection of the Directing Point *L*, and conſequently *ll* the Interſection of the Reflected Plane with the Plane EFGH; whereby the Reflection of any Figure in the Original Plane may be had, by making it ſimilar to the propoſed Figure, and alike ſituated with reſpect to *l*, as the propoſed Figure, as ſeen on the Backſide, is with regard to *L*[b], the proportional Meaſures of the Sides of the Reflection being taken on the Line *ll*[c].

COR. 3.

Fig. 158. N°. 5.

If the Original Plane be behind the Directing Plane, the Transprojective Image LL of its Interſection with the Plane EFGH will fall beyond EF in the Transprojective Part of that Plane, and OL and O*l* muſt be made to repreſent equal Lines from A, taken as a Vaniſhing Point[d]; and the Line *ll*, and conſequently the Reflected Plane *llk* being thence found, the Reflection of any Figure in the Original Plane may be had without its Image, by finding the Reflection of any one Point L in the Original Plane, whoſe Situation with reſpect to the propoſed Figure is known[e].

SCHOL.

And here, as the Transprojective Image LLK of the Original Plane is inverted, its Reflection *llk* is upright; nevertheleſs the Reflection of any Line in the Original Plane will ſtill be parallel to its Transprojective Image, this laſt being parallel to the Original Line, although the Reflections of all Figures in the Original Plane will be in an inverted Poſition with reſpect to their Transprojective Images, they having in all Caſes the ſame Situation with the Original Figures ſeen on the Backſide[f].

GENERAL COROLLARY.

The Reflection of any Original Plane being found, and in it the determinate Reflection of any Side or Line of a known Figure in the Original Plane being given; the intire Reflection of the propoſed Figure may be thence had, without any farther Aſſiſtance from its direct Image; by uſing the Reflected Plane as if it were an Original Plane, and thereon compleating a Figure, repreſenting a Figure ſimilar and equal to the Original Figure propoſed[g]; having regard to the contrary Poſition which the Reflection ought to have, to that of the Original Figure.

For the Reflection of any Plain Figure being ſimilar and equal to its Original[h], and the Reflected Image of any determinate Line repreſenting a Line equal to it[i]; if on the Reflected Image of any Side of a propoſed Figure, a Figure be compleated, which ſhall repreſent one ſimilar and equal to it, that will be the Reflection of the Figure propoſed.

Fig. 159.

Thus, let O be the Center, and OI the Diſtance of the Picture, EF and *ef* the Vaniſhing Lines of an Original and Reflecting Plane, and D*y* their common Interſection, *x* the Vaniſhing Point of Perpendiculars to the Reflecting Plane, and $\epsilon\phi$ the Reflected Vaniſhing Line; and let it be propoſed to find the Reflection of a Circle in the Original Plane which croſſes the Directing Line, and whoſe Image therefore forms two oppoſite Hyperbolas PPP, QQQ[k].

Let *AB* parallel to EF, be the Image of one Side of a Square, circumſcribing the propoſed Circle[l].

Produce *AB* to B its Interſection with D*y*, and from *t* (which biſects *xl*, parallel to EF) draw *t*B the indefinite Reflection of *AB*[m], the Extremities α and β of which, are found by *xA* and *xB*[n]; then on $\alpha\beta$ deſcribe the Image $\alpha\beta\gamma\delta$ of a Square in the Reflected Plane, ſubdivided as a Model for the Image of a Circle[o], which being inſcribed in it accordingly, it will be the Reflection of the Circle propoſed.

And here, y the Vaniſhing Point of the Reflections of Aa and its parallels, is found in $\epsilon\phi$, either by making it ſubtend a Right Angle with *t*, or by the Interſection of $\epsilon\phi$ with

[a] Cor. 2. Caſe 3. Prob. 1.

[b] Schol. Cor. 1.

[c] Prop. 38. B. IV.

[d] Cor. 1. and 6. Lem. 8. B. III.

[e] Cor. 1.

[f] Art. 16.

[g] Sect. 3. B. II.

[h] Art. 8.

[i] Gen. Cor. 3. Prob. 28.

[k] Con. Sect. Art. 15. B. III.

[l] Meth. 1. Prop. 18. B. III.

[m] Cor. 4. Prob. 29.

[n] Gen. Cor. 1. Prob. 28.

[o] Prob. 24. B. II.

with Ox, drawn through x, and O the Vaniſhing Point of Aa; and v the Vaniſhing Point of the Diagonal $\beta\gamma$ of the Reflected Square $\alpha\beta\gamma\delta$ is found, either by biſecting the Angle ſubtended by ty, or by drawing Ix from I the Vaniſhing Point of BC, the Complement of the correſponding Diagonal of the Original Square.

PROB. XXXIV.

The Center and Diſtance of the Picture, and the Vaniſhing Line of a Reflecting Plane, being given, together with the Vaniſhing Line of an Original Plane, and its Reflection; thence to find the Reflection of any other Vaniſhing Line propoſed.

CASE 1.

When the Reflecting Plane inclines to the Picture.

Fig. 160. N°. 1, 2. Let O be the Center of the Picture, ef and EF the Vaniſhing Lines of the Reflecting and Original Planes, x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, and $\varepsilon\varphi$ the Reflection of EF; and let Aa be another Vaniſhing Line whoſe Reflection is required.

METHOD 1.

Biſect the Vertical Line xo of the Reflecting Plane in T, and having through T drawn Tt parallel to ef, draw $x\tau$ parallel to the propoſed Vaniſhing Line Aa, cutting Tt in τ, and from a, the Interſection of Aa with ef, draw τa, which will be the Reflection ſought.

Dem. For if $x\tau$ be produced till it cut ef in m, it will be biſected in τ, and xm being parallel to Aa, τa is therefore the Reflection of Aa[a]. *Q. E. I.*

METHOD 2.

From x to A the Interſection of Aa with EF, draw xA cutting $\varepsilon\varphi$ in α, and the Point τ being found as before, $\tau\alpha$ drawn through τ and α will be the Reflection deſired.

Dem. For α being the Reflection of A, and τ a Point through which the Reflection of Aa paſſes[b], $\tau\alpha$ is therefore the Reflection of Aa. *Q. E. I.*

COR. 1.

Fig. 160. N°. 1. If any other Vaniſhing Lines Bb, Cc, parallel to Aa, be propoſed, their Reflections will all paſs through the ſame Point τ; which Point will be the Vaniſhing Point of the Reflections of all Original Lines whatſoever which are parallel to the Picture and to Aa[c].

For the Vaniſhing Lines of all Planes which can paſs through ſuch Original Lines, muſt be parallel to Aa, or coincide with it[d].

COR. 2.

If the Vaniſhing Lines Aa, Bb, Cc, be parallel to ef, the Point τ being then infinitely diſtant, their Reflections will alſo be parallel to ef, and paſs through α, β and γ the Reflections of A, B, and C.

COR. 3.

If EF paſs through O the Center of the Picture, and Aa, Bb, and Cc, be perpendicular to EF, then Aa, Bb, and Cc being Vaniſhing Lines of Planes perpendicular to the Plane EF[e], $\tau\alpha$, $\tau\beta$, and $\tau\gamma$ will be Vaniſhing Lines of Planes perpendicular to $\varepsilon\varphi$ the Reflection of EF[f]; and conſequently τ will be the Vaniſhing Point of Perpendiculars to the Reflected Plane $\varepsilon\varphi$[g].

COR. 4.

Fig. 160. N°. 2. If the Vaniſhing Lines Aa, Bb, Cc, meet in any Point z, draw xz cutting ef in v, and find y the Reflection of z[h]; and the Reflections αa, $b\beta$, $c\gamma$, of the propoſed Vaniſhing Lines will paſs through y.

For z being a Point common to the propoſed Vaniſhing Lines, y the Reflection of z, muſt be a Point common to their Reflections.

C O R.

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[a] Meth. 3. Prob. 29.

[b] Meth. 1.

[c] Cor. 4. Prob. 29.

[d] Cor. 1. Theor. 15. B. I.

[e] Cor. 3. Theor. 16. B. I.

[f] Art. 13.

[g] Cor. 3. Prop. 20. B. IV.

[h] Prob. 26.

Plate 78. Book 5.
Sect. 3.

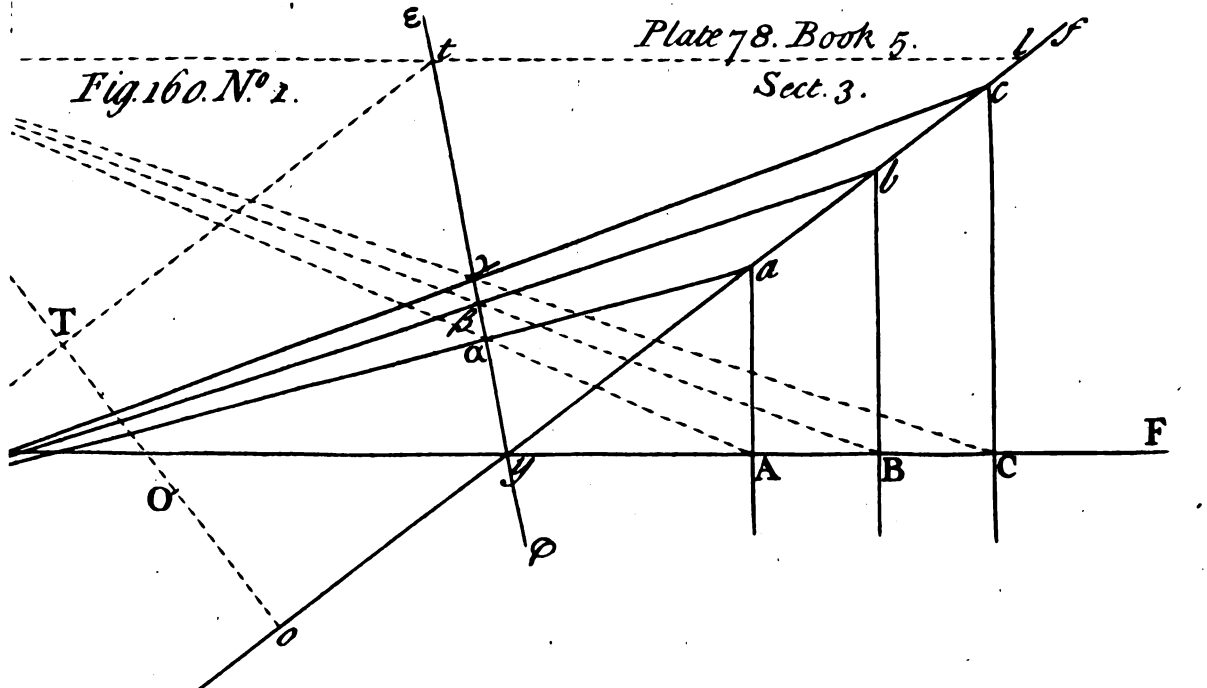
Fig. 160. N.° 1.

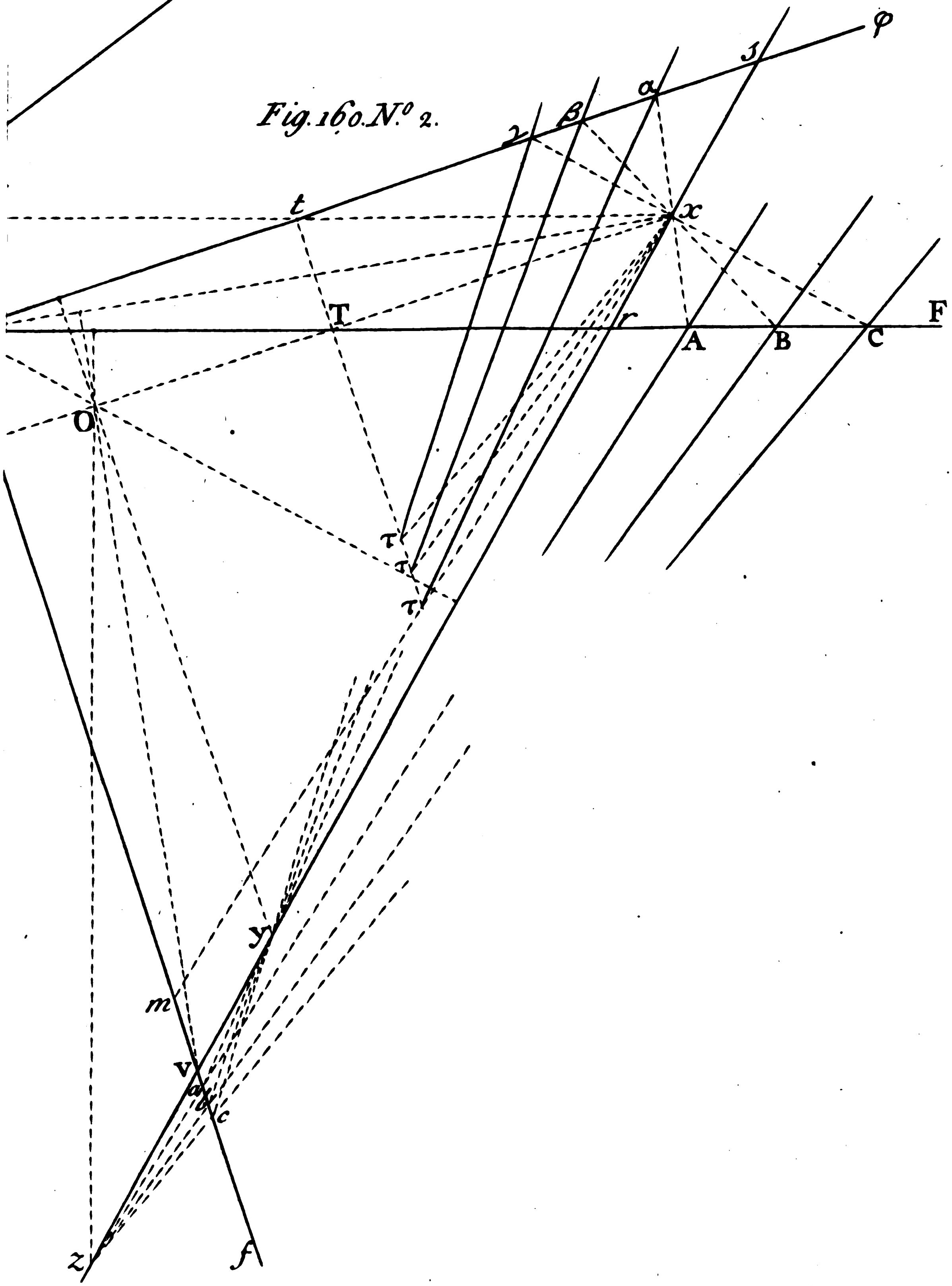
Fig. 160. N.° 2.

J. Mynde sc.

COR. 5.

If z be the Vaniſhing Point of Perpendiculars to the Plane EF, y will be the Vaniſhing Point of Perpendiculars to the Reflected Plane $\epsilon\phi$.

For as on this Suppoſition zA, zB, zC, are Vaniſhing Lines of Planes perpendicular to the Plane EF, ſo yα, yβ, yγ, are Vaniſhing Lines of Planes perpendicular to the Plane $\epsilon\phi$[a].

SCHOL.

That the Vaniſhing Point of Perpendiculars to any Reflected Plane, is the Reflection of the Vaniſhing Point of Perpendiculars to the Original Plane, may be likewiſe ſhewn in this manner.

Fig. 160. N°. 2. Let xz be the Vaniſhing Line of Planes perpendicular to ef, EF, and $\epsilon\phi$, the Reflecting, Original, and Reflected Planes, in which Line xz the Vaniſhing Points of Perpendiculars to all thoſe Planes muſt lye[b]; and let x, z and y be thoſe Vaniſhing Points.

Then, becauſe vx is Harmonically divided in v, r, x and s[c], and the Vaniſhing Points v and x being perpendicular[d], the Vaniſhing Points r and s ſubtend equal Angles with x[e]; and becauſe z and y are the Vaniſhing Points of Perpendiculars to the Original and Reflected Planes, the Vaniſhing Points z and r, and the Vaniſhing Points y and s are alſo perpendiculat; wherefore the Angles ſubtended by sx and yv are equal, as are alſo thoſe ſubtended by xr and vz, and conſequently the Angles ſubtended by yv and vz are equal; xz is therefore Harmonically divided in x, y, v and z[f], and y is therefore the Reflection of z[g].

COR. 6.

Fig. 160. N°. 3. If z biſect xv, the Point y will be infinitely diſtant, and the Reflections $a\alpha$, $b\beta$, $c\gamma$, of the Vaniſhing Lines zA, zB, zC, will be parallel to xv; and if z be alſo the Vaniſhing Point of Perpendiculars to the Plane EF, the Reflections of zA, zB and zC being parallel, and in this Caſe, repreſenting Vaniſhing Lines of Planes perpendicular to the Reflected Plane $\epsilon\phi$[h], they muſt be perpendicular to $\epsilon\phi$, and $\epsilon\phi$ muſt therefore paſs through O the Center of the Picture[i].

CASE 2.

When the Reflecting Plane is perpendicular to the Picture.

Fig. 160. N°. 4. Here, the Reflected Vaniſhing Lines $a\alpha$, $b\beta$, $c\gamma$, make the ſame Angles with $\epsilon\phi$, as Aa, Bb, and Cc, make with EF, but the contrary way.

Dem. For every Vaniſhing Line and its Reflection making equal Angles with ef[k], the propoſed Vaniſhing Lines make with each other Angles equal to thoſe made by their reſpective Reflections. *Q. E. I.*

CASE 3.

When the Reflecting Plane is parallel to the Picture.

Here likewiſe, the Reflected Vaniſhing Lines make the ſame Angles with $\epsilon\phi$, as their Originals make with EF; every Vaniſhing Line being parallel to its Reflection, and at an equal Diſtance with it from O the Center of the Picture[l]. *Q. E. I.*

GENERAL COROLLARY.

By the help of this Propoſition, if any one principal Original Plane and its Reflection be given, the Reflection of any other Plane may be found by its Relation to the given Plane; which may likewiſe be performed without the Image of the required Plane, if only its Interſection with the Principal Plane and its Inclination to that Plane be known; for if the Reflection of that Interſection be found in the Principal Reflected Plane, and thence the Vaniſhing Line of a Plane inclining to the Reflected Plane in the ſame Angle as the propoſed Plane doth to the Principal Original Plane[m], that will give the Reflection of the Plane required[n].

PROB. XXXV.

The Center and Diſtance of the Picture, and the Vaniſhing Lines of an Original Plane, and of its Reflection, being given; thence to find the Vaniſhing Line of the Reflecting Plane.

[a] Art. 13.

[b] Prop. 22. and Cor. 7. B. IV.

[c] Cor. Meth. 1. Prob. 29.

[d] Cor. 4. Prop. 20. B. IV.

[e] Cor. Lem. 7. B. III.

[f] Cor. Lem. 7. B. III.

[g] Prob. 26.

[h] Cor. 3. and 5.

[i] Cor. 3. Theor. 16. B. I. and Cor. 5. Prop. 20. B. IV.

[k] Prob. 31.

[l] Prob. 33.

[m] Prop. 25. B. IV.

[n] Art. 12.

CASE 1.

When the given Vaniſhing Lines interſect.

Fig. 160. N°. 2. Let O be the Center of the Picture, EF and $\varepsilon\varphi$ the given Vaniſhing Lines, and y their Interſection.

Having found zs the Vaniſhing Line of Planes perpendicular to the Vaniſhing Point y[a], cutting EF and $\varepsilon\varphi$ in r and s, biſect the Angle ſubtended by rs in x, and draw xy, and through y draw the Vaniſhing Line ef of Planes perpendicular to the Point x; then ef and yx will each of them be a Vaniſhing Line of a Reflecting Plane by which the Vaniſhing Line EF will be Reflected into $\varepsilon\varphi$.

Dem. For the Vaniſhing Points r and s ſubtending equal Angles with x, the Planes EF and $\varepsilon\varphi$ incline in equal Angles to the Plane yx[b]; and in regard the Plane yx is perpendicular to the Plane ef, the Planes EF and $\varepsilon\varphi$ alſo incline in equal Angles to the Plane ef, the Angles repreſented by φye and Fyf being each the Complement of the Angle $xy\varphi$ or its equal xyF to a Right Angle, and conſequently either yx or ef will anſwer the Problem. *Q. E. I.*

COR.

The Planes ef and yx which anſwer the Problem, are always perpendicular to each other.

CASE 2.

When the given Vaniſhing Lines are parallel.

If the given Vaniſhing Lines EF and $\varepsilon\varphi$ be parallel, the Line zs then becomes their common Vertical Line[c], in which the Points r, s, x, and v are to be found; but this makes no other Difference in the Practice or Demonſtration. *Q. E I.*

In theſe Figures, the Interſecting Lines of the propoſed Planes are not drawn, they no wiſe affecting the Demonſtrations of this or the preceeding Problem.

PROB. XXXVI.

The Center and Diſtance of the Picture, and a Reflecting and Original Plane, together with the Reflected Plane, being given, cutting each other in a Line not parallel to the Picture; and the Image of a Point out of the Original Plane, with its Seat on that Plane, being alſo given; thence to find the Reflection of that Point.

CASE 1.

When the Reflecting Plane inclines to the Picture.

Fig. 161. N°. 1. The ſame Letters marking the ſame things as uſual; let a be the propoſed Point, A its Perpendicular Seat, B its Oblique Seat, and C its Parallel Seat on the Original Plane EFGH with reſpect to the Reflecting Plane $efgb$.

METHOD 1.

By the Perpendicular Seat A of the propoſed Point.

Having found y the Vaniſhing Point of Perpendiculars to the Reflected Plane, from A draw Am parallel to EF, cutting Dy in m, and draw mt; and from x the Vaniſhing Point of Perpendiculars to the Reflecting Plane, draw xA cutting mt in a; then ya being drawn, $x\alpha$ will cut it in α the Reflection of a.

Dem. For mt being the Reflection of Am[d], a is the Reflection of A[e]; wherefore ya is the Reflection of the Perpendicular Support Aa of the propoſed Point a[f], and conſequently α is the Reflection of a. *Q. E. I.*

METHOD 2.

By the Oblique Seat B of the propoſed Point.

Having through t drawn Tt parallel to ef, draw $x\tau$ parallel to aB, cutting Tt in τ; and having drawn Bn parallel to EF, cutting Dy in n, draw nt and xB interſecting in b, and τb being drawn, xa will cut it in α the Reflection required.

Dem. For aB being a Line parallel to the Picture, τ is the Vaniſhing Point of its Reflection[g], and b being the Reflection of B, τb is the Reflection of aB, and conſequently α the Reflection of a. *Q. E. I.*

METHOD

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[a] Prop. 21. B. IV.

[b] Prop. 25. B. IV.

[c] Cor. 1. Prop. 22. B. IV.

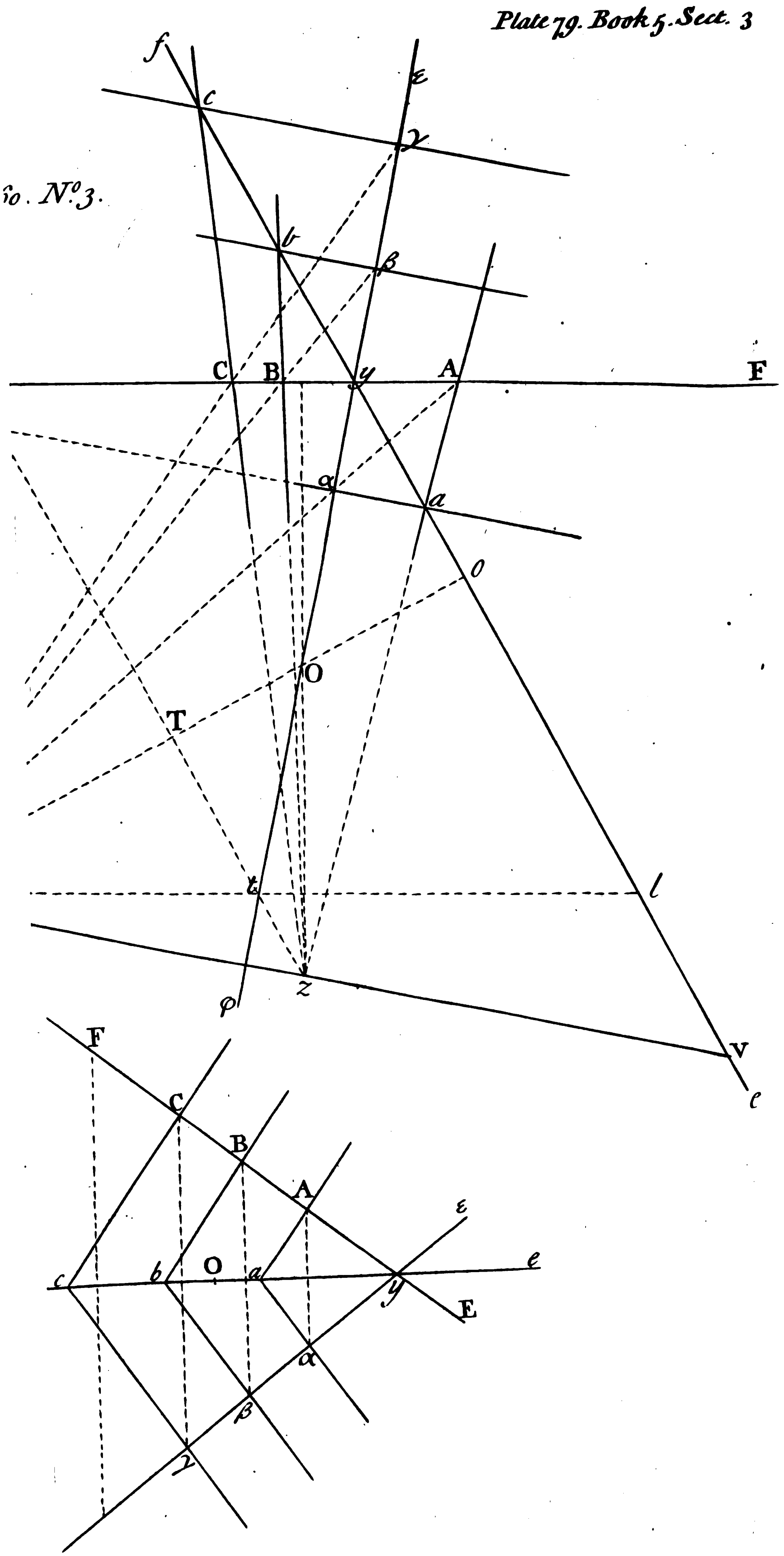
[d] Cor. 4. Prob. 29.

[e] Gen. Cor. 1. Prob. 28.

[f] Cor. 5. Prob. 34.

[g] Cor. 4. and 5. Prob. 26.

io. N°. 3.



J Mynde sc.

METHOD 3.

By the Parallel Seat C of the propoſed Point.

Having drawn Cn and nt as before, draw xC cutting nt in c; and $c\alpha$ drawn parallel to ef will be cut by $x\alpha$ in α the Reflection ſought.

Dem. For aC being a Line parallel both to the Picture and to the Reflecting Plane, it is parallel to its Reflection[a], and c being the Reflection of C, $c\alpha$ is the Reflection of Ca, and α the Reflection of a. *Q. E. I.*

COR. 1.

Hence the Reflection of the Parallel Support of any Point on the Original Plane with reſpect to the Reflecting Plane, is alſo the Parallel Support of the Reflected Point on the Reflected Plane with reſpect to the Reflecting Plane; and is therefore more convenient for Practice than the other Supports, the Vaniſhing Points of the Reflections of which may often be at an inconvenient Diſtance.

COR. 2.

The Original of the Oblique Support of the Reflected Point α on the Reflected Plane, is a Line drawn through the propoſed Point a to a Vaniſhing Point in Tt, where it is cut by a Perpendicular to $\varepsilon\varphi$ drawn from x.

For a Line drawn from x parallel to the required Reflected Support (which in this Caſe is to be perpendicular to $\varepsilon\varphi$) will cut Tt in the Vaniſhing Point of the Original Line which produces that Reflection[b].

COR. 3.

If the Original Plane be perpendicular to the Reflecting Plane, but not to the Picture; the Original and Reflected Planes being then the ſame[c], the Point y coincides with z; but this makes no other Difference in the Practice.

COR. 4.

Fig. 161. N°. 2.

If the Original Plane be perpendicular to the Picture, but not to the Reflecting Plane; the Point z being infinitely diſtant, the Point y biſects xv, and therefore coincides with τ, which is then the Vaniſhing Point of the Reflection of the Perpendicular as well as Oblique Support of a on the Original Plane, which are here the ſame, the Point A in this Caſe coinciding with B.

COR. 5.

Fig. 161. N°. 3.

If the Original Plane be perpendicular both to the Picture and to the Reflecting Plane; the Points z and y are both infinitely diſtant; and the Vaniſhing Lines ef and EF being then perpendicular, the Point x falls in EF, and the Point t coincides with w, the Interſection of tT with EF; the Points A and C coincide in B, and the Reflection of aB is parallel to it.

SCHOL.

Fig. 161. N°. 1.

Although the Lines Am, Bn, in the Original Plane, which paſs through the Seats of the propoſed Point, are directed to be drawn parallel to EF, yet any other Lines paſſing through A, B, or C may be uſed; for their Reflections being found, and in them the Reflection of A, B, or C, the Indefinite Reflection of the propoſed Support may be thence found in the ſame manner as before.

Thus if w be taken as the Vaniſhing Point of the Lines which paſs through A, B, and C, the Reflections of thoſe Lines will be parallel to $\varepsilon\varphi$[d]; or if the Lines through A, B, and C, be drawn parallel to Dy, their Reflections will alſo be parallel to Dy[e]; either of which, or any others may be uſed as may be moſt convenient, according to the Situation of the given Planes.

CASE 2.

When the Reflecting Plane is perpendicular to the Picture.

Fig. 161. N°. 4.

In this Caſe, x being infinitely diſtant, the Line Tt, and the Points depending on it, are alſo infinitely diſtant; the Reflections nb and ma of nB and mA are parallel to $\varepsilon\varphi$, and nb, nc, ma, are equal reſpectively to nB, nC, and mA[f]; zy is perpendicular to ef and biſected in v, and the Reflections of the Perpendicular, Oblique, and Parallel Supports of the propoſed Point a on the Original Plane, are the Perpendicular, Oblique, and Parallel Supports of the Reflected Point α on the Reflected Plane; and

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[a] Cor. 6. Prob. 26.

[b] Cor. 3. and 5. Prob. 26.

[c] Art. 11.

[d] Cor. 5. Prob. 29.

[e] Cor. 7. Prob. 29.

[f] Cor. 1. Caſe 1. Prob. 31.

and the Oblique and Parallel Supports of the propoſed Point, are reſpectively equal to their Reflections.

Dem. For the Triangle aBC being in a Plane parallel to the Picture, its Reflection abc is in the ſame Plane[a], and conſequently ſimilar and equal to it[b]. *Q. E. I.*

[a] Cor. 1. Prob. 31.

[b] Art. 8.

COR. 1.

If the Original Plane be perpendicular to the Reflecting Plane, but not to the Picture; the Point y will coincide with z[c], and the Vaniſhing Lines ef and EF being then perpendicular, the Point C coincides with B, and the Reflection of aB is parallel to it.

[c] Cor. 3. Caſe 1.

COR. 2.

If the Original Plane be perpendicular to the Picture, but not to the Reflecting Plane; the Point y being then in O the Center of the Picture, the Points z and y are both infinitely diſtant, the Point A coincides with B, and the Reflection of aB will be perpendicular to $\epsilon\varphi$.

COR. 3.

If the Original Plane be perpendicular both to the Picture and to the Reflecting Plane; the Vaniſhing Lines ef and EF interſecting in O, and being perpendicular, the Points A and C coincide in B, and the Reflection of aB is parallel and equal to it, $\epsilon\varphi$ coinciding with EF.

PROB. XXXVII.

The Center and Diſtance of the Picture, and a Reflecting and Original Plane, together with the Reflected Plane, being given, cutting each other in a Line parallel to the Picture; and the Image of a Point out of the Original Plane, with its Seat on that Plane, being alſo given; thence to find the Reflection of that Point.

CASE 1.

When the Reflecting Plane inclines to the Picture.

Fig. 162. N°. 1. Let a be the given Point, A its Perpendicular Seat, B its Oblique Seat, and C its Parallel Seat on the Original Plane EFGH with reſpect to the Reflecting Plane $efgh$.

SCHOL.

Here, the Point C is not ſuch a Parallel Seat as that deſcribed at Def. 15. Book IV. ſeeing a Line drawn from the given Point a, parallel to ef, will alſo be parallel to the Plane EFGH, ſo that in this Poſition of the given Planes no ſuch Parallel Seat can be; but the Point C is here the Interſection of the Original Plane EFGH with a Line from the given Point a, parallel to the Plane $efgh$, and alſo to the common Vertical Plane Oo of the given Planes, and which therefore has o the Center of the Vaniſhing Line ef for its Vaniſhing Point.

METHOD 1.

By the Perpendicular Seat A of the propoſed Point.

From A to w the Center of the Vaniſhing Line EF, draw Aw, cutting MM the Interſection of the given Planes in m, and from v the Center of the Reflected Vaniſhing Line $\epsilon\varphi$, draw vm the Reflection of wA[d], and in it find a the Reflection of A; then from y the Vaniſhing Point of Perpendiculars to the Reflected Plane, draw ya, which will be cut by xa in α the Reflection of a.

Dem. For ya is the Reflection of zA[e]. *Q. E. I.*

[d] Prob. 30.

[e] Cor. 5. Prob. 34.

METHOD 2.

By the Oblique Seat B of the propoſed Point.

From B draw Bw, and having found its Indefinite Reflection vm, and in it b the Reflection of B as before, biſect the Vertical Line xo of the Reflecting Plane in T, and draw Tb which will cut xa in the ſame Point a.

Dem. For Tb is the Reflection of Ba[f]. *Q. E. I.*

[f] Cor. 4. and 5. Prob. 26.

METHOD 3.

By the Parallel Seat C of the propoſed Point.

The Vaniſhing Point of the Parallel Support aC of the propoſed Point a here uſed, being q.161 N°.3.

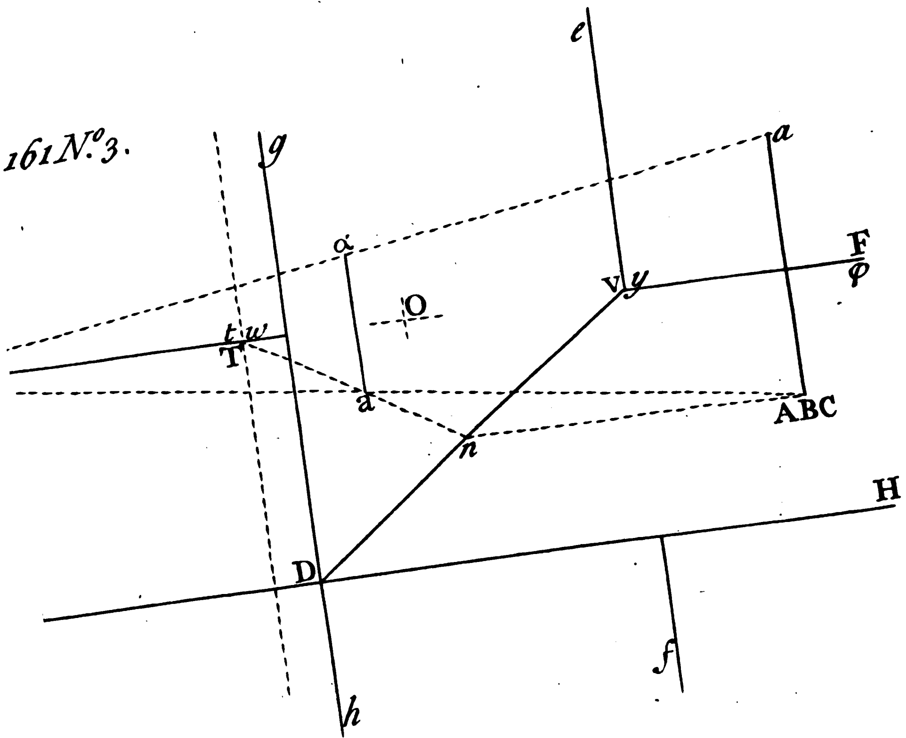
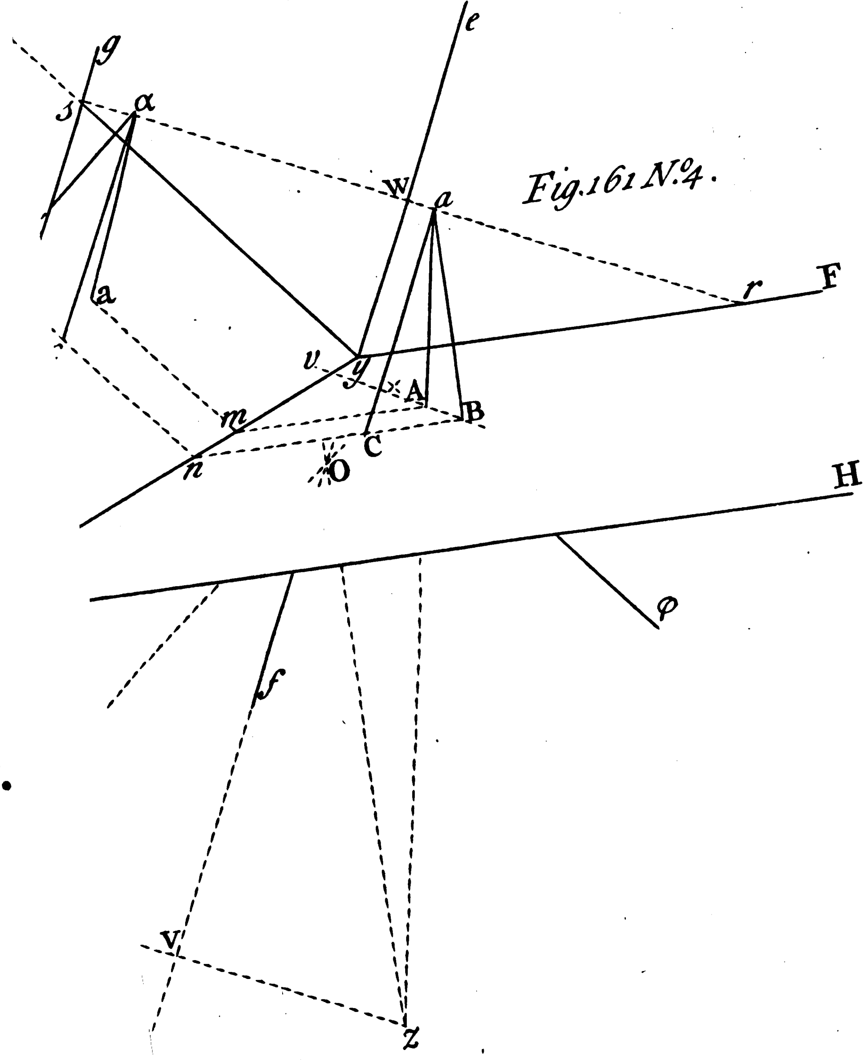


Fig.161 N°.4.



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being at *o* the Center of the Vaniſhing Line *ef*[a], the ſame Point *o* is alſo the Vaniſhing Point of its Reflection[b]; having therefore from C drawn C *w*, and in its Reflection *vm*[c], found *c* the Reflection of C, a Line *oc* will cut *xa* in the ſame Point *α* as before.

Dem. For *oc* is the Reflection of *o*C. *Q. E. I.*

SCHOL.

And here, the Points A, B, and C are all in the ſame Line B*w*; for B*w* is the Interſection of the Original Plane EFGH with the Plane *ow*B*a* paſſing through the propoſed Point *a*, parallel to the Vertical Plane, in which Line B*w* all the Seats of the Point *a* on the Original Plane muſt lie[d].

COR. 1.

If the Original Plane be perpendicular to the Reflecting Plane, the Points *v* and *w* coinciding in *x*, *z* and y will coincide in *o*, the Point A will coincide with C, and conſequently a with *c*; but T which biſects *xo* will ſtill be the Vaniſhing Point of the Reflection of B*a*.

COR. 2.

If the Original Plane be perpendicular to the Picture, the Point *z* being then infinitely diſtant, y will biſect *ox* and coincide with T, the Point A will coincide with B, and conſequently a with b, but *o* will ſtill be the Vaniſhing Point of *a*C, and of its Reflection *αc*.

COR. 3.

If the Original Plane be parallel to the Picture, the Point *w* being then infinitely diſtant, *v* will fall in T, through which *εφ* muſt therefore paſs[e]; the Point *z* will coincide with O, and the Oblique Seat of *a* on the Original Plane, will be the ſame with C its Parallel Seat, *o* being the Vaniſhing Point of its Support.

COR. 4.

If the Vaniſhing Line of the Original Plane paſs through T, the Reflected Plane will be parallel to the Picture[f], and the Point y will coincide with O; and *v* being infinitely diſtant, *vm* muſt be drawn through *m* parallel to *xo*; but T will continue the Vaniſhing Point of the Reflection of *a*B, which Reflection will therefore be parallel to the Original Plane.

COR. 5.

If the Original Plane be parallel to the Reflecting Plane, the Reflected Plane being alſo parallel to them, the Points *v* and *w* will coincide in *o*, and *z* and y in *x*. And as in this Caſe, a Line from *x* to the propoſed Point coincides with its Perpendicular Support on the Original Plane, the Reflection of the propoſed Point (when its Perpendicular Seat is only given) muſt be found by the Interſection of its Support with the Reflecting Plane[g]; but when the Oblique Support is given, the Reflection of *a* may be found as in the other Caſes[h], T being the Vaniſhing Point of the Reflection of *a*B. But here, the propoſed Point can have no Parallel Seat on the Original Plane, ſeeing a Line from that Point parallel to the Reflecting Plane is alſo parallel to the Original Plane.

CASE 2.

When the Reflecting Plane is perpendicular to the Picture.

Here, the Points *x* and T being infinitely diſtant, Bb, Aa, and C*c* are drawn parallel to the Vertical Line *z*y, and the Oblique Support B*a*, and its Reflection b*α* are equal, and in the ſame ſtraight Line: as to the reſt, the Practice is the ſame as before. *Q. E. I.* Fig. 162. N°. 2.

COR. 1.

If the Original Plane be perpendicular to the Reflecting Plane, it being then parallel to the Picture, and coinciding with the Reflected Plane, the Points *z* and y coincide in O, and *v* and *w* are infinitely diſtant; the Seats A and B coincide in C, C*c* perpendicular to *ef* is biſected by MM, and *aα* parallel to C*c* cuts O*c* in *α* the Reflection of *a*.

COR. 2.

If the Original Plane be parallel to the Reflecting Plane, the Points *v* and *w* coincide in O, and *z* and y are infinitely diſtant; the Point A coincides with B, and b*α* the Reflection of B*a* is found as before.

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[a] Schol. of this Prob.

[b] Art. 3.

[c] Prob. 30.

[d] Prop. 3. B. IV. and laſt Schol.

[e] Gen. Cor. 2. Prob. 30.

[f] Gen. Cor. 1. Prob. 30.

[g] Cor. 7. Prob. 26.

[h] Caſe 3. Prob. 30.

CASE 3.

When the Reflecting Plane is parallel to the Picture.

Fig. 162. N°. 3. Here, o being infinitely diſtant, vw is biſected in O with which x coincides; a and b are found in mv the Reflection of Aw[a] by O A and O B; bα is parallel to Ba, and y a is drawn from y the Vaniſhing Point of Perpendiculars to the Reflected Plane, and both are cut by Oa in α the Reflection of a, and the Seat C coincides with B. *Q. E. I.*

COR. 1.

If the Original Plane be perpendicular to the Reflecting Plane, and conſequently to the Picture; the Points w and v coincide in O, and z and y are infinitely diſtant; the Seat A coincides with B, and Bm and mb make one continued ſtraight Line, whoſe Vaniſhing Point is O, and repreſent equal Lines; and bα parallel to Ba is cut by Oa in α, the Reflection ſought.

COR. 2.

If the Original Plane be parallel to the Reflecting Plane, and conſequently to the Picture; the Reflected Plane being alſo parallel to the Picture[b], the Points z and y coincide in O, and w and v are infinitely diſtant; the Perpendicular Support of the propoſed Point on the Original Plane is perpendicular to the Picture, and coincides with its Reflection, and the Reflection of any Point in that Support is found as before ſhewn[c].

But in this Caſe, it may be more convenient to make uſe of ſome ſubſtituted Plane perpendicular to the Picture, on which the Seat of the propoſed Point being found, its Reflection may be thence more readily determined[d].

PROB. XXXVIII.

The Center and Diſtance of the Picture, and a Point with its Seat on an Original Plane, being given; thence to find a Reflecting Plane, by which the propoſed Point may be reflected into any aſſigned Point, whoſe Seat on the Original Plane is alſo given.

Fig. 163. Let O be the Center of the Picture, EFGH the Original Plane, a the propoſed Point, and B its Oblique Seat on that Plane; and let it be required to find a Reflecting Plane, by which the propoſed Point a may be reflected into the Point α, whoſe Seat on the Plane EFGH is b.

Draw Bb cutting EF in q, and draw qx parallel to aB, cutting $a\alpha$ in x; find ef the Vaniſhing Line of Planes perpendicular to x, cutting EF in y, and having in xa found a Point p, ſo that pa and $p\alpha$ may repreſent equal Lines, find P the Oblique Seat of p on the Line Bq; then draw Pπ parallel to EF, cutting $p\pi$ parallel to ef in π, and draw $y\pi$ cutting GH in D, through which gb being drawn parallel to ef, $efgb$ will be the Reflecting Plane required.

Dem. For x being the Vaniſhing Point of $a\alpha$, it is the Vaniſhing Point of Perpendiculars to the Reflecting Plane, ef is therefore the Vaniſhing Line of that Plane; and pa and $p\alpha$ repreſenting equal Lines, p is the Interſection of $a\alpha$ with that Plane, and πp being parallel to ef, is therefore a Line in the Reflecting Plane, and conſequently π is a Point in the Interſection of that Plane with the Original Plane; Dy is therefore that Interſection, and conſequently $efgb$ is the Reflecting Plane ſought. *Q. E. I.*

COR.

The Plane $efgb$ is the only Reflecting Plane which can ſatisfy the Problem. For no other Plane can be perpendicular to $a\alpha$, and paſs through the Point p.

PROB. XXXIX.

The Center and Diſtance of the Picture, and a Reflecting and Original Plane, together with the Reflected Plane, being given; and the Image of a Line out of the Original Plane, with its Seat on that Plane, being alſo given; thence to find the Reflection of that Line.

This may be done, by finding the Reflection of the Vaniſhing Line of the Plane of the Seat of the propoſed Line on the Original Plane[e], whence the Reflection of the Vaniſhing Point of that Line may be had, which, with the Reflection of any other Point of the propoſed Line[f] will give its entire Reflection. *Q. E. I.*

STEREO-

[a] Cor. 1. Prob. 33.

[b] Caſe 3. Prob. 33.

[c] Cor. 7. and 8. Prob. 26.

[d] Caſe 3. Prob. 33.

[e] Prob. 34.

[f] Prob. 36. and 37.

Plate 84. Book 6 Sect. 1.

ig. 164.

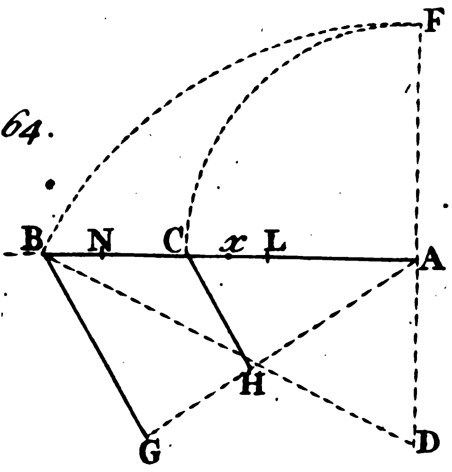
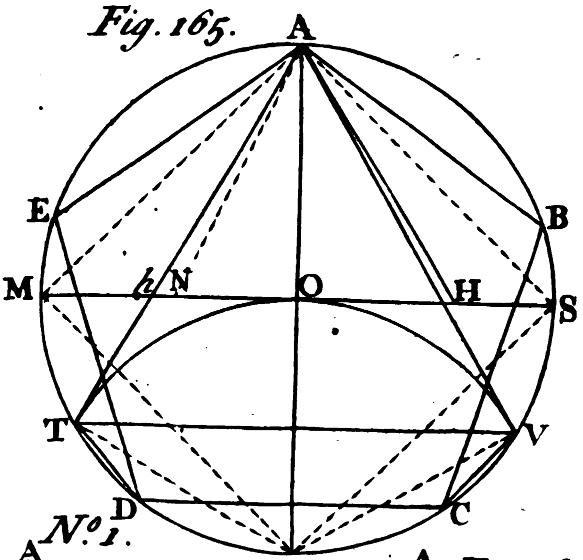


Fig. 165.



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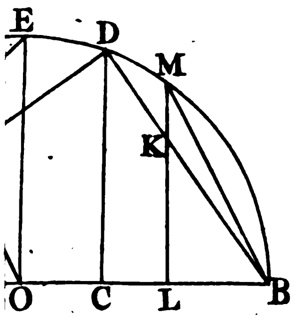
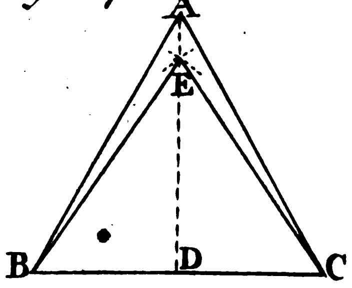


Fig. 167. No. 1.



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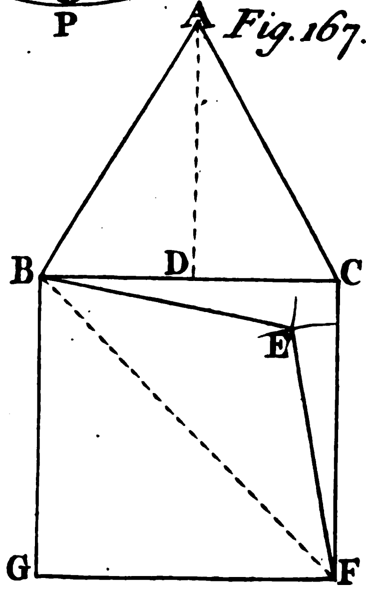

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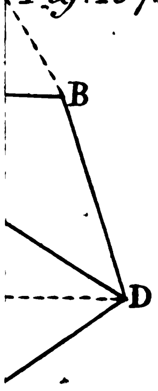

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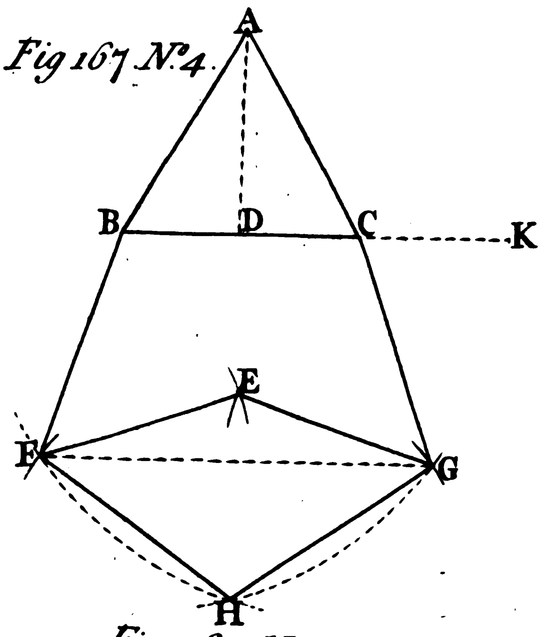


No. 1.

Fig. 168. No. 3.

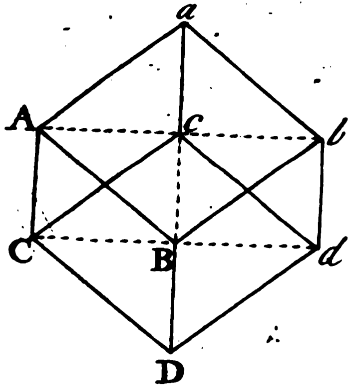
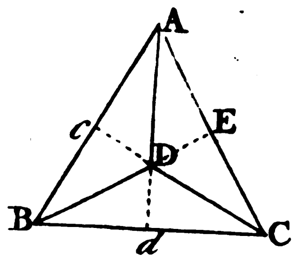


Fig. 169. No. 1.



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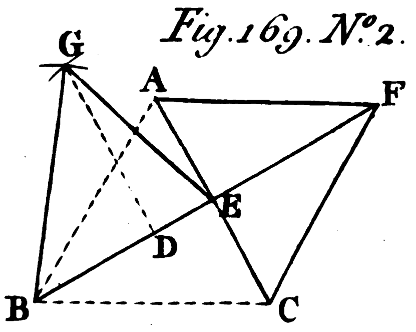


J. Mynde sc.

STEREOGRAPHY,
OR A
COMPLEAT BODY
OF
PERSPECTIVE,
In all its BRANCHES.

BOOK VI.
Of SOLID BODIES.

HAVING hitherto treated only of Lines, Planes, and Plain Figures, we ſhall now proceed to the Conſideration of ſolid Bodies, to the Deſcription of which the former Part of this Work is a full Introduction, it containing the Elements and Foundation of the whole Art of *Stereograpby*; and the Methods to be propoſed for deſcribing ſolid Bodies, will follow ſo naturally from what has been already taught, that the Problems to be ſolved, will appear only as ſo many Examples of putting in Practice the Rules already laid down, in ſeveral Circumſtances.

All ſolid Bodies are either contained within Plain Surfaces, or Surfaces partly Plain, and partly Curvilinear; or laſtly, within Surfaces wholly Curvilinear or otherwiſe irregular and uneven; of each of which we ſhall treat in their Order: and firſt begin with the *five Regular Solids*, in regard they afford ſo great a Variety in the Poſition of the Planes of their Faces to each other, that the Methods of deſcribing them will furniſh ſufficient Rules for the Deſcription of any other Bodies of this Claſs.

SECTION I.
Of the five Regular Solids.

DEF.

IF a Line A B be unequally divided in the Point C, in ſuch manner, that the larger Segment A C may be a mean Proportional between the whole Line A B and the leſſer Segment C B; then the Line A B is ſaid to be *divided in extreme and mean Proportion*. Fig. 164.

LEM. 1.

To divide a given Line AB in extreme and mean Proportion. Fig. 164.

Through either Extremity A of the given Line, draw F D perpendicular to it, and having biſected A B in *x*, take A D equal to A *x*, and from D as a Center with the *Radius* D B, deſcribe an Arch of a Circle cutting D F in F; then from A as a Center with the *Radius* A F, deſcribe another Arch cutting A B in C, and C will be the Point ſought [a]. *Q. E. I.*

COR. 1.

If from B and C any two parallel Lines B G and C H be drawn, and terminated in

[a] 11 El. 2.

in G and H by any Line drawn from A; then BG and CH will be to each other, as the Segments of a Line divided in extreme and mean Proportion.

For in the Similar Triangles ABG, ACH, $BG : CH :: AB : AC$
And by the *Lemma*, $AB : AC :: AC : CB$
Therefore $BG : CH :: AC : CB$.

COR. 2.

If from either Extremity A of the larger Segment CA, a Diſtance AL be ſet off on CA, equal to the ſmaller Segment BC; then CA will be divided by L in extreme and mean Proportion, and AL will be the larger Segment.

For by the *Lemma*, $AC : CB :: AB : AC$
Therefore by Diviſion, $AC : CB = AL :: AB - AC = AL : AC - CB = LC$.

COR. 3.

If to the Line AB there be added a part MB equal to the larger Segment CA; the whole Line AM will be divided by B in extreme and mean Proportion, and AB will be the larger Segment.

For by the *Lemma*, $AC : AB :: CB : AC$
Therefore by Compoſition, $AC + AB = AM : AB :: CB + AC = AB : AC = BM$.

COR. 4.

If either of the Segments AC or BC be given, the other may be found.

If the larger Segment AC be given, and the ſmaller CB required; divide AC in extreme and mean Proportion by the Point L[a], then add to AC a Part CB equal to AL the larger Segment of AC, and CB will be the ſmaller Segment deſired[b].

If the ſmaller Segment BC be given, and the larger AC required; divide BC in extreme and mean Proportion by the Point N, and take CL equal to NC the larger Segment of BC, then BL will be divided in extreme and mean Proportion in C, and BC will be its larger Segment[c]; then to BL add a Part LA equal to BC, and BA will be divided in extreme and mean Proportion by L, and LA and BC being equal, BA is therefore alſo divided in the ſame Proportion by C, wherefore CA is the Segment deſired.

LEM. 2.

Fig. 165. A Circle AMPS being given, therein to inſcribe any of the Regular Polygons.

1. To inſcribe an equilateral Triangle.

Draw any Diameter AP, and from either of its Extremities P as a Center, with the *Radius* PO, deſcribe an Arch, cutting the Circle in T and V; then ATV will be the Triangle deſired.

For TP and PV are two Sides of a Hexagon[d]. *Q. E. I.*

2. To inſcribe a Square.

Draw the Diameter MS perpendicular to AP, and join the Extremities A, M, P, and S. *Q. E. I.*

3. To inſcribe a Pentagon, Hexagon, and Decagon.

Biſect the Semidiameter MO in N, and ſet off from N the Diſtance NA at H in the Diameter MS, and draw AH; then AH will be equal to the Side of the Pentagon, AO to the Side of the Hexagon, and OH to the Side of the Decagon; and theſe Lengths being reſpectively ſet off round the Circumference of the Circle, will give the Angles of the Figures deſired.

For AO and NH being perpendicular, and NO being the half of OA, NH taken equal to NA gives OH equal to the larger Segment of OA divided in extreme and mean Proportion[e]; OS equal to OA is therefore divided in that Proportion by H; and if O*b* be taken equal to OH, *b*S will be divided in the ſame Proportion, and OS will be the larger Segment[f]; but OS is the Side of a Hexagon inſcribed in the Circle[g]; therefore O*b* or its equal OH is the Side of a Decagon[h]; now in the Rectangular Triangle AOH, the Square of AH is equal to the Squares of AO or SO and OH taken together[i], and SO being the Side of the Hexagon, and OH the Side of the Decagon, AH is therefore the Side of the Pentagon inſcribed in the ſame Circle[k]. *Q. E. I.*

4. To inſcribe an Octagon.

Biſect each of the Arches of the Square. *Q. E. I.*

5. To inſcribe a Dodecagon.

Biſect each of the Arches of the Hexagon. *Q. E. I.*

6. To

[a] Lem. and Cor. 2.
[b] Cor. 3.
[c] Cor. 3.
[d] 15 El. 4.
[e] Lem. 1.
[f] Cor. 3. Lem. 1.
[g] 15 El. 4.
[h] 9 El. 13.
[i] 47 El. 1.
[k] 10 El. 13.

6. To inſcribe a Quindecagon.

Having inſcribed the Pentagon ABCDE, and the equilateral Triangle ATV, having each one Angle at A, DT or CV will then be the Side of a Quindecagon[a]. *Q. E. I.*

LEM. 3.

The Diameter AB of a Sphere being given; thence to find the Sides of the five Regular Solids inſcribed in that Sphere. Fig. 166.

On AB deſcribe the Semicircle AEB, and having taken CB equal to one third Part of AB, erect the Perpendicular CD, and the *Radius* OE parallel to it, and draw AE, AD, and BD; from A erect AF perpendicular and equal to AB, and draw OF, cutting the Semicircle in G, from whence let fall the Perpendicular GH, and draw AG; laſtly, divide BD in extreme and mean Proportion by the Point K[b].

1. Then AD will be the Side of a Pyramid or Tetraedron; and $2AB^2 = 3AD^2$.
2. BD is the Side of a Cube or Hexaedron; and $AB^2 = 3BD^2$.
3. AE is the Side of an Octaedron; and $AB^2 = 2AE^2$.
4. BK the greater Segment of BD is the Side of a Dodecaedron; and BK is to BD as the Side of a Pentagon is to its Diagonal.
5. AG is the Side of an Icoſaedron, and alſo the Side of a Pentagon in the ſmaller Circle of the Sphere which ſubtends the ſolid Angle of the Icoſaedron; of which Circle, GH is the *Radius*, and AH is the Side of a Decagon in that Circle; and the Diameter AB of the Sphere is equal to GH and 2AH *Q. E. I.*

If LB be taken equal to one fifth of AB; erect the Perpendicular LM, and draw BM, which will be equal to GH[c].

LEM. 4.

To find the Angles of Inclination of the Planes of the Faces of the five Regular Solids.

1. For the Cube.

The Planes of the Faces are all perpendicular to each other. *Q. E. I.*

2. For the Pyramid or Tetraedron.

Let the equilateral Triangle ABC be one of the Faces. Having from either of the Angles A, drawn a Diameter AD perpendicular to the oppoſite Side BC; from B and C as Centers with the *Radius* AD, deſcribe two Arches interſecting in E, and BEC will be the Angle ſought. *Q. E. I.* Fig. 167. N°. 1.

3. For the Octaedron.

Let ABC be one of the Faces. Having drawn the Diameter AD as before, on any Side BC deſcribe a Square BGFC, and draw its Diagonal BF; then from B and F as Centers with the *Radius* AD, deſcribe two Arches interſecting in E, and the Obtuſe Angle BEF will be the Complement to two Rights of the Angle required, or the Angle contained between any two adjoining Faces within the Solid. Fig. 167. N°. 2.

Or, if an Iſoſceles Triangle be made, having its Sides equal to AD, and its Baſe equal to BC, the Angle at the Vertex of this Triangle will be the Angle deſired; and is the ſame with the Angle of Inclination of the Faces of a Tetraedron[d]; the Angles made by the Faces of the Tetraedron and Octaedron within the Solids, being together equal to two Rights. *Q. E. I.*

4. For the Dodecaedron.

Let the Regular Pentagon ABDFC be one of the Faces. Draw any Diagonal CD, and the Diameter FG perpendicular to it, cutting the Diagonal in H; then from C and D as Centers with the *Radius* HG, deſcribe two Arches interſecting in E, and the Obtuſe Angle CED will be the Complement to two Rights of the Angle ſought. Fig. 167. N°. 3.

Or, if on AB as a Baſe, an Iſoſceles Triangle AIB be drawn, having its Sides AI, BI, equal to GH, the Angle AIB will be the Angle deſired. *Q. E. I.*

5. For the Icoſaedron.

Let ABC be one of the Faces. Having drawn the Diameter AD, on any Side BC deſcribe a Regular Pentagon BCGHF, and draw its Diagonal FG; then from F and G as Centers with the *Radius* AD, deſcribe two Arches interſecting in E, and the Obtuſe Angle FEG will be the Complement to two Rights of the Angle required. Fig. 167. N°. 4.

Or, if an Iſoſceles Triangle be deſcribed, having its Sides equal to AD, and its Baſe equal to the Difference between BC and FG; the Angle at the Vertex of that Triangle will be the Angle ſought[e]. *Q. E. I.*

---

[a] 16 El. 4.

[b] Lem. 1.

[c] 18 El. 13.

[d] Art. 2.

[e] 7 El. 15.

SCHOL.

The Regular Pentagon BCGHF is deſcribed on BC in this manner: Produce BC to K until BC be to CK, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion[a]; then from B and C as Centers with the *Radius* BK, deſcribe two Arches interſecting in H, and from H as a Center with the *Radius* BC, draw two other Arches cutting the former in F and G; and thereby the Angles H, F, and G, and thence the Pentagon BCGHF will be determined[b].

And here, CK is equal to the Difference between BC and FG; BK and FG being equal.

Of the Ichnography and Elevation of the five Regular Solids.

The Ichnography and Elevation may be deſcribed on any two Planes whoſe Situations with reſpect to the propoſed Solid are known; but for the greater Regularity of the Projections, ſuch Planes are generally choſen as are either parallel or perpendicular to ſome Face, Side, or Diameter of the Solid; and for the like Reaſon, the Plane of the Elevation is conſtantly taken perpendicular to the Plane of the Ichnography: both the Ichnography and Elevation are the Perpendicular Seats of the ſeveral Sides and Angles of the propoſed Solid on the reſpective Planes[c], and conſequently either of them being taken as the Ichnography, the other may be the Elevation.

LEM. 5.

To deſcribe the Ichnography and Elevation of a Cube.

CASE 1.

If the Plane of the Ichnography be parallel to one of the Faces of the Cube; the Ichnography is a Square equal to that Face: and the Elevation on a Plane parallel to either of the Sides of that Face, and perpendicular to the Plane of the Ichnography, is alſo a Square equal to it. *Q. E. I.*

Note, *This Ichnography and Elevation of a Cube are the moſt Simple and uſeful.*

CASE 2.

1. To deſcribe the Ichnography of a Cube on a Plane perpendicular to either of its Diagonals.

Fig. 168. N°. 1. Let $AbCd$ repreſent a Cube, ABCD one of its Faces in its true Dimenſions, and BC the Diagonal of that Face; and let Da repreſent the Diagonal of the Solid, to which the Plane of the Ichnography is propoſed to be perpendicular; the Cube being ſuppoſed to reſt with its ſolid Angle D on that Plane.

Fig. 168. N°. 2. Deſcribe an equilateral Triangle Abc having its Sides equal to the Diagonal BC of the Face of the Cube, and circumſcribe this Triangle with a Circle, in which draw a Regular Hexagon $ABbdcC$, having three of its Angles in A, b, and c; and having drawn its Diameters Ad, Bc, Cb, the Hexagon thus divided will be the Ichnography ſought.

And here, $aACc$, $aABb$, and $abdc$, are the Ichnographies of the three upper Faces; and DBAC, $DBbd$, and $DdcC$, are thoſe of the three lower Faces of the Cube: the Diameters of the Hexagon are equal to the Diagonals of the Solid, and the Ichnography of its Diagonal Da is only a Point, ſo that the Ichnographies of the ſolid Angles D and a coincide in the Center of the Hexagon. *Q. E. I.*

2. To deſcribe the Elevation.

The Elevation on a Plane parallel to either of the Sides of the equilateral Triangles, formed by the Diagonals of the three upper or lower Faces of the Cube, which contain its ſolid Angle at a or D, is found in the following manner:

Let the Plane of the Elevation be ſuppoſed parallel to the Side Ab of the equilateral Triangle Abc.

Fig. 168. N°. 3. In the Plane of the Elevation, draw Da perpendicular to the Plane of the Ichnography, and equal to the Diameter of the Hexagon; divide Da into three equal Parts in B and c, through which Points draw Cd and Ab perpendicular to Da, each equal to the Side of the Triangle Abc, and biſected by Da; then having joined the Points D, C, A, a, b, d, and drawn Cc, dc, AB and bB, that will be the Elevation deſired.

And

[a] Cor. 4. Lem. 1.

[b] 10 and 11 El. 4.

[c] Art. 9. and 10. Sect. 3. B. I.

Fig. 170. N°2.

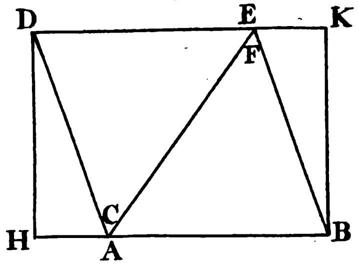


Fig. 170. N°3.

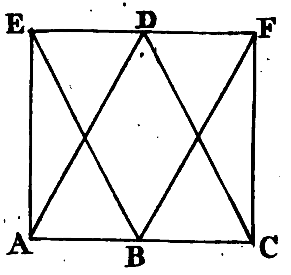


Fig. 170. N°5.

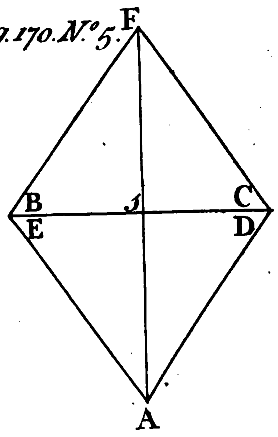
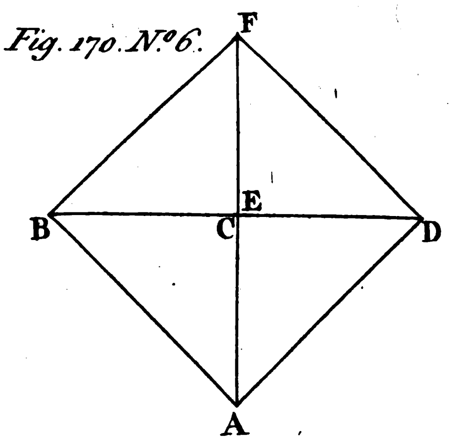


Fig. 170. N°6.



°1.

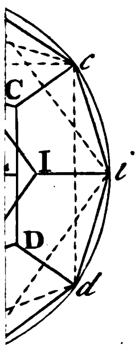
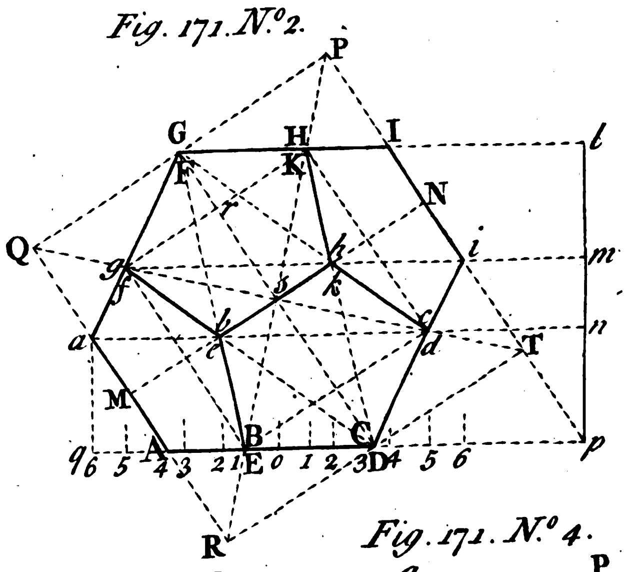


Fig. 171. N°2.



. N°3.

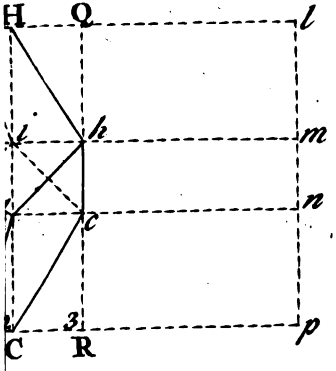
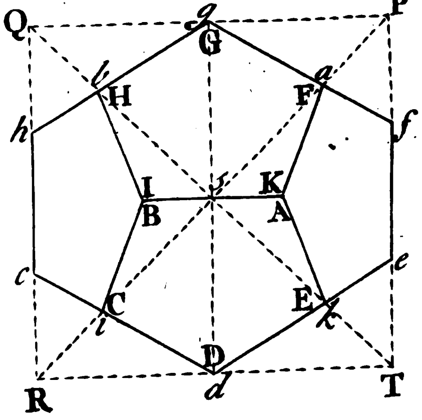


Fig. 171. N°4.



And here, *a*AB*b*, *a*AC*c*, and *abdc*, are the Elevations of the three upper Faces of the Cube, which contain the ſolid Angle at *a*; and DC*cd*, DCAB, and D*db*B, are the Elevations of the three lower Faces which contain the ſolid Angle at D. *Q. E. I.*

S C H O L.

Fig. 168. N°. 2.

If a Circle be inſcribed within the Hexagon AB*bdc*C of the Ichnography, touching its Sides; that Circle will contain a Square equal to the Face of the Cube, the ſame with the Ichnography of the propoſed Cube on a Plane parallel to one of its Faces; and if a Square $\alpha\beta\gamma\delta$ be accordingly inſcribed in that Circle, having two of its Sides parallel to either of the Sides AB of the Hexagon, that Square will fall wholly within it, and neither touch nor cut any of its Sides; whence it appears, that if all the ſolid Part of the Cube which hath $\alpha\beta\gamma\delta$ for its Ichnography, be cut out, it may be done without cutting either of the Sides of the Cube, whoſe Ichnographies form the Sides of the Hexagon, and conſequently without breaking the Cube aſunder; and that therefore ſuch a Paſſage may be cut out of the Solidity of a Cube, without deſtroying its continuity, as to admit another Cube of the ſame Dimenſions to paſs through it.

L E M. 6.

To deſcribe the Ichnography and Elevation of a Tetraedron.

1. To deſcribe the Ichnography.

Fig. 169. N°. 1.

If the Ichnography be deſired on the Plane of one of the Faces; let the equilateral Triangle ABC be the propoſed Face.

Find the Center D, by the Interſection of any two Diameters A*d* and C*c*, and draw BD; and ABCD will be the Ichnography ſought, and D the Seat of the ſolid Angle oppoſite to the given Face ABC. *Q. E. I.*

2. To deſcribe the Elevation.

Fig. 169. N°. 2.

Deſcribe an equilateral Triangle ABC equal to the given Face, and in it draw the Diameter BE; from B as a Center with the *Radius* AB, deſcribe an Arch, and from E as a Center with the *Radius* EB, deſcribe another Arch, cutting the former in G, and draw GD perpendicular to BE; then BEGD will be the Elevation of the Tetraedron on a Plane parallel to the Diameter BE of the given Face ABC, and DG will be the Height of the Vertex above the Center D.

Or, if the Elevation be deſired on a Plane parallel to any Side AC of the given Face; having found the Length of GD as before, produce BE to F, till EF be equal to GD, and draw AF and CF, and ACFE will be the Elevation ſought. *Q. E. I.*

L E M. 7.

To deſcribe the Ichnography and Elevation of an Octaedron.

C A S E 1.

1. To deſcribe the Ichnography.

Fig. 170. N°. 1.

Let ABC be one of the Faces of the Solid; and let the Ichnography be required on the Plane of that Face.

Draw the Regular Hexagon AEBFCD, having three Angles in A, B and C; and the contrary Points D, E, and F being joined, will give the Ichnography deſired.

And here, DEF is the Ichnography of the upper Face of the Solid, which is parallel to its Face ABC, but in a ſubcontrary Poſition; the Triangles AEB, BFC, and CDA, are the Ichnographies of the three lower Faces which adjoin to the Face ABC; and the Triangles DAE, EBF, FCD, are the Ichnographies of the three other Faces adjoining to the Face DEF; and the ſix angular Points of the Hexagon are the Seats of the ſix ſolid Angles of the Octaedron on the Plane of the Face ABC. *Q. E. I.*

2. To deſcribe the Elevation.

Having in the Hexagon, drawn a Diameter DB perpendicular to any Side AC of the given Face, cutting it in G, draw DH parallel to AC, and from G as a Center with the *Radius* BG, deſcribe an Arch cutting DH in H; then in the Plane of the Elevation ', draw a Rectangular Parallelogram DKHB, having its Sides DH, HB, equal to HD and DB in the Ichnography, and from D and B ſet off DE and BA each equal to GB, and draw DA, AE and EB; then DEAB will be the Elevation of

' Fig. 170. N°. 2.

of the Octaedron on a Plane parallel to the Diameter DB, and DH will be the Diſtance between its upper and lower Faces.

Fig. 170. N°. 3. Or, if a Rectangular Parallelogram EFAC be drawn, having its Sides EA and AC equal reſpectively to HD and AC in the Ichnography; biſect EF and AC in D and B, and draw DA, DC, BE, and BF, and that will give the Elevation of the Octaedron on a Plane parallel to AC one of the Sides of the given Face ABC. *Q. E. I.*

COR.

Fig. 170. N°. 1. If a Tetraedron be deſcribed on the Face ABC; the Height DH will be the ſame with the Height of the Vertex of the Tetraedron above that Baſe, and may therefore be found in the ſame manner[a].

SCHOL.

And here, in Fig. N°. 2, 3. the Elevation of every Point and Line of the propoſed Solid, is marked with the ſame Letters as its Ichnography in Fig. N°. 1. and ſuch of the Points as are marked with two Letters, ſhew that the ſame Point is the Elevation of both thoſe to which thoſe Letters relate; the ſame is to be underſtood of all the ſubſequent Figures.

CASE 2.

1. To deſcribe the Ichnography.

If the Ichnography be required on a Plane perpendicular to any one of the Diameters AF of the Octaedron, it is thus found.

Fig. 170. N°. 4. In the Plane of the Ichnography, deſcribe a Square BCED, having its Sides equal to the Side of the propoſed Solid, and draw the Diagonals EC, BD, Interſecting in A, and that will be the Ichnography deſired; in which the Ichnography F of the uppermoſt ſolid Angle coincides with A, over which it is ſuppoſed to be perpendicular. *Q. E. I.*

2. To deſcribe the Elevation.

Fig. 170. N°. 5. Make a Parallelogram FEAD with its Diagonals FA and ED perpendicular to each other, and equal to BD and ED in the Ichnography; and that will be the Elevation of the propoſed Solid on a Plane parallel to its Side BC.

Fig. 170. N°. 6. Or, if a Square FBAD be made equal to the Ichnography; that will be the Elevation of the Octaedron on a Plane parallel to one of its Diameters. *Q. E. I.*

LEM. 8.

To deſcribe the Ichnography and Elevation of a Dodecaedron.

CASE 1.

1. To deſcribe the Ichnography.

Fig. 171. N°. 1. Let the Regular Pentagon ABCDE be a Face of the propoſed Solid, on the Plane of which the Ichnography is deſired.

Having deſcribed another Regular Pentagon FGHIK equal and in a ſubcontrary Poſition to the given Face; from their common Center S, draw a *Radius* SA and produce it to *a*, making SA to A*a*, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion[b]; then with the Center S and *Radius* S*a*, deſcribe a Circle, and having from S through each of the Angles A, G, B, *&c.* of the Pentagons, drawn A*a*, G*g*, B*b*, *&c.* theſe will cut the Circle in the ten Angles of a Decagon inſcribed in it; and which being drawn accordingly, the Ichnography required will be compleated.

And here, the Pentagon FGHIK is the Ichnography of the upper Face of the Solid, which is parallel and in a ſubcontrary Poſition to its Face ABCDE; the Figures A*agb*B, B*bhc*C, C*cid*D, D*dke*E, and E*efa*A, are the Ichnographies of the five lower Faces adjoining to ABCDE; and the Figures F*fag*G, G*gbh*H, H*hci*I, I*idk*K, and K*kef*F, are the Ichnographies of the five remaining Faces which adjoin to FGHIK; and the twenty angular Points of the outward and inward Decagons, are the Seats of the twenty ſolid Angles of the Dodecaedron on the Plane of the Ichnography. *Q. E. I.*

COR. 1.

If the inner Decagon be compleated, by joining the Angles of the Pentagons; the Sides of the outward Decagon will be to thoſe of the inner, as the Segments of a Line divided in extreme and mean Proportion[c]; and if two Regular Pentagons be formed,

by

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[a] Lem. 6.

[b] Cor. 4. Lem. 1.

[c] Cor. 1. Lem. 1.

Plate 86. Book 6. Sect. 1.

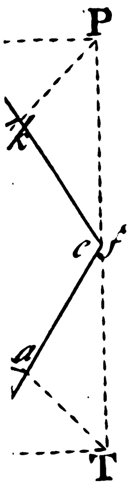


Fig. 171 N°6.

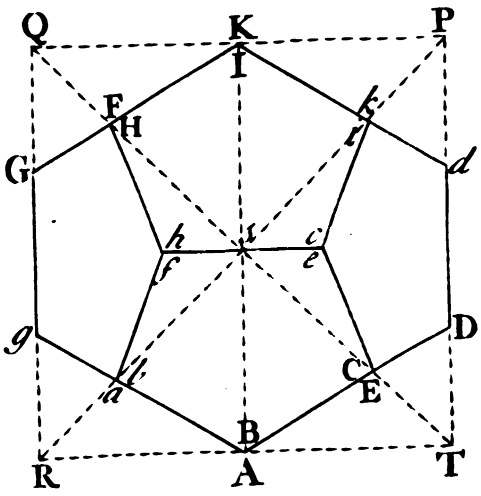
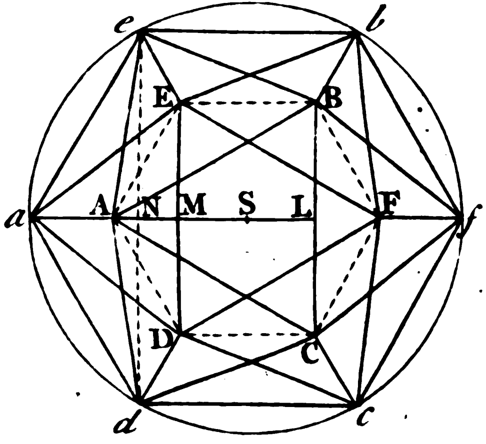


Fig. 172. N°1.



N°2.

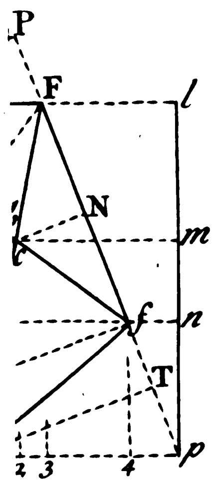


Fig. 172. N°3.

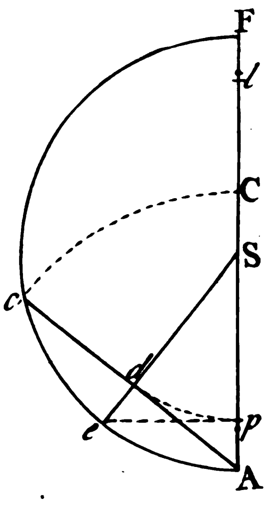


Fig. 172. N°4.

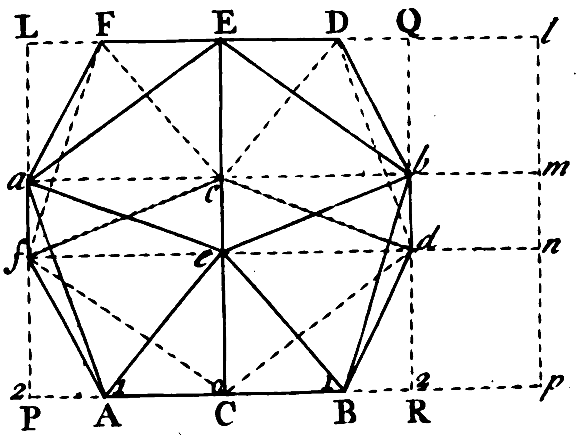


Fig. 172. N°6.

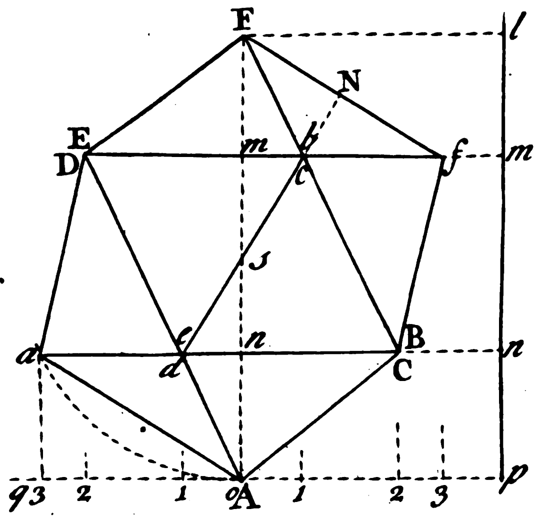
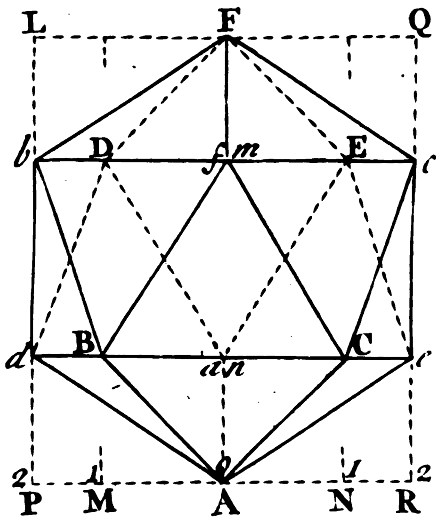


Fig. 172. N°7.



by joining the alternate Angles of the outward Decagon; the Sides of the outward Pentagons will be to thoſe of the inner, in the ſame Proportion; and conſequently the Side of the outward Pentagon will be equal to a Diagonal BE of the inner[a].

COR. 2.

If a Diagonal *gc* be drawn in the outward Decagon, leaving out two of its Angles *b* and *b*, that Diagonal will paſs through B and H two of the Angles of the inner Decagon.

For SA the Side of the Hexagon, is to AG, the Side of the Decagon inſcribed in the Circle which contains the Face ABCDE of the Dodecaedron, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion[b]; wherefore AG and A*a* are equal; if then B*c* and GC be drawn, GB being equal and parallel to C*c*, B*c* and GC will be parallel, but BH is parallel to GC, wherefore H is in the Line B*c*. In the ſame Manner it may be ſhewn that B is in the Line H*g*; and conſequently *gc* paſſes through B and H.

2. To deſcribe the Elevation.

The Elevation of this Ichnography may be deſcribed, either on a Plane parallel to a Diameter AL, or to a Side AB of the given Face ABCDE.

1. The Elevation on a Plane parallel to AL is found in this manner.

METHOD 1.

On the given Face ABCDE draw the Diameter AL, and the Diagonal BE perpendicular to it, cutting it in M.

Then in the Plane of the Elevation, draw any Line A*p*, repreſenting the Interſection of that Plane with the Plane of the given Face, and in it take AC and C*p* each equal to the Diameter AL; on C*p* deſcribe an Iſoſceles Triangle *p*C*i*, having its Baſe *pi* equal to the Diagonal BE; produce *pi* to I, and make *i*I equal to AB the Side of the Pentagon; then draw A*a*, *a*G, GI, parallel and equal reſpectively to I*i*, *i*C, and CA, and make CB, C*c*, GH, and G*g*, each equal to LM, that part of the Diameter AL which is intercepted between the Side DC and the Diagonal BE in the Ichnography, and draw C*g*, CH, GB, and G*c*, interſecting in *e* and *k*; laſtly draw *ek*, which will compleat the Elevation A*a*GI*i*C of the Dodecaedron propoſed. *Q. E. I.* Fig. 171. N°. 2.

COR.

And here, the Angle *i*C*p* is the Angle of Inclination of the Faces of the Dodecaedron to each other; and AI or *ai* is equal to the Diameter of the Sphere which circumſcribes that Solid.

METHOD 2.

The ſame Elevation may be alſo found in this manner:

Decribe two concentrical and parallel Squares QPRT, and *g*HB*c*, the Side of the outward Square being made equal to the Diagonal *eb* of the outward Pentagon, and the Side of the inner Square equal to BE the Diagonal of the inner Pentagon in the Ichnography, and conſequently in proportion to each other as the Segments of a Line divided in extreme and mean Proportion; then through *s* the common Center of the Squares, draw GC parallel to either of their Sides QR, and by the help of G and C thus found, the whole Elevation may be compleated, as in the Figure.

If the Line AC be given, the Poſition of the Squares with reſpect to that Line, may be thus found.

Take CB in AC equal to LM (Fig. N°. 1.) and on CB deſcribe an Iſoſceles Triangle BC*c*, having its Baſe B*c* equal to the Diagonal BE of the inner Pentagon, whereby the Angle BC*c*[c], and the Side B*c* of the inner Square are found; and the Square *g*HB*c* being drawn, its Diagonal HB will cut RT, drawn through C parallel to B*c*, in R, an Angle of the outward Square; whence the entire Elevation may be deſcribed. *Q. E. I.*

COR. 1.

The inner Square *g*HB*c* is equal to the Face of a Cube inſcribed in the ſame Sphere with the Dodecaedron[d].

COR. 2.

The Line *ek* is equal to the Side of the Pentagon ABCDE.

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[a] 8 El. 13. and Cor. 3. Lem. 1.

[b] 9 El. 13.

[c] Lem. 4.

[d] Lem. 3.

For C*s*, half the Side of the outward Square, is to *sr* half the Side of the inner Square, as the larger Segment is to the leſs, of a Line divided in extreme and mean Proportion, that is, $Cs : sr :: Cr : Cs$,

But, becauſe of the ſimilar Triangles, $Cr : Cs :: Cg : Ce :: gH : ek$, therefore *g*H is to *ek*, as the larger Segment to the leſs, of a Line divided in extreme and mean Proportion; and *g*H being equal to the Diagonal of the Pentagon ABCDE, *ek* is therefore equal to its Side.

COR. 3.

Hence, QP, *g*H, and *ek*, or their equals *eb*, EB, and DC, in the Ichnography, are to each other continually, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion.

SCHOL.

Here, the Points *a*, *b*, *c*, *d*, and *e*, which are the Elevations of the five angular Points of the outward Pentagon *abcde* in the Ichnography, are all in the ſame ſtraight Line *ad*; and the Elevations *f*, *g*, *h*, *i*, and *k*, of the Angles of the contrary outward Pentagon *fghik*, are in the ſtraight Line *gi*; both which Lines are parallel to A*p*, the Angles of thoſe two Pentagons lying in two Planes parallel to the given Face ABCDE of the Dodecaedron, and being contained in two parallel Circles, whoſe *Radii* are equal to S*a*.

COR. 4.

Fig. 171. N°. 2. If a Line *pl* be drawn perpendicular to A*p*, and GI, *gi*, and *ad*, be produced till they cut *pl* in *l*, *m*, and *n*; then *pl* will be the Height of the uppermoſt Face, and *pm* and *pn* the Heights of the two Planes of the outward Pentagons, above the given Face ABCDE; and the whole Line *pl* will be divided in extreme and mean Proportion by each of the Points *m* and *n*, and either extreme *ml* or *np* will be to the middle Part *mn*, as the greater Segment to the leſs, of a Line divided in that Proportion.

For the Lines C*g*, CH, GB, and G*c*, are all divided in that Proportion by the Points *e* and *k*[a].

COR. 5.

The Part *pn* or *lm* of the Line *pl*, is equal to SA the *Radius* of the Circle which contains the Face ABCDE of the Dodecaedron; and *pm* or *ln* is equal to S*a* the *Radius* of the Circle which contains the outward Pentagons; and conſequently the whole Line *pl* is equal to I*a* in the Ichnography, SI and SA being equal.

Fig. 171. N°. 2. Produce CA till it be cut in *q* by a perpendicular to it drawn from *a*.

Then A*a* being equal to the Side of the inner Pentagon, and *q* being the Perpendicular Seat of the Point *a* on the Plane of the Ichnography, A*q* is therefore equal to A*a* in the Ichnography, or to the Side of a Decagon inſcribed in the Circle which contains that Pentagon[b]; but in the Rectangular Triangle *aq*A, $Aa^2 = Aq^2 + qa^2$, wherefore A*a* being the Side of the Pentagon, and A*q* the Side of a Decagon, *qa* is the Side of a Hexagon inſcribed in the ſame Circle[c]; and conſequently *qa*, or its equal *pn*, is equal to SA the *Radius* of the Circle which contains the Pentagon ABCDE; and *pm* being to *pn*, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion[d], and S*a* being to SA in the like Proportion, *pm* is therefore equal to S*a* the *Radius* of the Circle which contains the outward Pentagons.

METHOD 3.

The ſame Elevation may be alſo found in this manner.

Fig. 171. N°. 2. Having drawn any Line *qp*, and *pl* perpendicular to it, and equal to I*a* in the Ichnography, and found in it the Points *m* and *n* as before[e]; through *n*, *m*, and *l*, draw *na*, *mg*, *l*G, parallel to *qp*; and having in *qp* taken any Point *o* to repreſent the Seat of the Center S of the Ichnography on the Plane of the Elevation, ſet off on each Side of *o*, the Diviſions *o*1, *o*2, *o*3, *o*4, *o*5, and *o*6, equal reſpectively to SM, SN, SP, SA, SQ, and S*a*, in the Ichnography; then from each of theſe Diviſions draw Perpendiculars to *qp*, which will cut *na*, *mg*, and *l*G, in the reſpective Points *a*, *e*, *d*; *g*, *h*, *i*; and G, H, I; by which the Elevation may be deſcribed, as in the Figure. *Q. E. I.*

Note, the Points 2 *and* 2 *may be omitted, ſeeing the Points e and k may be had without them*[f].

2. The Elevation of the ſame Ichnography on a Plane parallel to either of the Sides AB of the Pentagons, is thus deſcribed.

Fig. 171. N°. 3. Having drawn any Line P*p*, and *pl* perpendicular to it, and the Parallels *na*, *mg*, and

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[a] Cor. 2.

[b] Cor. 2. Ichnog.

[c] 10 El. 13.

[d] Cor. 4.

[e] Cor. 5. Meth. 2. Elev.

[f] Meth. 1. Elev.

and *l*G, at the ſame Diſtances from P*p*, as before directed; take in P*p* any Point *o*, on each Side of which ſet off *o* 1 equal to LC half the Side of the inner Pentagon, *o* 2 equal to MB half the Diagonal of that Pentagon, and *o* 3 equal to N*b* half the Diagonal of the outward Pentagon; from each of which Diviſions draw Perpendiculars to P*p*, which will cut *na*, *mg*, and *l*G, in the reſpective Points *e*, *a*, *d*, *b*, *c*; *f*, *k*, *g*, *i*, *b*; and F, K, G, I, H, the Seats of the correſponding angular Points of the outward and inward Pentagons on the Plane of the Elevation, by which the Elevation may be deſcribed as in the Figure. *Q. E. I.*

CASE 2.

If the Dodecaedron be ſuppoſed to reſt on one of its Sides on the Plane of the Ichnography, ſo as to have its oppoſite Side perpendicularly over it; then the Ichnography and Elevation will both be ſimilar to the Elevation in Fig. N°. 2.

Thus, if the Dodecaedron reſt on its Side A B on the Plane of the Ichnography, its oppoſite Side IK being perpendicularly over it; the Ichnography will be as in Fig. N°. 4. where IK and AB coincide: then if the Plane of the Elevation be parallel to AB, the Elevation of the Dodecaedron will be as in Fig. N°. 5. and if that Plane be perpendicular to AB, the Elevation will be as in Fig. N°. 6. all which Figures are equal and ſimilar, and differ only in their Poſition on their reſpective Planes. *Q. E. I.* Fig. 171. N°. 4, 5, 6.

COR. 1.

If the Line AB in the Ichnography be alone given, the intire Ichnography is thence found in this manner: Fig. 171. N°. 4.

Conſider AB as the ſmaller Segment of a Line divided in extreme and mean Proportion, and add to it its larger Segment[a]; then the whole Line will be equal to the Side of the outward Square, and its larger Segment to the Side of the inner Square[b], and the Line AB being biſected in *s*, will give their common Center; if then through *s* a Line DG be drawn perpendicular to AB, having each Moiety *s*G, *s*D, equal to half the Side of the outward Square, the Square QPRT is thence found, and thereby the inner Square, by which the intire Ichnography may be deſcribed.

COR. 2.

If the Side AB of the Elevation be given, the whole Elevation is thence found after the like manner. Fig. 171. N°. 5.

For by AB, the Sides of both the Squares are found as before, and AB being biſected in M, and MN being drawn perpendicular to it, and equal to the Side of the outward Square; that Square, and of conſequence the whole Elevation is thence found.

COR. 3.

Laſtly, when the Point A of the Elevation is given, if the Length of the Side AB of the Dodecaedron be known, the Length of the Sides of the Squares are thence found as before; and AK being drawn in the Plane of the Elevation, perpendicular to the Plane of the Ichnography, and equal to the Side of the outward Square, all the reſt is thence determined. Fig. 171. N°. 6.

LEM. 9.

To deſcribe the Ichnography and Elevation of an Icoſaedron.

CASE 1.

1. To deſcribe the Ichnography.

Let the equilateral Triangle ABC be a Face of the propoſed Solid, on the Plane of which the Ichnography is to be deſcribed. Fig. 172. N°. 1.

Having deſcribed another equilateral Triangle DEF, equal and in a ſubcontrary Poſition to the given Face, which will give the ſix Angles of a Regular Hexagon; from the Center S draw a *Radius* SA, and produce it to *a*, making SA to A*a*, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion[c]; then with the Center S and *Radius* S*a* deſcribe a Circle, and in it deſcribe a Regular Hexagon *aebfcd* parallel to the inner Hexagon; laſtly from each of the Angles of the outward Hexagon, draw Lines to the three neareſt Angles of the inner, as in the Figure, and that will compleat the Ichnography required.

And here, the Triangle DEF is the Ichnography of the upper Face of the Solid which is parallel and in a ſubcontrary Poſition to its Face ABC; the Triangles AB*e*, BC*f*, CA*d*, are the Ichnographies of the three Faces adjoining to the Face ABC;

and

[a] Cor. 4. Lem. 1.

[b] Cor. 3. Meth. 2. Elev. Caſe 1.

[c] Cor. 4. Lem. 1.

and the Triangles DE*a*, EF*b*, FD*c*, are the Ichnographies of thoſe adjoining to the Face DEF; and between each of theſe ſix Faces there fall two others, whoſe Ichnographies are A*ad*, A*ae*, B*be*, B*bf*, C*cf*, C*cd*; and D*dc*, D*da*, E*ea*, E*eb*, F*fb*, F*fc*; and the twelve angular Points of the outward and inward Hexagons are the Seats of the twelve ſolid Angles of the Icoſaedron on the Plane of the Ichnography. *Q. E. I.*

COR. 1.

If the inner Hexagon be compleated, by joining the Angles of the Triangles; the Sides of the outward Hexagon will be to thoſe of the inner, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion[a]; and if two equilateral Triangles *abc*, *def*, be formed in the outward Hexagon, their Sides will be to thoſe of the inner Triangles in the ſame Proportion.

COR. 2.

If the Dodecaedron and Icoſaedron be inſcribed in the ſame Sphere, the *Radii* SA, *sa*, of the inner and outward Decagons of the Ichnography of the one, will be reſpectively equal to the *Radii* of the inner and outward Hexagons of the Ichnography of the other.

For the ſame Circle in a Sphere contains the Faces of a Dodecaedron and Icoſaedron[b].

2. To deſcribe the Elevation.

The Elevation of this Ichnography may be deſcribed either on a Plane parallel to a Diameter AL, or to a Side AB of the given Face ABC.

1. The Elevation on a Plane parallel to AL is found in this manner:

METHOD 1.

On the given Ichnography draw *de* the Side of an equilateral Triangle in the outward Hexagon, and from any Angle A of the given Face ABC, draw a Diameter AL perpendicular to the oppoſite Side BC.

Fig. 172. N°. 2. Then in the Plane of the Elevation, draw any Line A*p* to repreſent the Interſection of that Plane with the Plane of the given Face, and in it take AC and C*p*, each equal to AL; on C*p* deſcribe an Iſoſceles Triangle *p*C*f*, having its Baſe *pf* equal to the Difference between *de* and DE, the Sides of the outward and inward Triangles in the Ichnography; produce *pf* to F, making *f*F equal to AB in Fig. N° 1. and draw A*a*, *a*E, and EF, parallel and equal reſpectively to F*f*, *f*C, and CA; laſtly draw EA, E*f*, C*a*, and CF, interſecting in *e* and *b*, and draw *eb*, which will compleat the Elevation A*a*EF*f*C of the Icoſaedron required. *Q. E. I.*

COR.

And here, the Angle *f*C*p* is the Angle of Inclination of the Faces of the Icoſaedron to each other, and AF is equal to the Diameter of the Sphere which circumſcribes that Solid.

METHOD 2.

Deſcribe a Square QPRT, having its Sides equal to a Side *ed* of the outward Triangle in the Ichnography; biſect the Sides of this Square by the Perpendiculars MN and CD, and from M and N ſet off M*a*, MA, NF, and N*f*, each equal to half the Side AB of the given Face ABC; and thereby the ſix Angles A, *a*, D, F, *f*, and C of the Elevation will be found, by which the whole may be compleated as in the Figure.

If the Line AC be given, the Poſition of the Square with reſpect to that Line may be thus found.

On AC deſcribe an Iſoſceles Triangle AC*f*, having its Baſe A*f* equal to *ed* the Side of the outward Triangle, and thereby the Angle AC*f*, and the Poſition of the Square QPRT will be found, its Sides RT and QP being equal and parallel to A*f*.

For A*f* equal to the Side of the outward Triangle, being to A*a* equal to the Side of the inner Triangle, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion, A*f* is the Diagonal of a Pentagon whoſe Side is A*a*; and AC and C*f* being each equal to AL in the Ichnography, the Angle AC*f* is therefore the Complement to two Rights of the Angle of Inclination of the Faces of the propoſed Icoſaedron[c]. *Q. E. I.*

COR.

[a] Cor. 1. Lem. 1.

[b] 2 El. 14.

[c] Lem. 4.

COR. 1.

The Line eb is equal to a Side of the given Face ABC.

For Es half the Side of the Square, is to sr the half of Aa, as the larger Segment to the leſs, of a Line divided in extreme and mean Proportion, that is, $Es : sr :: Er : Es$.

But becauſe of the ſimilar Triangles, $Er : Es :: AE : eE :: Af : eb$, therefore A$f$ is to eb, as the larger Segment is to the leſs, of a Line divided in extreme and mean Proportion, and conſequently eb and Aa are equal.

COR. 2.

The Lines QR, Aa, and fp, are to each other continually, as the greater Segment to the leſs of a Line divided in extreme and mean Proportion.

For fp is by Conſtruction equal to the Difference between QR and Aa[a].

SCHOL.

Here, the Elevations abc, def, of the outward Triangles, are in the ſtraight Lines ab and ef, both parallel to Ap; the Angles of thoſe two Triangles lying in two Planes parallel to the given Face ABC of the Icoſaedron, and being contained in two Parallel Circles whoſe *Radii* are equal to Sa.

COR. 3.

If a Line pl be drawn perpendicular to Ap, and EF, ab and ef be produced till they cut pl in l, m, and n; then pl will be the Height of the uppermoſt Face, and pm and pn the Heights of the two Planes of the outward Triangles, above the given Face ABC; and the Line pl will be divided in extreme and mean Proportion by each of the Points m and n, and conſequently in the ſame Proportion as the correſponding Line pl in the Elevation of the Dodecaedron[1].

For the Lines EA, Ef, Ca, and CF, are all divided in that Proportion by the Points e and b[b].

COR. 4.

If the Icoſaedron and Dodecaedron be contained in equal Spheres, the Lines pl relating to each of them, and conſequently their Diviſions will be equal.

For the Circles which contain the oppoſite and parallel Faces of thoſe Solids in the ſame Sphere, are equal[c], and conſequently equally diſtant from its Center. And hence, the Circles which contain the outward Pentagons and Triangles of theſe two Solids reſpectively, are alſo equal[d].

COR. 5.

The Part pn or ml of the Line pl is equal to SA the *Radius* of the Circle which contains the Face ABC of the Icoſaedron; and pm or nl is equal to Sa the *Radius* of the Circle which contains the outward Triangles[e]; and are therefore reſpectively equal to the Sides of the inward and outward Hexagons; and conſequently the whole Line pl is equal to Fa in the Ichnography, SA and SF being equal.

COR. 6.

The Diſtance of the Circles, which contain the Faces of the Icoſaedron and Dodecaedron, from the Center of the Sphere which circumſcribes thoſe Solids, and conſequently the Length of the Line pl relating to either of them, may be found in the following manner, the Diameter of the Sphere being given.

On the given Diameter AF of the Sphere, deſcribe a Semicircle AcF, and having found the Side of an Icoſaedron in that Sphere[f], deſcribe on that Side an equilateral Triangle, and find the Diameter of the Circle which contains it; from A with a *Radius* AC, equal to the Diameter thus found, draw an Arch cutting the Semicircle in c, and draw Ac; which being biſected in d by the *Radius* Se of the Sphere, Sd will be the Diſtance between its Center and the Plane of the Circle which contains the Face of either Solid; and conſequently Sp and Sl being made each equal to Sd, pl will be the Diſtance between the oppoſite Faces of the propoſed Solid. Fig. 172. N°. 3.

METHOD 3.

The ſame Elevation may be alſo deſcribed in this manner:

Having drawn any Line qp, and pl perpendicular to it, and equal to Fa in the Ichnography, and found in it the Points m and n as before; through n, m, and l, draw Fig. 172. N°. 2.

[a] Meth. 1. Elev.

[1] Fig. 171. N°. 2.

[b] Cor. 1.

[c] 2 El. 14.

[d] Cor. 4. Meth. 2. Elev. 1. Lem. 8.

[e] Cor. 5. Meth. 2. Elev. 1. Lem. 8.

[f] Lem. 3.

Parallels to qp, and having in qp taken any Point o to repreſent the Seat of the Center S of the Ichnography on the Plane of the Elevation, ſet off on each Side of o the Diviſions o 1, o 2, o 3, and o 4, equal reſpectively to SM, SN, SA, and Sa, in the Ichnography; then from each of theſe Diviſions draw Perpendiculars to qp, which will cut the Parallels nd, ma, and lD, in the reſpective Points f, e; b, a; and F, D; by which the Elevation may be deſcribed as in the Figure. *Q. E. I.*

And here, as in the Dodecaedron, the Points 2 and 2 may be omitted, the Points e and b being determined without them[a].

2. The Elevation of the ſame Ichnography on a Plane parallel to either of the Sides AB of the given Face, is thus deſcribed.

Fig. 172. N°. 4. Having drawn any Line Pp, and pl perpendicular to it, and the Parallels nf, ma, and lL, at the ſame Diſtance from Pp as before directed; take any Point o in Pp, and on each Side of it ſet off the Diſtances o 1, o 2, equal reſpectively to ME and Ne, half the Sides of the inward and outward Triangles in the Ichnography; and from each of theſe Diviſions draw Perpendiculars to Pp, which will cut nf, ma, and lL, in the reſpective Points f, e, d; a, c, b; and F, E, D; the Seats of the correſponding angular Points of the outward and inward Triangles on the Plane of the Elevation; by which it may be deſcribed as in the Figure. *Q. E. I.*

CASE 2.

Fig. 172. N°. 2. If the Icoſaedron be ſuppoſed to reſt on one of its Sides dc on the Plane of the Ichnography, its oppoſite Side eb being perpendicularly over it; then the Ichnography and Elevation will both be equal and ſimilar to the Elevation in Fig. N°. 2. only with this Difference as ~~to their Poſition, that~~ if the Plane of the Elevation be parallel to dc, the Elevation will have ſuch a Poſition as if this Figure reſted with its Side Aa on the Plane of the Ichnography, which Side Aa will then repreſent the Perpendicular Seat of dc on the Plane of the Elevation; but if this Plane be perpendicular to dc, the Elevation will ſtand ſo as to reſt with its Angle C on the Plane of the Ichnography, to which its Diameter CD will be perpendicular, and then the Point C will repreſent the intire Seat of dc on the Plane of the Elevation. *Q. E. I.*

COR. 1.

If the Line dc in the Ichnography be given, the whole Ichnography may be found, by biſecting dc in s, and producing dc to M and N, until sc be to cN, and sd to dM, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion; for CD being drawn through s perpendicular to MN, until sD and sC are each equal to sM, the Square QPRT is thence found, and thereby the whole Ichnography may be deſcribed; Aa and Ff being each equal to dc, and biſected in M and N.

COR. 2.

If the Side Aa of the Elevation be given, biſect it in M by the Perpendicular MN; and MN being made to Aa, as the greater Segment to the leſs, of a Line divided in extreme and mean Proportion, the intire Elevation may be thence found as before.

COR. 3.

Laſtly, when the Point C of the Elevation is given, if the Length of the Side dc be known, the Length of the Side of the Square QPRT is thence found as before; and CD being drawn in the Plane of the Elevation, perpendicular to the Plane of the Ichnography, and equal to the Side of the Square, all the reſt is thence determined.

CASE 3.

1. To deſcribe the Ichnography.

If the Icoſaedron reſt on one of its ſolid Angles A on the Plane of the Ichnography, its oppoſite Solid Angle F being perpendicular over it, the Ichnography is deſcribed in the following manner:

Fig. 172. N°. 5. Deſcribe two concentrical Regular Pentagons BCdae, and fcDEb, in contrary Poſitions, having their Sides equal to the Sides of the Icoſaedron; and having from thence deſcribed the Regular Decagon, and drawn its Diameters, the Ichnography will be compleated as in the Figure. *Q. E. I.*

2. To deſcribe the Elevation.

The Elevation of this Ichnography may be deſcribed, either on a Plane parallel to any

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[a] Meth. 1. Elev.

any of the Diameters af, or to any of the Sides BC of the Pentagons in the Ichnography.

1. The Elevation on a Plane parallel to af is found in this manner.

METHOD 1.

Fig. 172. N°. 6. Having drawn any Line qp, to repreſent the Interſection of the Planes of the Ichnography and Elevation; from any Point A in qp, draw the Perpendicular AF, and in it take An equal to aE the Side of the Decagon, nm equal to Aa the *Radius* of the Circle which circumſcribes the Ichnography, and mF equal to An; then through n and m draw aB and Ef parallel to qp, and from m and n ſet off mf and na contrarywiſe, each equal to mn, and mE and nB each equal to AM the Perpendicular Diſtance between the Center A and the Side DE of the Pentagon DEbfc in the Ichnography; and drawing AE and BF, cutting aB and Ef in d and b, the reſt of the Elevation is finiſhed as in the Figure. *Q. E. I.*

COR.

And here, AF is the Diameter of the Sphere which circumſcribes the Solid; mn or na is the *Radius* of the Circle which contains the Pentagon that ſubtends the ſolid Angle; Aa is equal to the Side of that Pentagon, and An to the Side of the Decagon inſcribed in the ſame Circle; and AF is equal to na and 2An[a].

SCHOL.

It is evident, this Elevation is equal and ſimilar to that in Fig. N°. 2. ſave that here it reſts with its Angle A, and there with its Side AB on the Plane of the Ichnography.

METHOD 2.

The ſame Elevation may alſo be thus deſcribed:

Fig. 172. N°. 6. Having drawn qp, and AF perpendicular to it, and found the Points n, m, and F, and drawn the Parallels aB and Ef as before; ſet off on each Side of A or o, the Diſtances $o1$, $o2$, $o3$, equal reſpectively to AN, AM, and Aa, in the Ichnography; and Perpendiculars to qp drawn from each of theſe Diviſions, will cut aB and Ef in the reſpective Points a, e, B; E, b, f, by which the Elevation may be compleated. *Q. E. I.*

And here, the Points 1 *and* 1 *may be omitted, the Points e and b being found without them*[b].

2. The Elevation of the ſame Ichnography on a Plane parallel to either of the Sides BC of the Pentagons, is thus deſcribed.

Fig. 172. N°. 7. Having drawn PR, and the Perpendicular AF, and found its Diviſions m and n, and drawn the Parallels Ef and aB as before; on each Side of A or o, ſet off the Diſtances $o1$, $o2$, equal reſpectively to ME and Ne, half the Side and half the Diagonal of the Pentagon DEbfc in the Ichnography; and Perpendiculars to PR drawn from each of theſe Diviſions, will cut the Parallels aB and Ef reſpectively in d, B, a, C, e, and b, D, f, E, c, the Seats of the correſponding Angles of the Pentagons in the Ichnography on the Plane of the Elevation; by which the Elevation may be compleated, as in the Figure. *Q. E. I.*

COR.

In both theſe Elevations, the Pentagons CBead and DEbfc, which ſubtend the ſolid Angles A and F of the Icoſaedron, are each in a Plane parallel to the Plane of the Ichnography, the Height of which Planes above the ſolid Angle A being marked at n and m; and nm is to nA or mF, as the larger Segment to the leſs, of a Line divided in extreme and mean Proportion[c].

GENERAL COROLLARY.

Here it may be obſerved, that although the Elevations of the Dodecaedron and Icoſaedron, on Planes parallel to a Diameter of their given Faces, are more Simple, moſt of the Angles and Sides of thoſe Elevations anſwering to two of the Solid; yet the Elevations on Planes parallel to one of the Sides of thoſe Solids, are both more eaſily drawn, and more conveniently put into Perſpective.

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[a] Lem. 3. Art. 5.

[b] Meth. 1. Elev. 1.

[c] 9 El. 13.

Of the Images of the five Regular Solids.

PROB. I.

The Center and Diſtance of the Picture, and one Face of a Cube, together with the Vaniſhing Line of its Plane, being given; thence to deſcribe the intire Image of the Cube.

Fig. 173. N°. 1.

Let O be the Center of the Picture, and IO its Diſtance, ABCD the given Face of a Cube, and EFGH the Plane of that Face.

METHOD 1.

By the Vaniſhing Lines of the Planes of the Faces.

Produce any Side AB of the given Face to its Vaniſhing Point z, through which draw zw a Vaniſhing Line of Planes perpendicular to the Plane EFGH[a]; then in the Plane zwA compleat the Image of a Square AaBb on the given Side AB[b], by which, and the Vaniſhing Point y of the Side AC, the intire Image of the Cube may be deſcribed, as in the Figure.

[a] Prop. 25. B. IV.

[b] Prob. 20. B. II.

Dem. Through y draw a Vaniſhing Line yu of Planes perpendicular to the Plane EFGH; then becauſe the Angle CAB repreſents a Right Angle, the Vaniſhing Points y and z are perpendicular, and conſequently the Planes zw and yu, which are perpendicular to the Plane EFGH, are alſo perpendicular to each other; the Lines EF, zw, and yu, are therefore the Vaniſhing Lines of the Faces of the Cube, *viz.* EF of the given Face ABCD ~~and its~~ oppoſite $abcd$ which is parallel to it, zw of the Face ABab and its oppoſite CDcd, and yu of the Faces ACac and BDbd. The reſt is evident. *Q. E. I.*

SCHOL.

Fig. 173. N°. 1.

In Fig. N°. 1. the Plane EFGH being perpendicular to the Picture, the Vaniſhing Lines zw and yu are perpendicular to EF[c]; and the Sides Aa, Bb, Cc, Dd, of the Cube, being parallel to the Picture, are alſo parallel to the Vaniſhing Lines zw, yu, of the Planes which paſs through them[d].

[c] Cor. 5. Prop. 22. B. IV.

[d] Cor. 1. Theor. 15. B. I.

Fig. 173. N°. 2.

In Fig. N°. 2. the Plane EFGH inclines to the Picture, the Vaniſhing Lines of the Planes of the elevated Faces of the Cube muſt therefore paſs through x the Vaniſhing Point of Perpendiculars to the Plane EFGH[e]; and as the Side BD of the given Face ABCD tends to o the Center of the Vaniſhing Line EF, xo is the Vaniſhing Line of the Faces BbdD and AacC; and the Side AB being parallel to EF, and conſequently to the Picture, the Vaniſhing Line xw of the Face ABab and its oppoſite, paſſes through x parallel to AB or EF.

[e] Cor. 3. Prop. 20. B. IV.

Here alſo, all the upright Sides Aa, Bb, &c. of the Cube, tend to x; and the Face ABba is found by the help of the Diagonal Ba, whoſe Vaniſhing Point is w the Point of Diſtance of the Vaniſhing Line xw; or the Face ACca is found by the Diagonal aC, whoſe Vaniſhing Point u biſects the Angle oIx, and which laſt is here the more convenient, by reaſon of the too great Diſtance of the Point w.

Fig. 173. N°. 3.

In Fig. N°. 3. the Plane EFGH alſo inclines to the Picture, and z and y are the Vaniſhing Points of the Sides AB and AC of the given Face ABCD; the Vaniſhing Lines of the Faces ABab and ACca which are elevated on thoſe two Sides, do therefore paſs through their Vaniſhing Points z and y, and the Vaniſhing Point x of Perpendiculars to the Plane EFGH[f]; the Vaniſhing Point v of the Diagonal AD, biſects the Angle ſubtended by y and z; w the Vaniſhing Point of the Diagonal Ba, biſects the Angle ſubtended by z and x; and u the Vaniſhing Point of the Diagonal Ca, biſects the Angle ſubtended by y and x; laſtly, the Centers and Radials of the Planes xz and xy are found by Prop. XXVIII. Book IV. and the Points w and u are determined by *Lem.* 3. Book II.

[f] Cor. 3. Prop. 20. B. IV.

GENERAL COROLLARY.

The Planes of all the Faces of the Cube being thus determined; it will be eaſy to divide thoſe Faces in any propoſed manner, or to deſcribe any required Figures in any of them, by the Rules in Book II. and the ſame Vaniſhing Lines ſerve for the Deſcription of any other Bodies of the Form of a Parallelepiped, ſuch as Rectangular Buildings, Walls, Door or Window Caſes or Frames, &c. in a like Situation with

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Fig. 173 No. 1.

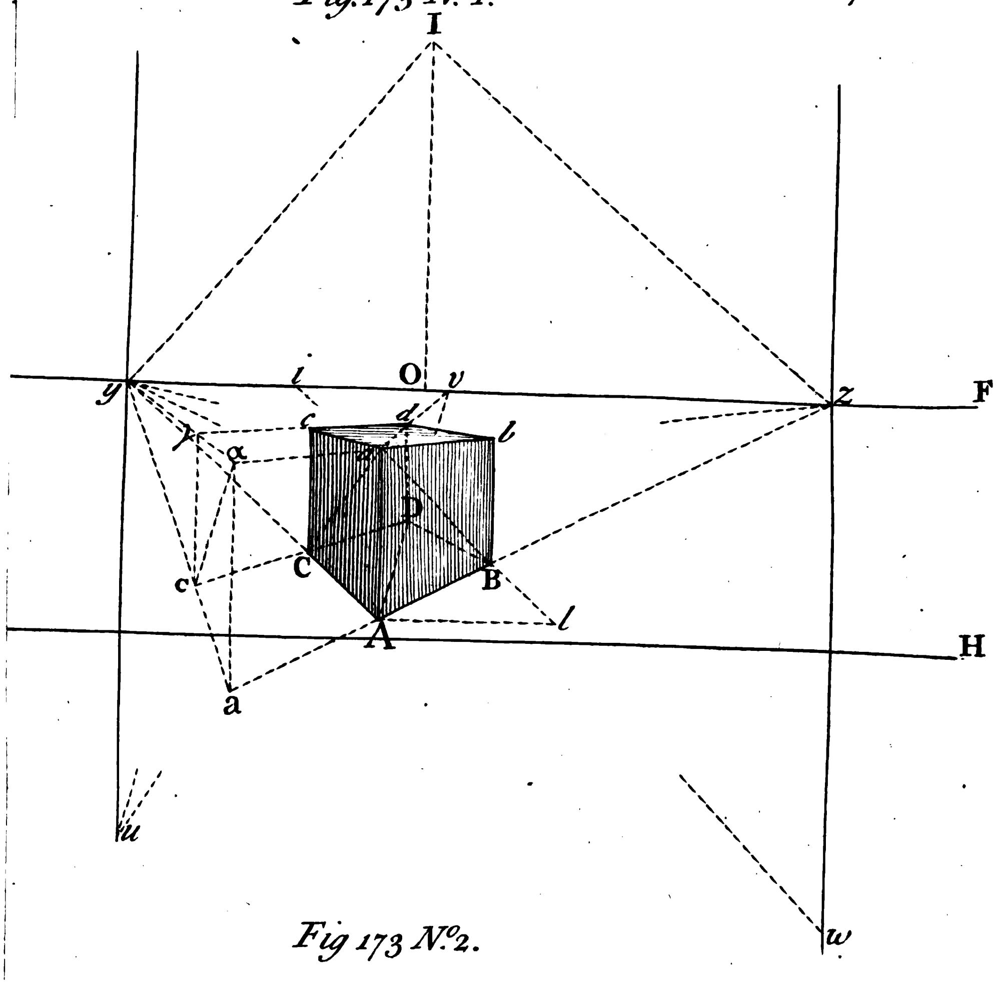
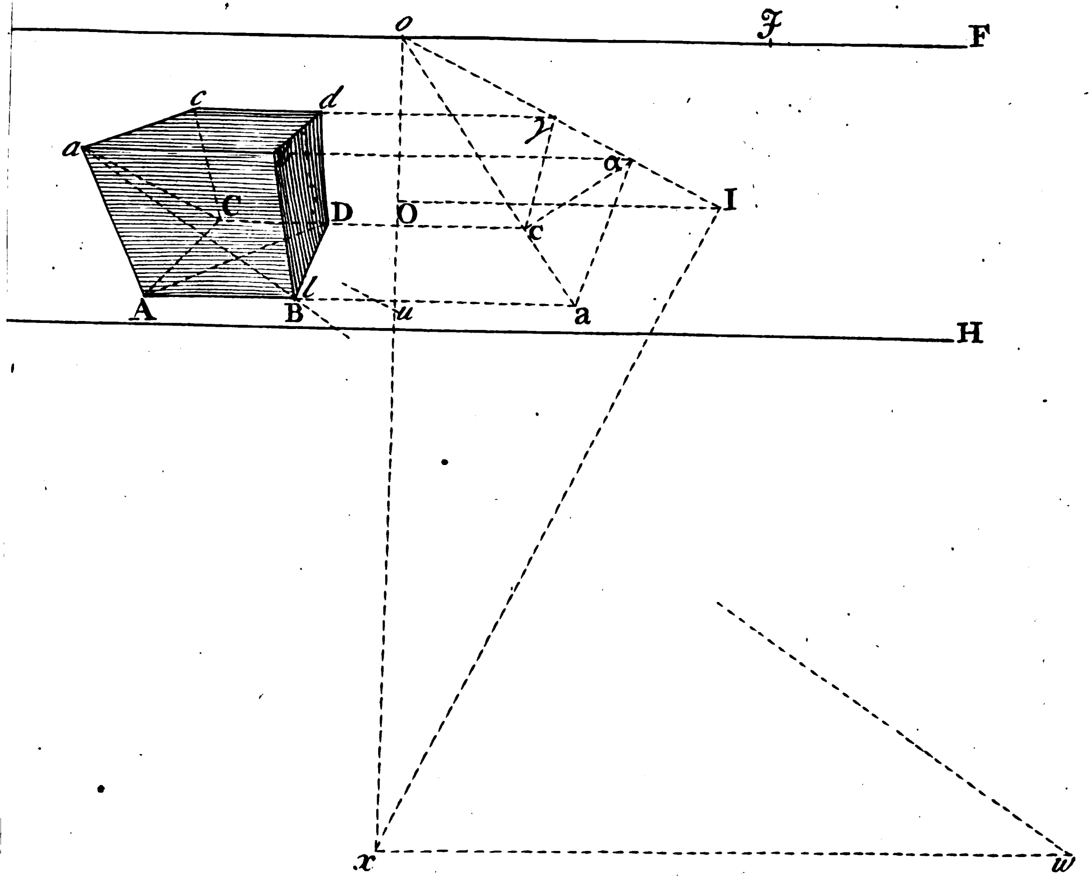


Fig 173 No. 2.



J. Mynde sc.

reſpect to the Planes of their Faces, whatever different Thickneſs, Length, or Breadth they may have; all which is ſufficiently evident from the Figure[i], where the Cube Ad is ſuppoſed to have its two Faces AB*ab* and CD*cd* parallel to the Picture, and all the reſt perpendicular to it, EF and IO being their Vaniſhing Lines; and the Walls, Windows, and other Apertures and Projections in the Room, being ſuppoſed to have their ſeveral Faces in the like Situation.

METHOD 2.

By the Ichnography and Elevation.

Produce either Side AC of the given Face to its Vaniſhing Point *y*, through which draw a Vaniſhing Line *yu* of Planes perpendicular to the Plane EFGH, which will therefore be the Vaniſhing Line of the Plane of the Elevation[a]; from *y* draw any Line *y*a in the Plane EFGH, to repreſent the Interſection of that Plane with the Plane of the Elevation, and produce the Sides AB, CD, of the given Face, till they cut *y*a in a and c; then in the Plane *yu*a, on the Side ac, compleat a Square a$\alpha\gamma$*c*, and having drawn the upright Sides A*a*, B*b*, C*c*, and D*d*, of the Cube, repreſenting Indefinite Perpendiculars to the Face ABCD, two Lines drawn from α and γ in the Elevation, to *z* the Vaniſhing Point of AB, will cut them in their Extremities *a*, *b*, *c*, and *d*, by which the intire Image of the Cube may be compleated. Fig. 173. N°. 1.

Dem. For here, the given Face ABCD may be alſo taken as the Ichnography of the propoſed Cube, and the Vaniſhing Point *z* being perpendicular to the Plane *yu*a ac is the Perpendicular Seat of AC, on the Plane of the Elevation; and AC and ac having the ſame Vaniſhing Point *y*, they repreſent equal Lines. And thus a$\alpha\gamma$c repreſenting a Square equal to the Face ABCD, in a Plane *yu*a perpendicular to that Face, and parallel to its Side AC, it is therefore the Elevation of the propoſed Cube[b]; laſtly, becauſe of the Vaniſhing Points *z* and *y*, the Sides A*a*, B*b*, C*c* and D*d* repreſenting Lines equal to the Originals of aα and cγ, the Length of thoſe Sides, and conſequently the Image of the Cube, are rightly determined. *Q. E. I.*

The like Method is purſued in Fig. N°. 2, 3, 4. which needs no farther Explanation.

SCHOL.

The Figure of a Cube is ſo Simple, that the Method of drawing its Image by its Ichnography and Elevation, doth not render the Work either eaſier or ſhorter; ſeeing the Vaniſhing Points of all the Sides muſt be ſtill uſed, and likewiſe the Vaniſhing Point *u* of the Diagonal αc of the Elevation muſt be found, in order to compleat that Square; which Vaniſhing Point is alſo that of the Diagonal C*a*, ſo that nothing is hereby ſaved: but the principal Uſe of this Method, is for finding the Diviſions of any Face of an Object of this Sort, when the Plane of that Face has not ſufficient Depth to give room for expreſſing them; or when the Lines which ought by their Interſections to determine thoſe Diviſions, cut each other too obliquely; or laſtly, to avoid incumbering the propoſed Figure with uſeleſs Lines: in which Caſes a Subſtituted Elevated Plane of a greater Depth becomes more convenient, and the propoſed Diviſion being found in that Subſtituted Plane, they may be thence transferred to the proper Face of the Figure, by Lines perpendicular to its Plane.

The ſame Method may be alſo uſed on the like Occaſion, with reſpect to the Ichnography; which may be transferred from the Plane of the given Face, to any other Plane parallel to it, as may be more convenient.

Thus, if *abcd* were the given Face, on which the Image of the Cube was to be erected; the Ichnography might be transferred to ABCD in the lower Plane EFGH, this being the the Perpendicular Seat of *abcd* on that Plane; but it muſt be obſerved, that a$\alpha\gamma$c would not, in this Caſe, be the Elevation, but the Elevation muſt be raiſed on *y*α the Interſection of the Plane of the Elevation with the Plane of the given Face *abcd*, Fig. 173. N°. 1.

METHOD 3.

The Image of the Cube may be likewiſe found in this manner:

Through any Angle A of the given Face, draw A*l* parallel to EF, and find the proportional Meaſure A*l* of the Side AB on that Line[c], and having from A drawn A*a*, repreſenting a Perpendicular to the Plane EFGH, equal to the Original of A*l*[d], thereby the Point *a* will be determined, and thence the intire Image of the Cube may be compleated without the help of the Diagonals of either of the Faces. Fig. 173. N°. 1.

[i] Fig. 173. N°. 4.

[a] Lem. 5.

[b] Lem. 5.

[c] Cor. 1. Prob. 9. B. II.

[d] Prop. 39. B. IV.

S C H O L.

The Deſcription of a Cube reſting on one of its ſolid Angles, by the help of its Ichnography and Elevation in that Poſture, is not here ſhewn, it being more laborious than uſeful; however, if it ſhould be required, it may be eaſily deduced from what will be taught in the following Propoſitions relating to the Deſcription of the more complicated Solids.

P R O B. II.

The Center and Diſtance of the Picture, and one Face of a Tetraedron, together with the Vaniſhing Line of its Plane, being given; thence to deſcribe the intire Image of the Tetraedron.

Fig. 174. Let O be the Center of the Picture, IO its Diſtance, ABC the given Face of the Tetraedron, and EFGH the Plane of that Face.

M E T H O D 1.

By the Vaniſhing Lines of the Planes of the Faces.

Produce any convenient Side AB of the given Face to its Vaniſhing Point *y*, and thence find the Vaniſhing Points *z* and *v* of the other two Sides AC and BC[a]; then through *y*, draw a Vaniſhing Line *yu* of Planes inclining to the Plane EFGH in the ſame Angle as the Faces of the Tetraedron incline to each other[b]; and in *yu* find two Points *u* and ζ, ſubtending each with *y*, an Angle of ſixty Degrees; then draw *u*A, ζB, interſecting in D, and draw DC, which will compleat the Image of the Tetraedron deſired.

Dem. For the Plane *uy*A inclining to the Plane EFGH in the ſame Angle as the Faces of the Tetraedron do to each other, *uy*A is the Plane of the Face adjoining to AB; and the Points *u* and ζ ſubtending each with *y* an Angle of ſixty Degrees, ABD repreſents an equilateral Triangle, and is therefore the Image of that Face: the reſt is evident. *Q. E. I.*

S C H O L.

Here, the Angle of Inclination of the Faces of the Solid is ſo large, that the Interſection of *yu* with *w*w the Vaniſhing Line of Planes perpendicular to the Vaniſhing Point *y*[c], is out of reach; nevertheleſs the Indefinite Radial *J*w, which ſhould determine that Interſection, being found, (by making on *Jw*, a Triangle *Jhk* ſimilar to the Triangle BEC in Fig. 167. N°. 1[d], with its Angle correſponding to E at *J*) the Vaniſhing Line *yu* is had, by making it tend to the ſame inacceſſible Point w with *J*w and *w*w[e]: and after the ſame manner, the Side BD of the Face ABD is found, by making it tend to the ſame Point with *u*ζ, and *i*ζ the Indefinite Radial of the Vaniſhing Point ζ.

This Method, if rendered a little familiar, will appear very eaſy, exact, and expeditious, and will be of frequent uſe on the like Occaſions.

C O R. 1.

The Line *uz* drawn from *u* through *z*, is the Vaniſhing Line of the Plane of the Face ADC, *u* and *z* being the Vaniſhing Points of AD and AC, two Lines in that Plane[f]; and if in *uz*, the Vaniſhing Point v of DC be found, a Line drawn through v and the Vaniſhing Point *v* of the Side BC, or tending to it, will be the Vaniſhing Line of the Face DCB, and will likewiſe paſs through ζ the Vaniſhing Point of DB.

C O R. 2.

If from *w* the Vaniſhing Point of Lines in the Plane EFGH perpendicular to AB, a Line *w*C be drawn, and from *r* the Vaniſhing Point of Lines perpendicular to BC, a Line *r*A be drawn, cutting *w*C in *s*, *s* will repreſent the Center of the given Face ABC; and if from *s*, a Line *s*D be drawn, repreſenting a Perpendicular to the Plane EFGH, it will be cut by *u*A, ζB, or vC, in the ſame Point D, by which the intire Image of the Tetraedron may be compleated.

M E T H O D 2.

By the Ichnography and Elevation.

As for the Ichnography, it is found as in the preceeding Corollary.

The Elevation on a Plane parallel to the Side AB, is found in this manner.

Having

[a] Prob. 19. B. II.

[b] Prop. 25. or Cor. 6. Prop. 26. B. IV. and Lem. 4. Art. 2.

[c] Prob. 25. B. IV.

[d] Lem. 4. Art. 2.

[e] Prob. 18. B. II.

[f] Cor. 2. Theor. 10. B. I.

Having through y the Vaniſhing Point of AB, drawn yy a Vaniſhing Line of Planes perpendicular to the Plane EFGH, which will therefore be the Vaniſhing Line of the Plane of the Elevation; from y draw any Line ya for the Interſection of theſe two Planes, and by the help of the Vaniſhing Point w, tranſfer A, B, and the Center s, to a, b, and s, in that Line; then on ab deſcribe a Triangle abd in the Plane of the Elevation, repreſenting a Triangle ſimilar to ACF in Fig. 169. N°. 2[a]. having its Side ab correſponding to AC in that Figure[b], and draw sd; and abds will be the Elevation deſired.

If the Vaniſhing Point ξ or y of either of the Sides ad or bd be out of reach; draw sd repreſenting a Perpendicular to ab in the Plane of the Elevation, which will be cut by bd or ad in the ſame Point d.

The Elevation being thus found; from s the Center of the Ichnography, draw sD repreſenting a Perpendicular to the Plane EFGH, and dw will cut it in D the Vertex of the Tetraedron.

If the Elevation be deſired on a Plane parallel to any Diameter Ce of the Ichnography; through w the Vaniſhing Point of Ce, draw the Vaniſhing Line wv of the Plane of the Elevation, perpendicular to the Plane of the Ichnography; and having drawn $w\varepsilon$ for the Interſection of theſe two Planes, by the help of the Vaniſhing Point y which is perpendicular to w, tranſfer e, C, and s, to ε, γ, and σ, in that Line, and on $\varepsilon\gamma$ deſcribe a Triangle $\varepsilon\gamma\delta$ in the Plane of the Elevation, repreſenting a Triangle ſimilar to BEG in Fig. 169. N°. 2[c]. having its Side $\varepsilon\gamma$ correſponding to EB, and the Point ε to E in that Figure, and drawing $\sigma\delta$, the Ichnography will be compleated; and then a Line $y\delta$ will cut sD in the ſame Point D as before.

And here, v the Vaniſhing Point of CD, is alſo the Vaniſhing Point of $\gamma\delta$ in the Elevation; the Originals of the Triangles sCD, $\sigma\gamma\delta$, being ſimilar and parallel.

METHOD 3.

Through s the Center of the Ichnography, draw lm parallel to EF, and in it find lm the proportional Meaſure of either of the Sides BC of the given Face ABC[d]; and having found the Height of the Vertex above the Baſe, of a Tetraedron whoſe Sides are equal to lm[e], that Height will be the proportional Meaſure of sD on the Line lm which paſſes through its Extremity s; by which the Line sD, and conſequently the Point D may be had[f], whence the intire Image of the Tetraedron may be deſcribed.

GENERAL COROLLARY.

After this manner, the Image of any kind of Pyramid may be found, whatever Polygon it may have for its Baſe; the Image of the Baſe, and the Length of the Axe from the Vertex to the Center, and the Angle of Inclination of the Axe to the Plane of the Baſe being given; for all which, ſuch full Inſtructions have already been laid down[g], that it would be ſuperfluous to give farther Examples.

PROB. III.

The Center and Diſtance of the Picture, and one Face of an Octaedron, together with the Vaniſhing Line of its Plane, being given; thence to deſcribe the intire Image of the Octaedron.

Let O be the Center of the Picture, IO its Diſtance, ABC the given Face, and *EF*GH its Plane. Fig. 175. N°. 1.

METHOD 1.

By the Vaniſhing Lines of the Planes of the Faces.

Produce any convenient Side AB to its Vaniſhing Point y, and find the other Vaniſhing Points z and v of the given Face[h]; then through y, draw a Vaniſhing Line yu of Planes inclining to the Plane ***EFGH*** in the ſame Angle as the Faces of the Octaedron incline to each other[i]; and in yu find two Points u and ζ, each ſubtending with y an Angle of ſixty Degrees; by the help of which, find the Face ABE which adjoins to AB: from u the Vaniſhing Point of BE, through C, draw CD, which muſt be terminated in D, by ED drawn from E to v the Vaniſhing Point of BC; then zE and yD will give the upper Face EFD, and BF, CF, and AD being drawn, the intire Image of the Octaedron is compleated.

Demi.

[a] Lem. 6.

[b] Prob. 4. and 19. B. II.

[c] Lem. 6.

[d] Cor. 1. Prob. 9. B. II.

[e] Lem. 6.

[f] Prop. 39. B. IV.

[g] Prop. 39. and 40. B. IV.

[h] Prob. 19. B. II.

[i] Prop. 25. B. IV. and Lem 4. Art. 3.

Dem. For the Originals of EB and CD, and of ED and BC, being parallel, they have the ſame Vaniſhing Points u and v; and the upper and lower Faces of the Solid which are parallel, being equilateral Triangles in ſubcontrary Poſitions, the ſame Vaniſhing Points y, z, and v ſerve for both[a]; the reſt is ſufficiently evident. *Q. E. I.*

SCHOL.

The Angle of Inclination of the Faces of an Octaedron within the Solid, being Obtuſe, it will be more convenient to find the Angle of Inclination of the Sides of a Tetraedron, which is the Complement to two Rights of the other[b]; by drawing on Jw produced beyond J, a Triangle Jkb ſimilar to the Triangle BEC in Fig. 167. N°. 1[c]. with the Angle correſponding to E at J; for then kJw the Complement to two Rights of the Angle kJb, will be the Angle required; and the Vaniſhing Line yu will be found, by making it tend to the ſame Point w, with Jk and w w[d].

COR. 1.

The Line yu is the Vaniſhing Line of the Face ABE by the Conſtruction, and conſequently of the Face CDF which is parallel to it; and uz is the Vaniſhing Line of the Faces ACD and BEF, z and u being the Vaniſhing Points of AC and CD, two Sides of the Face ACD, to which the Face BEF is parallel: laſtly, the Vaniſhing Line of the Faces EDA and BCF, muſt paſs through the two inacceſſible Points ζ and v, the Vaniſhing Points of the Sides AE and ED; and may be found, if neceſſary, by producing the Radials Iv and iz to the Vaniſhing Lines yu and EF, till by their mutual Interſections they form a *Trapezium*, the Vaniſhing Line of which being found[e], it will be the Vaniſhing Line ſought; in regard that the Line thus determined muſt paſs through ζ and v.

COR. 2.

A Line uv is the Vaniſhing Line of BCDE, one of the Squares which biſects the Solid, and the Vaniſhing Point of Perpendiculars to that Plane, is the Vaniſhing Point of AF, the Diameter of the Solid which is perpendicular to that Square; a Line $z\zeta$ is the Vaniſhing Line of the Square ACFE, to which the Diameter BD is perpendicular; and a Line drawn from y, through the Vaniſhing Point of AD in the Line ζv, is the Vaniſhing Line of the Square ABFD, to which the Diameter EC is perpendicular.

METHOD 2.

By the Ichnography and Elevation.

1. To deſcribe the Ichnography.

Here, not to incumber the propoſed Figure with Lines, let the Ichnography be deſcribed on a Plane parallel to the Plane of the given Face ABC; for which purpoſe, from y the Vaniſhing Point of AB, draw any Line ab, and transfer the Side AB to ab in that Line, by the help of Aa and Bb, repreſenting Perpendiculars to the Plane EFGH, by which means a and b will be the Perpendicular Seats of A and B on the Plane of the Ichnography; then by the help of ab, and the Vaniſhing Points y, z, and v, compleat the Image $aebfcd$ of a Regular Hexagon, with two equilateral Triangles inſcribed, as in the Figure[f], and that will be the Ichnography required[g].

2. To deſcribe the Elevation.

The Elevation on a Plane parallel to either of the Sides ab of the inſcribed Triangles in the Ichnography, is found in this manner:

Having through y drawn yy the Vaniſhing Line of the Plane of the Elevation[h], from y draw any Line ab to repreſent the Interſection of that Plane with the Plane of the Ichnography, and by the help of w, transfer a, b, and c, to a, b, and c in that Line; and having drawn wA in the Plane EFGH, till it be cut in α by a α drawn perpendicular to that Plane, thereby α the Seat of a on the Plane EFGH will be found, and $y\alpha$ will be the Interſection of that Plane with the Plane of the Elevation; and the Points b and c being in like manner transferred to β and γ in that Line, produce aα, cγ, and bβ at pleaſure; then in the Vaniſhing Line yy, find a Point y ſubtending with y an Angle equal to the Angle EFB in Fig. 170. N°. 3[i]. and draw yγ till it cut aα in δ, then $y\delta$ will give ε and φ the Extremities of cγ and bβ, and the other Lines being drawn as in the Figure, the Elevation $\alpha\beta\varphi\delta$ will be compleated.

This

[a] Caſe 3. Prob. 25. B. II.

[b] Lem. 4 Art. 3.

[c] Lem. 4. Art. 2.

[d] Schol. Meth 1. Prob. 2.

[e] Cor 3. Prop. 18. B. IV.

[f] Caſe 3. Prob. 25. B. II.

[g] Lem. 7.

[h] Meth. 2. Prob. 2.

[i] Lem. 7.

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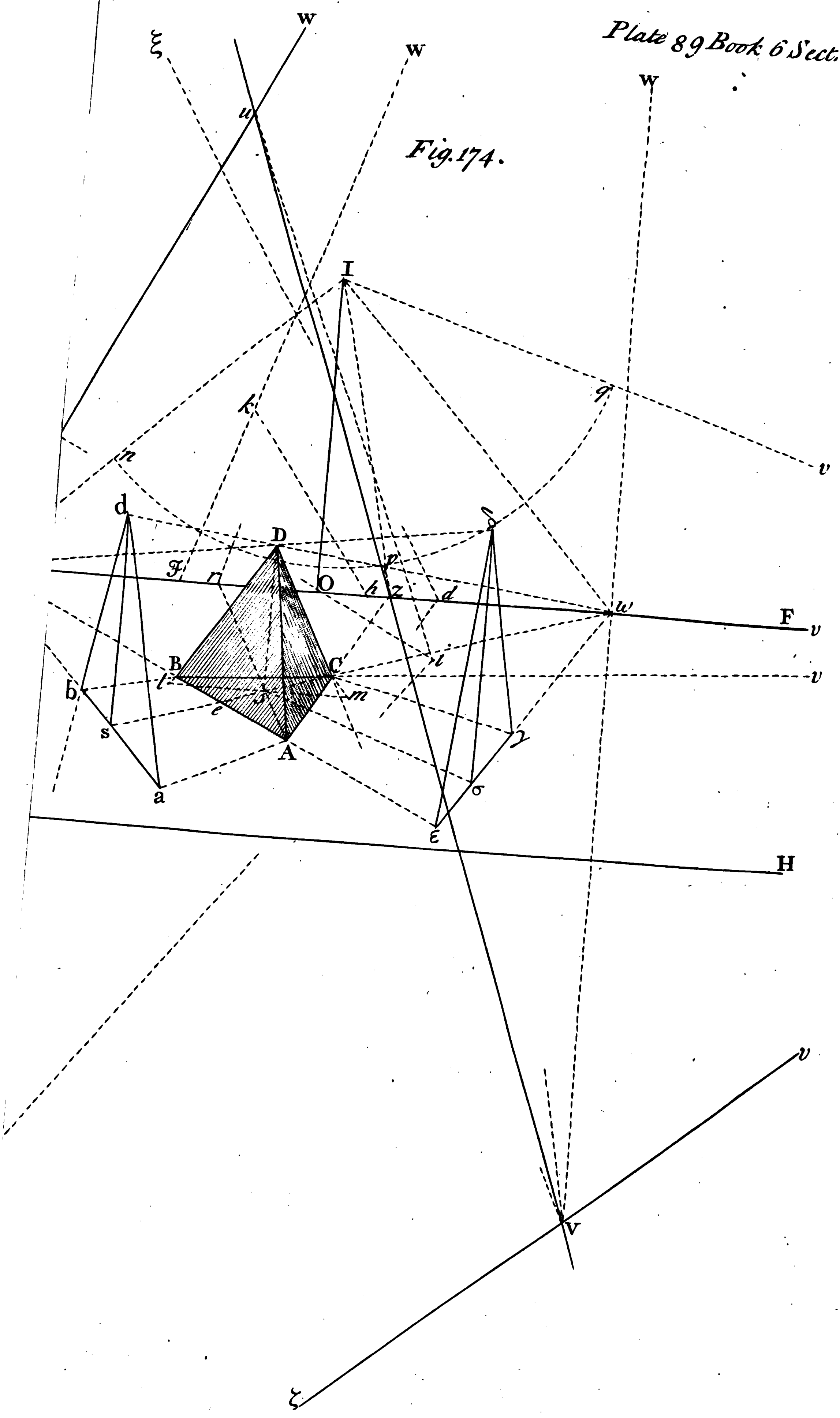
Fig. 174.



This being done, the Angles D, E, and F of the Octaedron are found by the Interſections of Lines perpendicular to the Plane *EFGH* drawn from their Seats *d*, *e*, and *f* in the Ichnography, with other Lines drawn from δ, ϵ, and ϕ their correſponding Seats in the Elevation to *w* the Vaniſhing Point of Perpendiculars to that Plane. *Q. E I.*

SCHOL.

After the like manner, the Elevation on a Plane parallel to either of the Diameters of the Ichnography may be found, whereby the Angles E, D, F, of the upper Face may be determined as before; the Elevation in this Caſe being made to repreſent a Figure ſimilar to Fig. 170. N°. 2[a]. But the Method of drawing the other Elevation being already ſo fully explained, it will be unneceſſary to add this.

COR.

Inſtead of finding the Vaniſhing Point y, thereby to determine the Point δ in the Elevation, which will be but inaccurate, by reaſon of the Obliquity of the Interſections of the Lines; the Length of $\alpha\delta$ may be better aſcertained, by finding the Proportional Meaſure of $\alpha\beta$ on a Line drawn through α parallel to EF[b], and finding the Height of the Vertex of a Tetraedron above its Baſe, whoſe Sides are equal to that Proportional Meaſure; which will give the Proportional Meaſure of $\alpha\delta$[c], by which $\alpha\delta$, and conſequently the intire Elevation may be found.

Or if the Ichnography be compleated on the Plane of the given Face ABC, whereby the Seats of the Points D, E, F, on that Plane are had; the Proportional Meaſure of the Height of either of thoſe Points above its Seat being found, the reſt are thence eaſily obtained.

CASE 2.

If the Image of either of the Squares which biſect the Octaedron be given, its intire Image may be thence deſcribed in this manner:

Let O be the Center, and OI the Diſtance of the Picture; and let BCDE be the given Square, and *EF* the Vaniſhing Line of its Plane. Fig. 175. N°. 2.

Draw the Diagonals EC and BD interſecting in S the Center of the Solid, and through S draw AF perpendicular to the given Plane; then through *x* and *v* the Vaniſhing Points of AF and CE, draw *xv* the Vaniſhing Line of the Plane in which thoſe two Lines lie, and by the help of the Diagonal EC, and of the Vaniſhing Point *w* which biſects the Angle ſubtended by *x* and *v*, compleat the Square FEAC, by drawing *w*E, *w*C, cutting AF in A and F; laſtly drawing AB, AD, FB, and FD, the intire Image of the Octaedron is deſcribed.

For it is evident, that BCDE and FEAC repreſent two concentrical Squares, in Planes perpendicular to each other, and having one Diagonal EC common to both; and that therefore B, C, D, E, F, and A, repreſent the Solid Angles of the Octaedron, and the Diagonals AF, EC, and BD, the Diameters of that Solid. *Q. E. I.*

COR.

If through *v* and A, a Line A*e* be drawn till it be cut in *e* by *x*E; *e* will be the Seat of E on a Plane paſſing through A parallel to the Plane BCDE, by which the Ichnography *bcde* of the Octaedron on that Plane may be deſcribed, if deſired.

PROB. IV.

The Center and Diſtance of the Picture, and one Face of a Dodecaedron, together with the Vaniſhing Line of its Plane, being given; thence to deſcribe the intire Image of the Dodecaedron.

Let O be the Center of the Picture, and *J*O its Diſtance, ABCDE the given Face, and *EF* the Vaniſhing Line of its Plane. Fig. 176. N°. 1.

METHOD 1.

By the Vaniſhing Lines of the Planes of the Faces.

Produce any convenient Side AB to its Vaniſhing Point *y*, and having on the Radial *Jy* drawn the Semicircle *lnq*, and divided it into five equal Parts, thereby the Vaniſhing Points 2, 3, 4, and 5, of the reſt of the Sides of the given Face, or ſuch of them as are within reach, will be found, and which will alſo be the Vaniſhing Points of the upper Face of the Solid, oppoſite to that which is given[d].

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[a] Lem. 7.

[b] Cor. 1. Prob. 9. B. II.

[c] Cor. Lem. 7.

[d] Caſe 2. Prob. 27. B. II.

Then through *y* draw a Vaniſhing Line *y*7 of Planes inclining to the Plane *EF* in the ſame Angle as the Faces of the Dodecaedron incline to each other[a]; and having by the help of *y*, found the other neceſſary Vaniſhing Points 6, 7, 8, in that Line, for the Deſcription of a Pentagon whoſe Side is AB, deſcribe the Pentagon AB*bga*, which will repreſent the Face of the Solid which is elevated on AB; through 7 the Vaniſhing Point of A*a*, and 4 the Vaniſhing Point of AE, draw 7, 4, which will be the Vaniſhing Line of the Face elevated on AE, A*a* and AE being two Sides of that Face; and compleating the Vaniſhing Points 9 and 10 in that Line, the three remaining Sides of the Pentagon *a*AE*ef* are thereby obtained: then by 3 the Vaniſhing Point of BC, and 8 the Vaniſhing Point of B*b*, the Vaniſhing Line 3, 8, of the Face elevated on BC is found, BC and B*b* being two Sides of that Face; which Vaniſhing Line alſo paſſes through the Point 9 in the Line 7, 4, before found, that Point being the Vaniſhing Point of *fe*, whoſe Original is parallel to that of the Side *bc* of the Face *b*BC*cb*, which Face is therefore found by ſupplying one more Vaniſhing Point 11, in the Line 3, 8; then by 2 the Vaniſhing Point of DE, and 10 the Vaniſhing Point of E*e*, the Vaniſhing Line 2, 10, of the Face elevated on ED is had, which likewiſe paſſes through 6 the Vaniſhing Point of *bg*, and 11 the Vaniſhing Point of *bh*, whoſe Originals are reſpectively parallel to thoſe of *dk* and *ke* of the Face *e*ED*dk*, which is therefore deſcribed by the help of thoſe four Vaniſhing Points; which Points alſo ſerve for the Deſcription of the Face *g*GH*hb*, which is parallel to *e*ED*dk*, as the Line 3, 8, with its two other Points 11 and 9, ſerve to deſcribe the Face *f*FK*ke*, which is parallel to *b*BC*cb*; and thus four Sides of the Face *g*GF*fa* being obtained, the Line GF compleats it, and likewiſe gives a third Side in the uppermoſt Face FGHIK, whoſe remaining Sides KI, HI, are found by 4 and *y* in its Vaniſhing Line *EF*: laſtly, I*i* being drawn to 7 the Vaniſhing Point of A*a*, and terminated in *i*, either by H*i* drawn to 9 the Vaniſhing Point of *bc*, or by K*i* drawn to 6 the Vaniſhing Point of *kd*, two Lines *ci* and *di* will give the only two remaining Sides which were wanting to compleat the Dodecaedron.

Or in caſe the Obliquity of H4 and K*y* render the Point I uncertain, the Interſection of H9 with K6 gives *i*, and 7*i* will by its Interſection with K*y* determine the Point I more accurately, and *ci* and *di* will compleat the Dodecaedron as before. *Q. E. I.*

SCHOL.

Here, the Line *y*7 being the only Vaniſhing Line which is to be determined by the Inclination of its Plane to the Plane *EF*, all the other Vaniſhing Lines as they come ſucceſſively to be drawn, ſtill paſſing through at leaſt two Vaniſhing Points already found; the truth of the Figure depends on the Exactneſs with which that firſt Vaniſhing Line is drawn. And as the Angle of Inclination of the Faces of the Dodecaedron within the Solid is Obtuſe, it will therefore be more convenient to find its Complement to two Rights, as in the laſt Problem[b], by drawing on d*w* the Radial of the Plane *w*w, which is perpendicular to *y*[c], produced beyond d, a Triangle dtq ſimilar to the Triangle AIB in Fig. 167. N°. 3[d]. having the Angle correſponding to I at d; for then td will cut *w*w in a Point w, through which the Vaniſhing Line to be drawn from *y* muſt paſs, or to which it muſt be made to tend, if that Point be out of reach, the Angle td*w* being by this Conſtruction equal to the Angle propoſed[e].

COR.

Here, the Vaniſhing Lines of the Planes of all the Faces are found, except only of the Face *f*FG*ga* and its Oppoſite *c*CD*di*, the Vaniſhing Line of which two Faces ought to paſs through 12 the Vaniſhing Point of F*f* in the Vaniſhing Line 3, 8, and the Interſection of the Radial *J*5 with the Vaniſhing Line *EF*, which is the Vaniſhing Point of GF and its oppoſite DC.

METHOD 2.

By the Ichnography and Elevation.

Fig. 176. N°. 2. Let O be the Center, and *J*O the Diſtance of the Picture; and let *EF* be the Vaniſhing Line of the given Face ABCDE of the Dodecaedron, whoſe Side AB hath O for its Vaniſhing Point.

1. To deſcribe the Ichnography.

Here, as in the laſt Problem, let the Ichnography be deſcribed on a Plane parallel to the given Face; for which purpoſe having found the reſt of the Vaniſhing Points of

that

[a] Prop. 25. B. IV. and Lem. 4. Art. 4.

[b] Schol. Meth. 1. Prob. 3.

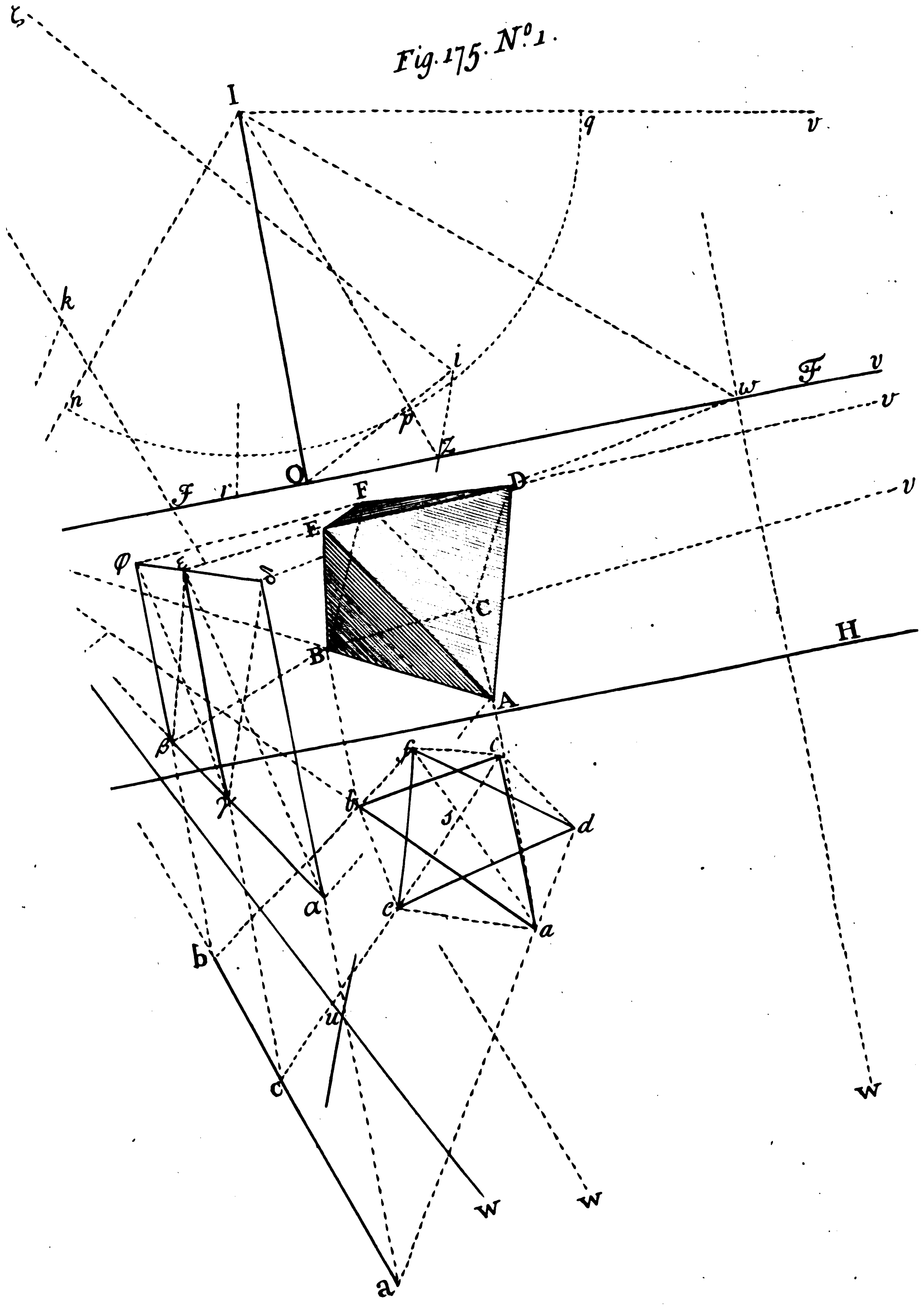
[c] Prop. 25. B. IV.

[d] Lem. 4. Art. 4.

[e] Lem. 4. Art. 4.

Plate 90. Book 6. Sect. 1.

Fig. 175. N°. 1.



J Mynde sc.

that Face, from *v* the Vaniſhing Point of the neareſt Side A E, draw any Line A E at a ſufficient Diſtance from it, ſo that the Ichnography, when drawn, may not interfere with the given Face, and transfer the given Side A E to A E in that Line by Perpendiculars to the Plane *E F*, and on the transferred Line A E as one Side, by the help of the Vaniſhing Points of the given Face, deſcribe the inner Pentagons of the Ichnography[a]; then having drawn any three adjoining Diameters *kb*, *eb*, and *fc*, from *t* the Vaniſhing Point of the middlemoſt *eb*, through DI and A G the oppoſite Sides of the inner Decagon, draw *ck*, *bf*, which will cut the Diameters *kb*, and *fc*, in *k*, *b*, *f*, and *c*, four Points of the outward Decagon[b], whereby two Sides *kf* and *cb* of the outward Pentagons are found, and thence the intire Ichnography may be compleated as in the Figure.

2. To deſcribe the Elevation.

Here, it will be moſt convenient to draw the Elevation on a Plane parallel to the Side A B of the given Face, that Side being perpendicular to the Picture.

For this purpoſe, from O the Vaniſhing Point of AB, draw any Line O P for the Interſection of the Planes of the Ichnography and Elevation, and transfer the Lines *ab*, A B, *fb*, and the Center S of the Ichnography, to O P, by Lines perpendicular to the Plane of the Elevation, cutting O P in 3, 2, 1, 0, 1, 2, 3[c]; and having found A the Seat of any Point A of the given Face on this Plane, through that Seat draw O*p* which will repreſent the Interſection of that Plane with the Plane of the given Face; then having on *f* P in the Plane of the Ichnography (which Line here coincides with *ef*, and is parallel to *E F*) found the Proportional Meaſures s a and s*e* of the Semidiameters S E, S*e*, of the Ichnography, from P draw P*l* perpendicular to the Plane of the Ichnography, cutting O*p* in *p*, and from *p* ſet off *pe* and *pf* equal to s a and s*e*, and *fl* equal to *pe*, and thereby the Height *pl* of the Elevation, which is equal to h*e* the Proportional Meaſure of H*e* in the Ichnography, will be got[d]; and the Diviſions of O P being transferred to O*p*, and the Image *pl*Q R of the Parallelogram which encloſes the Elevation, with its Subdiviſions, being thence deſcribed, the Elevation may be thereby compleated as in the Figure[e].

The Ichnography and Elevation being thus drawn, the Image of any angular Point of the Dodecaedron is obtained by the Interſection of Lines drawn from the Ichnography and Elevation of that Point, perpendicular to thoſe Planes reſpectively; in doing of which, to avoid Confuſion, it will be proper to find the Faces ſingly one after another, beginning with that which is moſt directly oppoſed to the Eye.

Thus, to find the Face A*afe*E, whoſe Ichnography and Elevation are marked with the ſame Letters; from *a*, *f*, and *e* in the Ichnography, erect Perpendiculars to its Plane (which are here alſo perpendicular to *E F*) and from the correſponding Points *a*, *f*, and *e* in the Elevation, draw Perpendiculars to that Plane (which are here parallel to *E F*) and the correſponding Interſections of thoſe Lines will give the Images *a*, *f*, and *e* of three Angles of the propoſed Face, by which and the given Side A E, it may be compleated; and after the ſame manner, any other Face may be found, till the intire Dodecaedron, or ſo many of its Faces as are viſible, be deſcribed. *Q. E. I.*

C O R. 1.

The Height *pl*, and the Diviſions *e* and *f* of the Elevation, may be alſo found in this manner.

Having transferred the Diviſions of O P to O*p* as before, from d the Point of Diſtance of the Vaniſhing Point O in the Vaniſhing Line *J*O of the Plane of the Elevation, through D and C in the Line O*p*, draw *d*D, *d*C, which will cut P*l* in *f* and *l*, two of the Points ſought, and the Line *d*D will alſo cut the Perpendicular 2 F in *a*, through which a Line drawn from O, will give the Point *e*.

For by this Conſtruction, *pf* and *pl* are the proportional Meaſures of D*p* and C*p* on the Line *pl*, whoſe Originals are equal to thoſe of H*e* and S*e* in the Ichnography; and E*a* is the proportional Meaſure of DE on the Line 2 F in the Elevation, and conſequently of S E in the Ichnography[f].

C O R. 2.

The Seat of any angular Point of the Dodecaedron on the Plane of the given Face, is found by the Interſection of a Perpendicular drawn from its Ichnography, with a Perpendicular drawn from the Seat of its Elevation on the propoſed Plane.

Thus,

[a] Caſe 2. Prob. 27. B. II.

[b] Cor. 2. Caſe 1. Ichnog. Lem. 8.

[c] Elev 2. Lem. 8.

[d] Cor. 5. Meth. 2. Elev. Lem. 8.

[e] Elev. 2. Lem. 8.

[f] Cor. 5. Meth. 2. Elev. Lem. 8.

Thus, if the Seat of the Point *f* of the Dodecaedron on the Plane of the given Face ABCDE be required; it is had by the Interſection of a Perpendicular to the Plane *EF*, drawn from *f* in the Ichnography, with a Perpendicular to the Plane of the Elevation, drawn from *p* the Seat of the Elevation *f* on O*p* the Interſection of that Plane with the Plane of the given Face.

SCHOL.

The Elevation on a Plane parallel to either of the Diameters of the Ichnography might be found after the ſame manner, by making it to repreſent a Figure ſimilar to Fig. 171. N°. 2. for doing of which, ſufficient Rules have been given[a]; but then, as the Plane of this Elevation would not be perpendicular to the Picture, the Perpendiculars drawn from any Point of the Elevation, muſt tend to the Vaniſhing Point of Perpendiculars to that Plane.

And here, it may be proper to obſerve, that when any Plane, not perpendicular to the Picture, is choſen for the Plane of the Elevation, it ought to be ſuch a one, the Vaniſhing Point of Perpendiculars to which, may be within reach; and it will be beſt, when it can be done, to place the Elevation on the oppoſite Side of the propoſed Solid from that Point; for then, the ſeveral Angles of the Solid will be more accurately determined, than when the Elevation is placed on the ſame Side with that Vaniſhing Point; the Elevation being in the firſt Caſe larger, and in the other leſs, than the Image of the Solid required.

The ſame is to be underſtood of the Choice of the Plane of the Ichnography; but when either of thoſe Planes is perpendicular to the Picture, it matters not on which Side of the propoſed Solid it is placed, ſo that it have a ſufficient Depth given it; ſeeing the Perpendiculars to that Plane will then be parallel to the Picture.

PROB. V.

The Center and Diſtance of the Picture, and one Face of an Icoſaedron, together with the Vaniſhing Line of its Plane, being given; thence to deſcribe the intire Image of the Icoſaedron.

Fig. 177. N°. 1. Let O be the Center, and IO the Diſtance of the Picture, ABC the given Face, and *EF* the Vaniſhing Line of its Plane.

METHOD 1.

By the Vaniſhing Lines of the Planes of the Faces.

Having produced any Side AB of the given Face to its Vaniſhing Point *y*, and thence found the other Vaniſhing Points 2 and 3 of that Face, through *y*, 3, and 2, draw three Vaniſhing Lines *y* 5; 3, 7; 2, 8; of Planes inclining to the Plane *EF*, in Angles equal to the Inclination of the Faces of the Icoſaedron[b]; and in each of thoſe Lines, from the Vaniſhing Point given, compleat the remaining Points 4, 5; 6, 7; and 8, 9, requiſite for deſcribing an equilateral Triangle; then *y* 5 will be the Vaniſhing Line of the Face AB*e* elevated on AB; 3, 7, will be the Vaniſhing Line of the Face BC*f* elevated on BC; and 2, 8, will be the Vaniſhing Line of the Face AC*d* elevated on the Side AC; which three Faces, by the help of the given Side in each, may be therefore deſcribed, and by this means three ſolid Angles, *e*, *d*, and *f*, are found.

Then, through 8 and 9 in the Vaniſhing Line 2, 8, of the Face AC*d*, draw two other Vaniſhing Lines 8, 11, and 9, 13, of Planes inclining in the like Angles to the Plane 2, 8, and compleat their Vaniſhing Points 10, 11, and 12, 13, and the Line 8, 11, will be the Vaniſhing Line of the Face C*dc* adjoining to the Side C*d*, and 9, 13, will be the Vaniſhing Line of the Face A*da* adjoining to the Side A*d* of the Face AC*d*; which two Faces C*dc*, A*da*, by the help of the given Sides C*d* and A*d*, may be therefore found, and thereby two more ſolid Angles *a* and *c* are determined.

From *f* through 12 the Vaniſhing Point of *ad*, draw *fb*, and from *e* through 10 the Vaniſhing Point of *dc*, draw *eb* cutting *fb* in *b*, and thereby another ſolid Angle *b*, and the Side *eb* of the Face *eb*E are obtained; the Originals of *ad* and *dc* which meet in *d*, being reſpectively parallel to thoſe of *fb* and *eb* which meet in *b*.

Then becauſe the Originals of the Faces C*dc* and *eb*E are parallel; by their Vaniſhing Line 8, 11, and the Side *eb*, the Face *eb*E, and another ſolid Angle E are found, whence alſo the Side E*a* common to the Faces E*ae* and E*a*D is had; the Original

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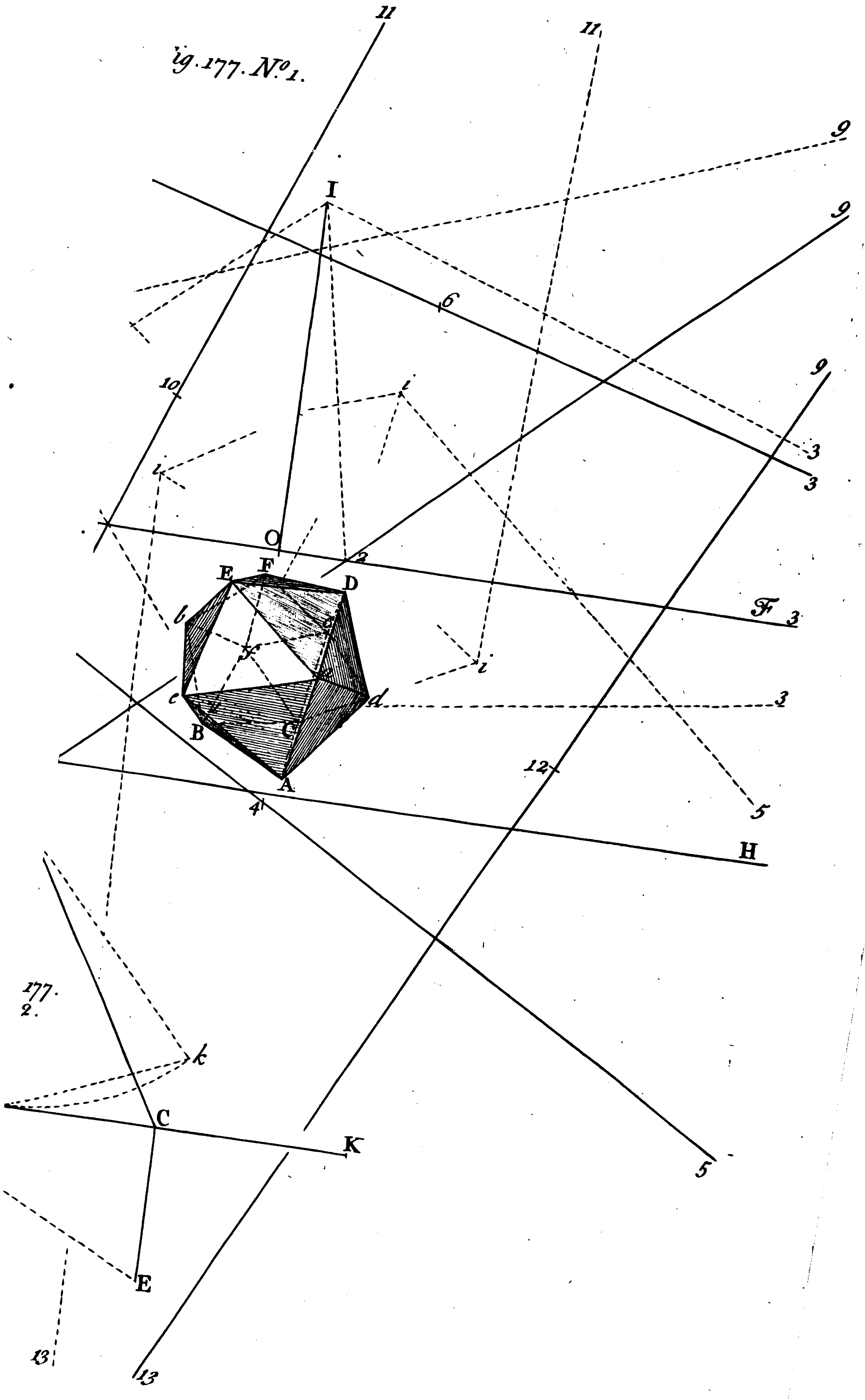
[a] Lem. 8.

[b] Prop. 25. and 26. B. IV. and Lem. 4. Art. 5.

Plate 94. Book 6. Sect. 1.



ig. 177. N°1.

177. 2.

J. Mynde sc.

Original of which laſt Face being parallel to that of BC*f*, it is found by its Vaniſhing Line 3, 7, and thereby the ſolid Angle D is got, and by the Side ED the upper Face EDF is deſcribed, and thereby the laſt ſolid Angle F is determined; or the Point F may be found by drawing *f*F from 13 the Vaniſhing Point of A*a*, the Originals of A*a* and F*f* being parallel: and thus all the ſolid Angles, and moſt of the Faces being determined, the intire Image of the Solid may be compleated by joining the remaining proper Points, as in the Figure. *Q. E I.*

SCHOL.

In regard that from any Point *y* in the Vaniſhing Line *EF*, there may be drawn two Vaniſhing Lines of Planes inclining to the Plane *EF* in the propoſed Angle, the one riſing above, and the other falling below *EF*[a]; care muſt be taken to chuſe the right, according to the way which the required Face inclines to that which is given.

Thus, the Face AB*e* inclining towards the Eye with reſpect to ABC, its Vaniſhing Line *y* 5 falls below *EF*; but the Face BC*f* inclining the contrary way to ABC, its Vaniſhing Line 3, 7, riſes above *EF*; the Angles 3*y*5 and *y*37 repreſenting the acute Angle formed by the Faces of the Icoſaedron, or the Complement to two Rights of the Angle they make together, within the Solid.

The ſame is to be obſerved in drawing the other Vaniſhing Lines.

Here alſo, it being required to draw ſeveral Vaniſhing Lines of Planes, by their Angle of Inclination to a given Plane, it may be convenient to draw that Angle in a Figure a-part, with a *Radius* ſufficiently large, which may be readily done in this manner.

On any Line BC deſcribe an equilateral Triangle BAC, and draw its Diameter AD, and having drawn CE perpendicular to BC and equal to CD, take the Diſtance BE in the Compaſſes, and ſet it off from D to K in the Line BC; then an Iſoſceles Triangle AD*k* with its Sides equal to AD, and its Baſe to CK, will give DA*k* the Angle required[b]. Fig. 177. N°. 2.

COR. 1.

1. Here, *y*, 2, 3, is the Vaniſhing Line of the Faces ABC, DEF.
2. *y*, 4, 5, is the Vaniſhing Line of the Faces AB*e*, FD*c*.
3. 3, 6, 7, is the Vaniſhing Line of the Faces BC*f*, DE*a*.
4. 8, 2, 9, is the Vaniſhing Line of the Faces CA*d*, EF*b*.
5. 8, 10, 11, is the Vaniſhing Line of the Faces C*dc*, E*be*.
6. And 9, 12, 13, is the Vaniſhing Line of the Faces A*da*, F*bf*.
7. And the Vaniſhing Line of the Faces A*ae*, F*fc*, paſſes through 5 and 13.
8. that of the Faces C*cf*, E*ea*, paſſes through 9 and 11.
9. that of the Faces B*be*, D*dc*, paſſes through 4 and 10.
10. And that of the Faces B*bf*, D*da*, through 6 and 12.

Laſtly, *abc*, and *def*, repreſent the angular Points of two equilateral Triangles parallel to, and alike poſited with the Faces ABC, and DEF.

COR. 2.

1. The Pentagons which ſubtend the ſolid Angle A, and its oppoſite F, are repreſented by BC*dae*, and ED*cfb*, and their Vaniſhing Line paſſes through the Points 8, 4, 12, and 3.

2. The Pentagons which ſubtend B, and D, are repreſented by AC*fbe*, and FE*adc*, and their Vaniſhing Line paſſes through 5, 12, 2, 10, and 7.

3. The Pentagons which ſubtend C, and E, are repreſented by AB*fcd*, and FD*a'b*, and their Vaniſhing Line paſſes through *y*, 10, 6, and 9.

4. The Pentagons which ſubtend *a*, and *f*, are repreſented by A*e*ED*d*, and F*c*CB*b*, and their Vaniſhing Line paſſes through 3, 5, 9, and 11.

5. The Pentagons which ſubtend *b*, and *d*, are repreſented by EF*f*B*e*, and AC*c*D*a*, and their Vaniſhing Line paſſes through 11, 6, 2, 4, and 13.

6. And the Pentagons which ſubtend *c*, and *e*, are repreſented by DF*f*C*d*, and BA*a*E*b*, and their Vaniſhing Line paſſes through *y*, 7, 8, and 13.

METHOD 2.

By the Ichnography and Elevation.

Let O be the Center, and IO the Diſtance of the Picture; and let *EFGH* be the Plane of the given Face ABC of the Icoſaedron, and *x* the Vaniſhing Point of Perpendiculars to that Plane. Fig. 177. N°. 3.

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[a] Prop. 25. B. IV.

[b] Lem. 4. Art. 5. and Schol.

1. To deſcribe the Ichnography.

Here, becauſe the Vaniſhing Point x is below the propoſed Solid, let the Ichnography be taken above it; and that its Plane may have a ſufficient Depth, let it be placed above the Eye, ſo as to be ſeen on the underſide.

For this purpoſe, from v the Vaniſhing Point of any Side AB of the given Face, draw any Line AB above the Vaniſhing Line EF, and transfer the Points A and B of the given Side, to A and B in that Line, by Lines drawn from x; and on the transferred Line AB, as one Side, by means of the Vaniſhing Points y, z, and v of the given Face, deſcribe the inner Triangles ABC, DEF, of the Ichnography[a]; and having from the Vaniſhing Points t and w, which are perpendicular to v and y, drawn two Indefinite Diameters ce, and fa, and thereby found the Center S of the Ichnography; through E the neareſt Angle to the Eye, draw ab parallel to EF, and in it, by the help of r the Point of Diſtance of t, find Ea the proportional Meaſure of SE; and having on ab taken Eb to Ea, as the ſmaller Segment to the greater, of a Line divided in extreme and mean Proportion[b], Eb will be the proportional Meaſure of Ee, by which the Point e of the outward Hexagon of the Ichnography is had, and thence the intire Ichnography may be compleated, as in the Figure[c].

2. To deſcribe the Elevation.

Here, it will be moſt convenient to draw the Elevation on a Plane parallel to the Diameter fa of the Ichnography, whoſe Vaniſhing Point is w, to which the Point y is perpendicular; but as the Elevation cannot here, for want of room, be placed ſo as to fall on the contrary Side of the propoſed Solid from y, it muſt be placed on the ſame Side, which is done in this manner.

From w, draw any Line wP, between the Ichnography and the Vaniſhing Line EF, to repreſent the Interſection of that Plane with the Plane of the Elevation, and by the help of the Vaniſhing Point y, transfer the Diviſions a, A, N, M, S, L, n, F, and f, of the Diameter fa to wP, cutting it in P, 3, 2, 1, o, 1, 2, 3, 4[d]; then having produced the given Side BC of the Icoſaedron, and its Ichnography BC, to their common Vaniſhing Point y, from x through 1 the Interſection of the Ichnography BC with wP, draw x1 cutting the given Side BC produced, in the Point marked BC; and the Point BC will be a Point in the Interſection of the Plane of the Elevation with the Plane of the given Face, and conſequently wp drawn through that Point, will be the Interſection of thoſe two Planes.

Then by the help of x, transfer the Diviſions of wP to wp, and having drawn the Vaniſhing Line xw of the Plane of the Elevation, find in it the Point v which biſects the Angle ſubtended by x and w; and from v through o, and the Point 3, fartheſt from p in the Line wp, draw vo, v3, which will cut Pp in m and l, two Points in the Height of the Elevation; and the ſame Line vo will alſo cut the upright Diviſion 3, 3, next to p, in a Point, through which, and the Points m and l, Lines being drawn to w, the Parallelogram plq4 with its Subdiviſions will be found, whereby the Elevation may be compleated, as in the Figure.

Dem. For it is evident the Points o, 3, 3, and p, in the Line wp, are the Seats on the Plane of the given Face, of the correſponding Points o, 3, 3, and P, in the Line wP, to which the Points S, F, A, and a, in the Ichnography, were transferred; and that by means of the Vaniſhing Point v, the Originals of op and 3p in the Line wp, are reſpectively equal to thoſe of pm and pl in the Line pP, and conſequently equal to the Originals of Sa and Fa in the Ichnography[e].

From the Ichnography and Elevation thus found, the Image of any Angle of the Solid is determined by the Interſection of a Line drawn from x to the Ichnography of that Angle, with a Line drawn from y through its Elevation; all which correſponding Points are here marked with the ſame Letters, as is likewiſe done in the former Figures. *Q. E. I.*

CASE 2.

If any Diameter AF of an Icoſaedron, with its Vaniſhing Point be given, and the Inclination of any Side Cd of the Pentagon which ſubtends either of the ſolid Angles A, to the Interſecting Line of the Plane of that Pentagon be known, and alſo whether that Side be viſible or not; the intire Image of the Icoſaedron may be thence deſcribed.

METHOD

[a] Caſe 3. Prob. 25. B. II.

[b] Cor. 4. Lem. 1.

[c] Lem. 9.

[d] Meth. 3. Elev. 1. Lem. 9.

[e] Cor. 5. Meth. 2. Elev. 1. Lem. 9.

METHOD 1.

By the Vaniſhing Line of the Pentagons which ſubtend the given ſolid Angles.

Let O be the Center, and OI the Diſtance of the Picture; and let the given Diameter AF be parallel to the Picture, and the Side Cd be ſuppoſed to be viſible. Fig. 177. N°. 4.

Through O draw a Vaniſhing Line EF perpendicular to the given Diameter AF, which will therefore be the Vaniſhing Line of Planes perpendicular to AF[a], and conſequently of the Planes of the Pentagons which ſubtend the ſolid Angles A and F.

Through A draw Ab perpendicular and equal to AF, and having on AF as a Diameter, deſcribed a Semicircle FkgA, from its Center s draw sb cutting it in g, and draw gn parallel to Ab; then gn will repreſent the *Radius*, and n the Center of the Circle which contains the lower Pentagon; and nm being taken on AF equal to ng, m will repreſent the Center of the Circle which contains the upper Pentagon, having its *Radius* mk equal to ng, and the remainder mF of the given Diameter will be equal to nA[b].

Then having from I, drawn a Radial Iy inclining to the Vaniſhing Line EF in the ſame Angle as the ſuppoſed Side Cd inclines to the Interſecting Line of the Plane of the Pentagon in which it lies, thereby y the Vaniſhing Point of that Side is found[c]; and having drawn another Radial Iw perpendicular to Iy, from I as a Center, with any *Radius*, deſcribe an Arch cutting Iy and Iw in t and r; then divide the Quadrant tr into five equal Parts, and ſet off one of thoſe Parts on the Arch beyond r, and drawing Radials through each of theſe Diviſions, thereby the Points y, v, 2, z, 3, w, and 4 will be found, of which, y, 2, 3, and 4, are Vaniſhing Points of the Sides of the Pentagons, and v, z, and w, Vaniſhing Points of their Diameters[d].

Then from v and w the Vaniſhing Points of the two extremes of any three adjoining Diameters, through n the Center of the lower Pentagon, draw two Indefinite Diameters vn, wn, and by the help of the *Radius* ng, which is parallel to the Picture, find their Extremities a and e which lye beyond the Center n[e], and thereby a Side ae of the lower Pentagon is got, whence the intire Pentagon aeBCd may be deſcribed: after the ſame manner, the Extremities c and f of the Diameters wm, vm, which lye on the hither Side of their Center m, being found, the Side cf of the upper Pentagon is obtained, and thence the intire Pentagon cfbED is found; and joining the proper Angles by ſtraight Lines as in the Figure, the Image of the Icoſaedron is thereby compleated.

Dem. For the Diameter wn being Perpendicular to the Side Cd whoſe Vaniſhing Point is y, and Cd being ſuppoſed to be viſible, and therefore to lye on the hither Side of the Center n, the Extremity e of the Diameter wn, which determines the Angle of the Pentagon oppoſite to Cd, muſt therefore be beyond n, conſequently e is a Point of the Pentagon required; and a Line joining the correſponding Extremities of any two alternate Diameters of a Regular Decagon, being the Side of an inſcribed Pentagon perpendicular to the intermediate Diameter, and the Diameter vn being alternate to the Diameter wn, in regard there lies a Diameter zn between them, the Extremity a of the Diameter vn is therefore another Point, and conſequently ae a Side of the Pentagon ſought; and the contrary Extremities c and f of the correſponding Diameters wm, vm, of the upper Pentagon, do therefore determine its Side cf which is oppoſite and parallel to ae: the reſt is evident. *Q. E. I.*

COR.

If the given Diameter be not parallel to the Picture, the Image of the Icoſaedron may be found in this manner:

Let O be the Center, and IO the Diſtance of the Picture, AF the given Diameter, and x its Vaniſhing Point. Fig. 177. N°. 5.

Having found EF the Vaniſhing Line of Planes perpendicular to the Lines x[f], through A draw Af parallel to xo, and by the help of v the Point of Diſtance of the Vaniſhing Point x, find Af the proportional Meaſure of AF on that Line[g]; and uſing Af as the given Diameter, find in it the Points n and m as before, and transfer thoſe Points by the help of v, to n and m in the Diameter AF; then, becauſe nm in Af, is equal to ng the *Radius* of the Circles which contain the Pentagons, through n and m in AF, draw Parallels to xo, terminated in μ and ν by the Lines drawn from v, and thereby $n\mu$ and $m\nu$ the proportional Meaſures of thoſe *Radii* at the Points n and m of the Diameter AF will be found; through which two Points, mk and ng being drawn parallel to EF, and equal reſpectively to $m\nu$ and $n\mu$, thereby the

[a] Cor. 1. Prop. 21. B. IV.

[b] Lem. 3. Art. 5.

[c] Caſe 3. Prob. 3. B. II.

[d] Prob. 27. B. II.

[e] Cor. 4. Meth. 2. Prob. 24. B. II.

[f] Prop. 21. B. IV.

[g] Caſe 2. Prop. 39. B. IV.

the Diviſions m and n of the given Diameter AF, and the proportional Meaſures mk, ng, of the *Radii* of the Circles in the Planes of both Pentagons, whoſe Vaniſhing Line is EF, are determined, whereby every thing elſe may be compleated as before.

METHOD 2.

By the Ichnography and Elevation.

1. To deſcribe the Ichnography.

Fig. 177. N°. 4. Here, in order to deſcribe the Ichnography, the Vaniſhing Line EF, and the ſeveral Vaniſhing Points in that Line, and alſo the proportional Meaſure of the *Radius* of the Circles which contain the Pentagons, muſt be found as before.

Which being done, produce the given Diameter AF at pleaſure, to the Point marked AF, denoting the Center of the Ichnography, and by the help of the proportional Meaſure Ak of the *Radius* of the circumſcribing Circle, and the ſeveral Vaniſhing Points in EF, the Ichnography may be deſcribed[a] as in the Figure.

2. To deſcribe the Elevation.

Here, becauſe the Vaniſhing Point w of the Diameter ce of the Ichnography, is the only Point that hath another Vaniſhing Point y perpendicular to it, within reach; it is moſt convenient to deſcribe the Elevation on a Plane parallel to that Diameter.

For this purpoſe, having drawn any Line wP for the Interſection of the Planes of the Ichnography and Elevation, at ſuch a Diſtance from the Ichnography, that the Elevation may not interfere with it; by the help of the Vaniſhing Point y, transfer the Points e, E, a, A, D, d, and c, to 3, 2, 1, o, 1, 2, P, in the Line Pw; and from o the Projection of the Center of the Ichnography on that Line, draw oF perpendicular to the Plane EF, and terminated in A and F by Lines drawn from y through the Extremities of the given Diameter AF; and thereby wp the Interſection of the Plane of the Elevation with a Plane EF paſſing through the Extremity A of the given Diameter, and wl the Line which terminates the Height of that Elevation, are found.

Then having in the given Diameter AF, or on its Projection AF on the Plane of the Elevation, determined the Points n and m by the Method already ſhewn, and transferred the Diviſions of wP to wp, by Perpendiculars to the Plane EF, the Parallelogram $plq3$ which incloſes the Elevation, with its ſeveral Subdiviſions, may be compleated, and the Elevation therein deſcribed[b] as in the Figure; and by the Ichnography and Elevation thus found, the Image of the Icoſaedron is obtained as in all other Caſes. *Q. E. I.*

SCHOL.

Here, in order to the right placing of the Elevation, it muſt be obſerved to which of the Pentagons the Extremities c and e of the Diameter ce of the Ichnography reſpectively belong: in the preſent Caſe, e is a Point of the lower Pentagon, and c of the upper; wherefore the Elevation e muſt be placed on the Line which paſſes through n, and the Elevation c on that which paſſes through m; for if they were placed on the contrary Lines, the Elevation would be the Reverſe of what it is, and the Poſition of the Pentagons in the Image of the Icoſaedron would be exchanged.

The other Ichnographies and Elevations of the Regular Solids, deſcribed in the foregoing Lemma's, being eaſily drawn by the like Methods as thoſe of the preceeding Problems, it is unneceſſary to inſert Examples.

PROB. VI.

The Image of either of the Regular Solids, being given; thence to deſcribe its Shadow or Projection on the Plane on which it reſts, from a given Luminous Point, whoſe Seat on that Plane is given.

Find the Ichnography of the given Solid on the Plane on which it reſts, whereby the Perpendicular Seats and Supports of all its Angles on that Plane will be determined; and the Perpendicular Seat of the Luminous Point on the ſame Plane being found, the Shadow or Projection of the Solid on that Plane may be thence obtained[c]. *Q. E. I.*

COR. 1.

If from the Seat of the Luminous Point on the propoſed Plane, two Lines be drawn touching the Ichnography on each Side, thoſe will determine the Bounds of the Shadow on the Sides, and will ſerve as a Guide to ſhew which of the Angles of the

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[a] Prob. 27. B. II.

[b] Meth. 2. Elev. Caſe 3. Lem. 9.

[c] Prob. 5. B. V.

Solid will be ſo projected, as to form Angles in the outline of the Shadow, which are all that are neceſſary to be found.

COR. 2.

If the Shadow be required on ſeveral different Planes; the intire Shadow on any one of the Planes being found, that Shadow may be conſidered as a given Figure in that Plane, from whence it may be projected on the other propoſed Planes, which will give the Shadow deſired[a].

[a] Prob. 14. B. V. and Gen. Cor. 2. of that Prob.

PROB. VII.

The Image of either of the Regular Solids, being given; thence to find its Reflection in a Reflecting Plane, whoſe Situation with reſpect to the Plane of either of the Faces of the given Solid, is known.

Find the Reflection of the Plane of the given Face, and in it the Reflection of that Face; and on the Reflected Face, deſcribe the Image of the propoſed Solid, by any of the Methods in the preceeding Problems, and that will be the Reflection deſired[b]. *Q. E. I.*

[b] Gen. Cor. Prob. 33. and Gen. Cor. Prob. 34. B. V.

SCHOL.

The various Ways of finding the Projections and Reflections of Lines and Planes, having been ſo fully treated of in the firſt and third Sections of Book V. and their Application to the preſent Purpoſe being ſo obvious, it is unneceſſary to draw Figures for theſe two laſt Propoſitions, the Trouble of which is therefore left for the Exerciſe of the Learner.

GENERAL COROLLARY.

The Figures of the Solids referred to in the five firſt Problems, are but lightly hatched, juſt to help the Appearance, and are in ſome Meaſure repreſented as if they were Tranſparent, that all their Sides, and Angles, and the Letters relating to them may appear, which would have been in a great Meaſure prevented, had their viſible Faces been more ſtrongly ſhadowed; as they are, the Methods uſed for deſcribing them, will be more clearly ſeen and underſtood, which will enable the Learner to apply thoſe Methods to the Deſcription of any other ſolid Bodies contained within Plain Surfaces, of which therefore it will be unneceſſary to give farther Examples: for it will be eaſy, from a right Knowledge of the Shape, Proportions, and Dimenſions of the ſeveral Parts of any propoſed Original Object (without which no Deſcription can be attempted) to chuſe ſuch of the foregoing Methods, as may be moſt convenient for deſcribing it; and Practice will naturally ſuggeſt other Expedients in particular Caſes, where neceſſary, if the Artiſt be fully Maſter of the Principles before taught.

It may be proper here, only to obſerve farther, that the Method of deſcribing Solid Bodies by the Vaniſhing Lines and Points of their Faces and Sides, is the moſt Extenſive; in regard, that when once thoſe Lines and Points are found, they ſerve alike for the Deſcription of any Number of ſimilar Bodies in a like Poſition with reſpect to the Plane of the Picture, whether they be bigger or ſmaller, nearer or farther, or more or leſs directly oppoſed to the Eye, than the Body firſt propoſed; beſides, that thereby the Faces and Sides of the Object become manageable in all reſpects: but the Method of deſcribing an Object by its Ichnography and Elevation, is confined to that particular Object in that one Situation alone; and it frequently takes up more Time and Labour to draw the Ichnography and Elevation, which are only Preparatory, and muſt afterwards be Effaced, than to deſcribe the Object itſelf by the other Method.

SECTION II.

Of the Image of the Cone and its Sections.

LEM. 10.

IF from the Eye at Σ, a Line be drawn through V the Vertex of a Cone ABV, till it cut the Plane of its Baſe in T, and from T there be drawn two Tangents to the Circular Baſe, meeting it in *l* and *m*; then, if from theſe Points of Contact, two Fig. 178.

Sides lV, mV, of the Cone be drawn, they will terminate the viſible Part of the Cone from Σ.

Dem. For lV being a Straight Line, and joining ΣT and Tl, thoſe three Lines are in the ſame Plane, which Plane touches the Cone in lV; in like manner, ΣTm is a Plane, touching the Cone in mV, and ΣT is the common Interſection of theſe two Planes; if then from Σ there be drawn any Line in either of theſe Planes towards the Cone, it will touch the Cone in ſome Point of lV or mV, and conſequently no Part of the Cone can be viſible from Σ beyond theſe two Lines; wherefore mAlV is the viſible Part of the Cone from Σ. *Q. E. D.*

COR. 1.

If ΣV be parallel to the Plane of the Baſe, then T being infinitely diſtant, the Tangents to the Baſe muſt be drawn parallel to ΣV, and will therefore paſs through the Extremities of a Diameter of the Baſe, and conſequently one Moiety of the Cone will be viſible; if the Point T fall beyond the Cone, more than one half of it will be viſible, but if T fall on the ſame Side of the Cone with the Eye, the viſible Part will be leſs than one half.

COR. 2.

If the Eye be moved any where in the Line ΣV on the ſame Side of V, the ſame Part of the Cone will remain viſible; if it be moved on the other Side of V towards T, the contrary Part lBmV of the Cone will become the viſible Part; but where-ever the Point Σ is taken in the Line ΣT, a Line drawn from thence to any Point in Vm, or Vl, will be a Tangent to the Cone in that Point.

COR. 3.

The Line ΣV is the common Interſection of all Planes whatſoever which paſs through any Side of the Cone and the Point Σ.

For the Point V being common to all the Sides of the Cone, if from A the Extremity of any Side VA of the Cone, a Line AΣ be drawn, a Triangle ΣVA will be formed, of which ΣV will always be one Side.

PROB. VIII.

The Center and Diſtance of the Picture, and the Image of any Diameter of the Circular Baſe of a Cone, and the Vaniſhing Line of its Plane, together with the Length of the Axe, and its Inclination to the Plane of the Baſe, being given; thence to deſcribe the Image of the Cone, and to determine its viſible Part.

Fig. 179. N°. 1.

Let O be the Center, and IO the Diſtance of the Picture, CD the given Diameter of the Baſe, and EF the Vaniſhing Line of its Plane; and let the Axe of the Cone be ſuppoſed perpendicular to its Baſe, and equal to a known Line.

By the help of the given Diameter CD, deſcribe the Image of the Circular Baſe ACBD[a]; and having from its Center S, erected SV perpendicular to the Plane EF, and made it to repreſent a Line equal to the propoſed Axe, from its Vertex V draw two Tangents to the Baſe[b], meeting it in l and m; then ACBDV will be the intire Image of the Cone, and lAmV its viſible Part.

Dem. For V being the Indefinite Image of a Line drawn from the Eye through the Vertex of the Cone, V alſo repreſents the Interſection of that Line with the Plane of the Baſe[c]; wherefore Tangents drawn from thence to the Image of the Baſe, meeting it in l and m, thereby determine its viſible Part[d]. *Q. E. I.*

[a] Prob. 24. B. II.

[b] Cor. 3. Prob. 3. B. III.

[c] Theor. 18. B. I.

[d] Lem. 10.

COR.

It is evident, that the Shadow of the Cone, N°. 1. on the Plane of its Baſe, from any Luminous Point Σ, is found by projecting the Vertex V on that Plane at n[e], and drawing from that Projection two Tangents np, nr, to the Baſe[f]; and that the Sides pV, rV, of the Cone, will terminate its enlightned Part from the Point Σ[g].

It is alſo clear, that the Reflection of the Cone in any Reflecting Plane, is had by finding the Reflections of its Baſe and Vertex[h], and drawing two Tangents from the Reflected Vertex to the Reflected Baſe.

[e] Prob 1. B.V.

[f] Cor. 3. Prob. 3. B. III.

[g] Lem. 10.

[h] Prob. 7.

CASE

Plate 97. Book 6 Sect. 2.

Fig. 178.

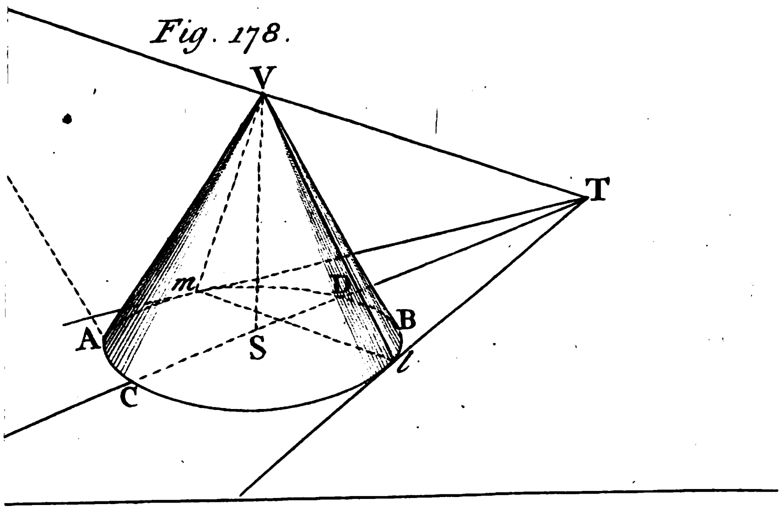
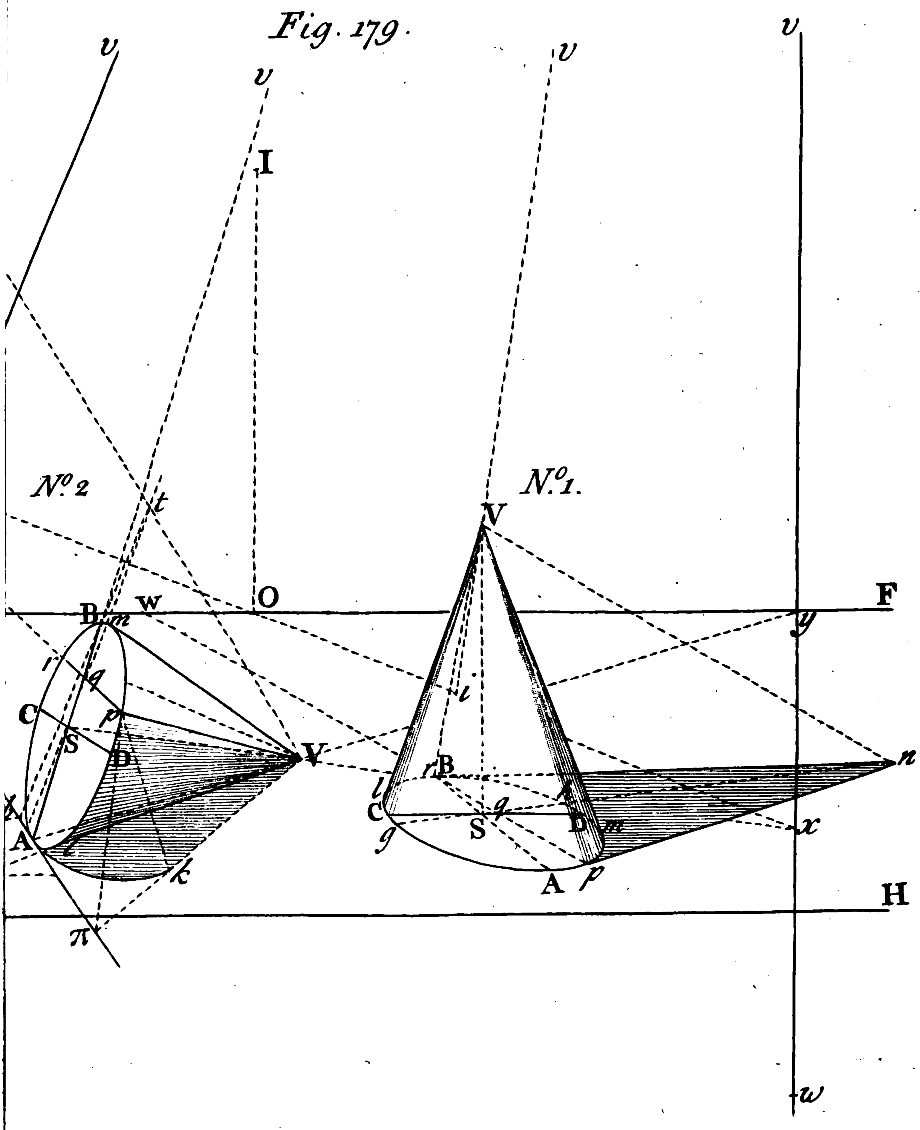


Fig. 179.



J. Mynde sc.

CASE 2.

The Determinate Image of one Side of a Cone reſting on a given Plane, and the Vaniſhing Point of its Axe, being given; thence to deſcribe the Image of the Cone.

Let A V be the given Side, reſting on the Plane E F G H, and y its Vaniſhing Point; and let x be the Vaniſhing Point of the Axe of the Cone, and V its Vertex. Fig. 179. N°. 2.

Having through x and y drawn a Vaniſhing Line xy, find zv the Vaniſhing Line of Planes perpendicular to the Vaniſhing Point x[a], cutting E F in z, and xy in v, and draw Av and xV interſecting in S; conſider S A as the *Radius* of a Circle in the Plane zv, having S for its Center, and by it deſcribe the Image A D B C of that Circle[b]; and from V draw the Tangents Vm, Vl, and thereby the Image lCmDV of the Cone will be compleated.

Dem. Becauſe x is the Vaniſhing Point of the Axe, and y the Vaniſhing Point of the given Side A V of the Cone, xy is the Vaniſhing Line of a Plane paſſing through the Axe and that Side, which Plane is perpendicular to the Plane E F G H, as well as to the Plane of the Baſe; and becauſe zv is the Vaniſhing Line of Planes perpendicular to the Axe, whoſe Vaniſhing Point is x, zv is the Vaniſhing Line of the Plane of the Baſe of the Cone, here ſuppoſed Perpendicular to its Axe; and A the Extremity of the given Side A V, being alſo a Point in the Baſe, Av drawn to v the Interſection of zv with xy, is the common Interſection of the Plane of the Baſe and the Plane xyV, and xV therefore cuts Av in S; wherefore V S repreſents the determinate Axe of the Cone, S its Center, and S A a Semidiameter of the Baſe; and conſequently the Image A C B D of a Circle, formed on S A as a *Radius*, in the Plane zvA, repreſents the Baſe of the propoſed Cone, and lCmDV is therefore its intire Image. *Q. E. I.*

COR. 1.

If the Cone be Scalene, and reſt on its longeſt Side, as it muſt naturally do on an Horizontal Plane; the Line Av muſt be ſo drawn, that its Vaniſhing Point v in the Line xy, may ſubtend with x, an Angle equal to the Inclination of the Axe of the Cone to its Baſe; and the Vaniſhing Line zv of the Plane of the Baſe, muſt paſs through that Point v, and the ſame Point z in the Line E F as before; the Plane of the Baſe being, in this Caſe, alſo Perpendicular to the Plane S V A, which laſt Plane is Perpendicular to the Plane E F G H.

For in this Poſition of the Cone, the Diameter C D of the Baſe, which is Parallel to the Plane E F G H, is Perpendicular to the Axe, as are the Vaniſhing Points x and z[c]; the reſt is evident from the Nature of a Cone.

COR. 2.

The Originals of the Diameters A B and C D of the Baſe, are always Perpendicular, whether the Cone be Right or Scalene.

For in either Caſe, the Planes zv and E F being both perpendicular to the Plane xy, the Interſection z of zv and E F, muſt be the Vaniſhing Point of Perpendiculars to the Plane xy[d]; the Vaniſhing Point z is therefore perpendicular to the Vaniſhing Point v, where-ever it falls in xy[e]; and conſequently the Originals of the Diameters A B and C D are always Perpendicular.

COR. 3.

The Interſection zA of the Plane of the Baſe with the Plane E F G H, is always Perpendicular to the Side A V of the Cone, on which its reſts.

For the Vaniſhing Points z and y are Perpendicular.

COR. 4.

The Shadow of the Cone, N°. 2. on the Plane E F G H, from any Luminous Point Σ, is determined in this manner.

From τ, the Parallel Seat of Σ on the Plane E F with reſpect to the Plane zv, draw τV cutting zA in b, and draw bt Parallel to zv; which, by its Interſection with ΣV, will give t the Projection of V on the Plane of the Baſe from the Point Σ, or the Interſection of ΣV with that Plane[f]; then find pr the Chord of the Tangents to the Baſe from t[g], and draw pV and rV, which will determine pVrB, that part of the Conick Surface which is enlightned from Σ[h]; and kV, the Projection of pV (found by drawing tp till it cut zA in π, and drawing πV and Σp, interſecting in k) will mark the Boundary of the Shadow on the Plane E F G H on the hither Side, as

[a] Prop. 21. B. IV.

[b] Prob. 24. B. II.

[c] Cor. 4. Prop. 20. B. IV.

[d] Cor. 3. Prop. 20. B. IV.

[e] Cor. 4. Prop. 20. B. IV.

[f] Converſe of Cor. Meth. 1. Prob. 6. B. V.

[g] Cor. 3. Prob. 3. B. III.

[h] Cor. 2. Lem. 10.

as the Projection of *r*V would mark its Boundary on the other Side, could it be ſeen; and the Remainder of the Shadow is found by the Projection of the Part *p*A*r* of the Circumference[a], when the Light falls upon the Baſe, as in the preſent Caſe; or by the Projection of the Part *p*B*r* of the Circumference, if the Light had fallen the contrary way.

PROB. IX.

The Center and Diſtance of the Picture, and the Image of a hollow Cone, with the Vaniſhing Line of the Plane of its Baſe, being given; thence to find the Boundary of the Light on its Concave Surface, which can enter it from a given Luminous Point, whoſe Seat on the Plane of the Baſe is given, and to determine the Species of the Curve thereby produced.

Fig. 180. N°. 1. Let O be the Center, and IO the Diſtance of the Picture, ADBEV the given Cone, ſuppoſed to be made of Tin or any other thin Subſtance, ſo that ADBE may repreſent the open Baſe of its Concave Surface; and let *EF* be the Vaniſhing Line of the Plane of the Baſe, Σ the given Luminous Point, and T its Seat on that Plane.

Draw ΣV, and find its Interſection *T* with the Plane of the Baſe, and having found DE the Chord of the Tangents to the Baſe from *T*[b], draw from *T* any Line *T*A, cutting the Baſe in A and B; and having drawn the Side VA of the Cone, draw ΣB cutting it in *b*, and *b* will be a Point of the Boundary of the Light required; and after the like manner, draw from *T* any other Line *T*F, cutting the Baſe in F and G, and ΣG will cut the Side VF of the Cone in *g*, another Point of the Boundary ſought; and thus, as many Points of that Boundary may be found, as are requiſite to deſcribe the whole, which muſt terminate at the Points D and E.

Dem. In the firſt place, it is evident, that ſo much of the Projection of the Baſe ADBE from the Point Σ, as can fall on the Concave Surface of the Cone, muſt form the Boundary of the Light which can enter it from that Point: now, DE being the Chord of the Tangents to the Baſe from *T*, it is apparent, that all Lines drawn from Σ to any Point of the Arch DBE, muſt, if produced, fall within the Cone, and cut it on the oppoſite Side; and that all Lines drawn from Σ to any Point of the Arch DAE, muſt paſs wholly without the Cone, the Lines ΣD, ΣE, being Tangents to the Cone in D and E[c]; and conſequently the Projection of the Arch DBE on the Concave Surface, will mark the intire Boundary of the Light which can enter the Cone from Σ, and which muſt terminate at D and E, thoſe Points coinciding with their Projections.

Now, becauſe AB and AV which meet in A, alſo meet ΣV in *T* and V, they are in the ſame Plane with ΣV, and conſequently with Σ; the Side AV of the Cone is therefore the Indefinite Projection of AB on the Concave Surface from Σ, wherefore *b* is the Projection of the Point B of the Arch DBE on that Surface; and in the ſame manner, it may be ſhewn that *g* is the Projection of G: the ſame Demonſtration will ſerve for any other Point correſponding to *b* or *g*, found after the like manner; and conſequently the Curve D*gb*E thus determined, is the Projection of DBE on the Concave Surface of the Cone from the Point Σ, and is therefore the Boundary of the Light required. *Q. E. I.*

COR. 1.

The Curve D*gbe* thus determined, is a Portion of a Conick Section, lying in a Plane paſſing through the Chord of the Tangents DE, and the Diagonal *ab* of the *Trapezium* B*b*A*a*, formed by the mutual Interſections of ΣA and ΣB, with the Sides VB and VA of the Cone.

Having biſected DE in C, draw *T*C, cutting the Baſe in A and B, and from *b* the Projection of B, draw *b*C, cutting the Side VB of the Cone produced, in *a*, and the Line ΣV in ſome Point σ, which it muſt do, if *b*C and ΣV be not parallel; in like manner, from *g* the Projection of G, through *q* the Interſection of FG and DE, draw *gq*, cutting the Side VG of the Cone in *f*, and the Line ΣV in ſome Point *v*.

Then, becauſe DE is the Chord of the Tangents from *T*, *T*A is Harmonically divided in *T*, B, C, and A[d]; wherefore VA, VC, VB, and V*T*, being Harmonical Lines, *b*σ, which cuts them all four, is Harmonically divided by them in *b*, C, *a*, and σ; and becauſe A*T* and *b*σ, which are both Harmonically divided, have their

Point

[a] Prob. 14. B. V.

[b] Cor. 3. Prob. 3. B. III.

[c] Lem. 10.

[d] Ellip. Art. 11. B. III.

Point C in common, and $\mathcal{T}\sigma$ joins their ſecond Points of Diviſion from C, bB and Aa which join their other Points, muſt meet in the ſame Point of $\mathcal{T}\sigma$[a], and conſequently in Σ where bB cuts that Line, wherefore ba is the Diagonal of the *Trapezium* $BbAa$; and by reaſon of the Harmonical Diviſion of $b\sigma$ in b, C, a, and σ, Ab, AC, Aa, and $A\sigma$, being Harmonical Lines, $V\Sigma$ which cuts them all four, is Harmonically divided by them in V, $\mathcal{T}$, Σ, and σ.

[a] Lem. 9. B. III.

In like manner, becauſe $\mathcal{T}F$ is Harmonically divided in $\mathcal{T}$, G, q, and F, VF, Vq, VG, and $V\mathcal{T}$ are Harmonical Lines, which therefore cut gf Harmonically in g, q, f, and v; and becauſe q is a Point of the Harmonical Diviſion of $F\mathcal{T}$ and gv, Ff and gG muſt meet $\mathcal{T}v$ in the ſame Point Σ, wherefore gf is the Diagonal of the *Trapezium* $GgFf$; and Fg, Fq, Ff, and Fv, being Harmonical Lines, the Line $V\Sigma$ is Harmonically divided by them in V, $\mathcal{T}$, Σ, and v; but $V\Sigma$ was before ſhewn to be Harmonically divided by Ab, AC, Aa, and $A\sigma$, in V, $\mathcal{T}$, Σ, and σ, and the Points V, $\mathcal{T}$, and Σ, in both theſe Diviſions, being the ſame, the fourth Points v and σ are therefore alſo the ſame[b]; and conſequently ba and gf which meet in σ, and paſs through C and q, two Points in the Line DE, are in the ſame Plane with DE, wherefore the Points b and g of the Curve $DgbE$ are in that Plane; the ſame may be ſhewn of any other Points in the Curve $DgbE$ found in the ſame manner, wherefore the whole of that Curve lies in a Plane paſſing through DE and the Diagonal ab of the *Trapezium* $bAaB$ formed by the Interſections of ΣA and ΣB with the Sides VB and VA of the Cone, in which Plane the Diagonal gf of the *Trapezium* $gFfG$ formed after the like manner alſo lies, and the Cone being thus cut by a Plane, the Section produced is therefore a Conick Section.

[b] Lem. 2. B. III.

COR. 2.

The Line ab is a Diameter of the Section $aDbE$, and DE is a double Ordinate to that Diameter.

For DE being the Chord of the Tangents to the Baſe from $\mathcal{T}$, σD and σE are Tangents to the Cone in D and E[c], and conſequently Tangents to the Section of that Cone by the Plane σDE; and DE being biſected in C, σC is therefore a Diameter of the Section, to which DE is a double Ordinate[d]; and a and b where σC cuts the Sides VB and VA of the Cone, are the Extremities of that Diameter, which Diameter being terminated both ways by the Sides of the Cone on the ſame Side of its Vertex, and being Harmonically divided in b, C, a, and σ, as already ſhewn, the Section here produced is therefore an Ellipſis.

[c] Cor. 2. Lem. 10.

[d] Ellip. Art. 11. B. III.

COR. 3.

Hence, if through a and b two Lines $\lambda\mu$ and nr be drawn parallel to DE, they will be Tangents to the Section in a and b, and if from σ, the Tangents σD, σE, be drawn, cutting $\lambda\mu$ and nr in λ, μ, n, and r, a *Trapezium* $\lambda\mu nr$ will be thereby formed, by the help of which the intire Section may be deſcribed[e].

But as here, the Part $DgbE$ of the Section is all of it that falls within the Cone, the other Part DaE being only imaginary, the Deſcription of this laſt Part may be ſaved, by uſing $DnrE$ as repreſenting half the Square which circumſcribes a Circle, and finding in it the Image $DgbE$ of a Semicircle whoſe Diameter is repreſented by DE[f].

[e] Meth. 3. Prop. 16. B. III.

[f] Cor. 2. Meth. 1. Prob. 24. B. II. and Schol.

COR. 4.

If $V\mathcal{T}$ and $\mathcal{T}\Sigma$ be equal, the Point σ will be infinitely diſtant, ab will be parallel to $V\Sigma$ and biſected in C, and DE will be a Diameter of the Section conjugate to the Diameter ab, to which the Tangents in D and E will be parallel, and the Section will ſtill be an Ellipſis.

Fig. 180. N°. 2.

The ſame Letters marking the ſame Points as before, through C draw ba parallel to $V\Sigma$; then, becauſe $V\Sigma$ is biſected in $\mathcal{T}$, ba parallel to $V\Sigma$ cutting AV and $A\Sigma$ in b and a, is biſected by $A\mathcal{T}$ in C; and becauſe of the Harmonicals ΣA, ΣC, ΣB, and $\Sigma\mathcal{T}$, ba parallel to $V\Sigma$ cuts the other three and is biſected by them in a, C and b[g], and the Points a and C being the ſame as before, the Point b is alſo the ſame, wherefore ba meets ΣB in b the Projection of B from the Point Σ; and becauſe of the Harmonicals VA, VC, VB, and $V\mathcal{T}$, ba is likewiſe biſected by VA, VC, and VB, in b, C and a, and the Points b and C being the ſame as in the former Diviſions, the Point a alſo remains the ſame, wherefore ba meets the Side of the Cone VB in a, and is therefore the Diagonal of the *Trapezium* $bAaB$; and ba being biſected in C, ba is a Diameter of the produced Section, and C is its Center: the reſt is evident.

[g] Lem. 7. B. III.

COR. 5.

If Σ be infinitely diſtant in the Line $V\mathcal{T}$, that is, if the Luminous Point be in the Directing Plane, and the Line $V\mathcal{T}$ be drawn parallel to the Direction of the Projecting Lines; then Bb being parallel to $V\mathcal{T}$, ba drawn through C will meet $V\mathcal{T}$ in a Point σ below or beyond V, and σV and $V\mathcal{T}$ will be equal; but the Section will ſtill be an Ellipſis.

Fig. 180. N°. 3. Draw Aa parallel to $V\mathcal{T}$, and produce $V\mathcal{T}$ to σ till σV be equal to $V\mathcal{T}$, and from σ through C draw σa cutting AV and Aa in b and a.

Then, becauſe $\sigma\mathcal{T}$ is biſected in V, $A\sigma$, AV, $A\mathcal{T}$, and Aa, are Harmonical Lines, wherefore σa is Harmonically divided by them in σ, b, C, and a; and $A\mathcal{T}$ which is alſo Harmonically divided, having its Point C in common with σa, and $\sigma\mathcal{T}$ and Aa which join their Points σ, $\mathcal{T}$, and A, a, being parallel, the Line Bb which joins their remaining Points B and b, is alſo parallel to them[a]; wherefore σC paſſes through b the Projection of B from the infinitely diſtant Point Σ: laſtly, VA, VC, VB, and $V\mathcal{T}$, being Harmonical Lines, σa is Harmonically divided by them in σ, b, C, and a, and the Points σ, b, and C, being here the ſame as before, the Point a is alſo the ſame; wherefore bC cuts the Side VB of the Cone in a, and ab is the Diagonal of the *Trapezium* $bAaB$, and conſequently the Section is ſtill an Ellipſis.

COR. 6.

The ſame Letters marking the ſame Things as before, let the Point $\mathcal{T}$ fall on the hither Side of the Cones Baſe.

Fig. 180. N°. 4. Then, if the Luminous Point Σ be ſo ſituated in the Line $V\mathcal{T}$, as that ΣB may be parallel to the Side VA of the Cone, the Diameter aC of the produced Curve will alſo be parallel to VA, and conſequently Indefinite at that Extremity which ſhould be determined by its Interſection with ΣB and VA, and the Section produced will be a *Parabola*.

Draw $C\sigma$ parallel to VA and ΣB, cutting VB and $V\Sigma$ in a and σ.

Then becauſe $\mathcal{T}B$ is Harmonically divided in $\mathcal{T}$, A, C, and B, $B\Sigma$, $C\sigma$, AV, and a Line through $\mathcal{T}$ parallel to them, are Harmonical Parallels; wherefore $\mathcal{T}\Sigma$ is alſo Harmonically divided in $\mathcal{T}$, V, σ, and Σ; and $\mathcal{T}B$ and $\mathcal{T}\Sigma$ being both Harmonically divided, and having their Point $\mathcal{T}$ in common, and the Line $C\sigma$ joining their Points C and σ, ΣA and VB which join their other Points, muſt croſs each other in the ſame Point a of $C\sigma$[b], wherefore $C\sigma$ parallel to VA paſſes through a the Projection of A from the Point Σ; and in regard that $V\mathcal{T}$, VA, VC, and VB, are Harmonical Lines, $C\sigma$ which is parallel to VA, is biſected by the other three in C, a, and σ[c]; now becauſe σ is the Point where the Tangents to the Section in the Extremities D and E of the double Ordinate DE which paſſes through C, meet the Indefinite Diameter aC produced beyond its Vertex a[d], and $a\sigma$ and aC being equal, the Section thus produced is therefore a *Parabola*[e], which may from theſe *Data* be deſcribed as formerly ſhewn[f].

COR. 7.

Fig. 180. N°. 5. If the Point Σ be ſo ſituated in the Line $\mathcal{T}V$, that ΣB may meet the Side VA of the Cone in a Point b beyond the Vertex V, the Diameter aC of the produced Section will paſs through the ſame Point b, and the Section will be a Portion of an *Hyperbola*.

Having drawn ΣB cutting AV produced beyond V, in b, draw Cb and $\mathcal{T}b$.

Then becauſe $b\mathcal{T}$, bA, bC, and bB, are Harmonical Lines, $\mathcal{T}\Sigma$ is Harmonically divided by them in $\mathcal{T}$, V, σ, and Σ; and $\mathcal{T}B$ and $\mathcal{T}\Sigma$ which are both Harmonically divided, having their Point $\mathcal{T}$ in common, ΣA and VB cut each other in the ſame Point a of the Line $C\sigma$[g], which Point is the Projection of A from the Point Σ: Now σ being the Point where the Tangents to the produced Section in the Extremities D and E of the double Ordinate DE which paſſes through C, meet the Diameter ab, and the Point σ falling between the Extremities a and b of that Diameter, and bC being Harmonically divided in b, σ, a, and C, by the Harmonicals $V\mathcal{T}$, VA, VC, and VB, the Section thus produced is therefore a Portion of an *Hyperbola*[h], which by the help of the Diameter ab, and of the double Ordinate DE, may be thence deſcribed[i].

SCHOL.

The Point $\mathcal{T}$ muſt always fall without the Baſe of the given Cone; for if $\mathcal{T}$ be within the Baſe, the Luminous Point will enlighten the whole Concave Surface of the

[a] Lem. 9. B. III.

[b] Lem. 9. B. III.

[c] Lem. 7. B. III.

[d] Cor. 2.

[e] Parab. 6. B. III.

[f] Prop. 17. B. III.

[g] Lem. 9. B. III.

[h] Hyp. Art. 26. B. III.

[i] Meth. 2. Prop. 18. B. III.

Plate 98. Book 6. Sect. 2

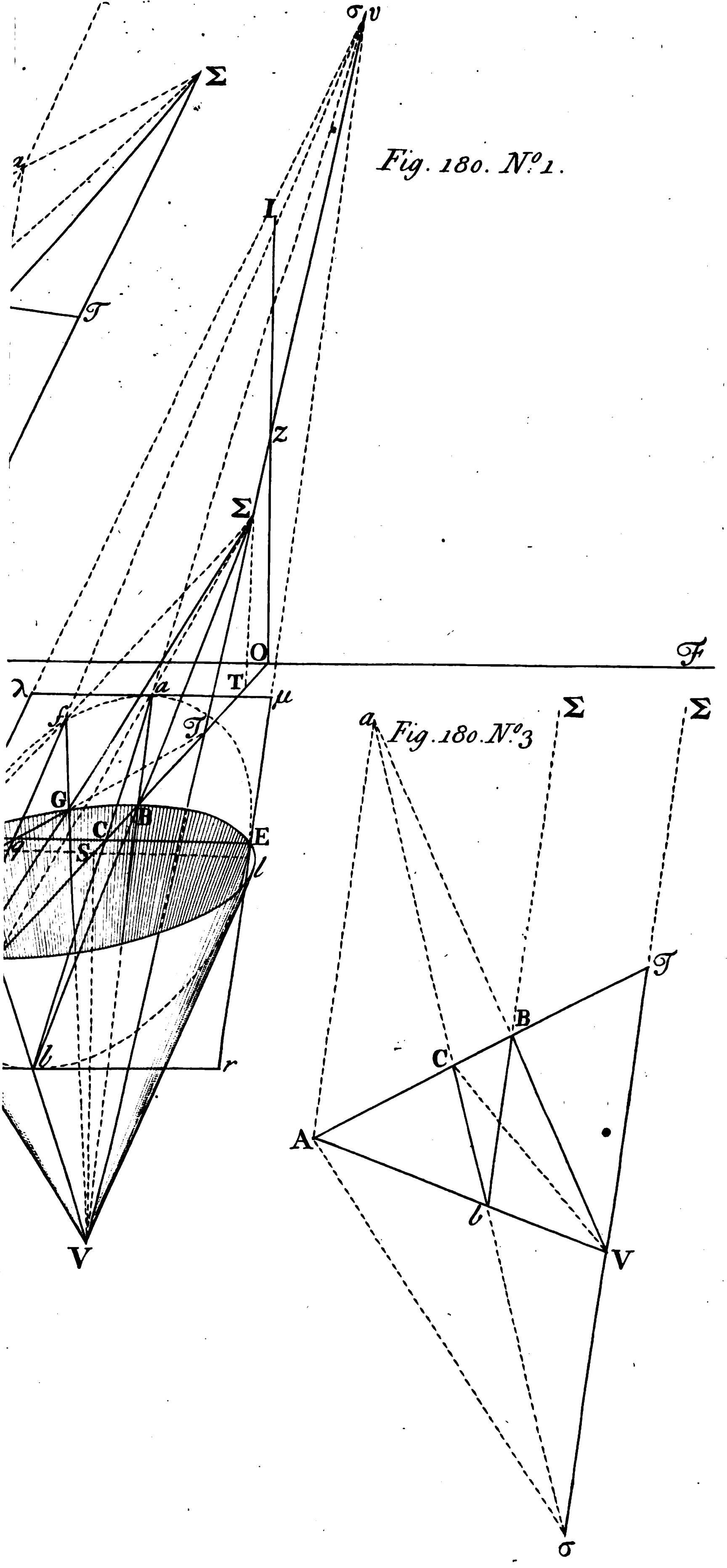


Fig. 180. No. 1.

Fig. 180. No. 3

J Mynde sc.

the Cone, ſo that no Part of the Baſe can be projected within that Surface, and the Light will therefore have no other Boundary but the Baſe of the Cone itſelf.

If the Point T lye beyond the Baſe of the Cone, the Point Σ muſt always fall ſomewhere between T and the Vaniſhing Point z of the Line VT, with which laſt Point it will coincide if the Luminous Point be ſuppoſed Infinitely diſtant before the Eye; the Part Tz of the Line VT, repreſenting the whole of the Original of VT indefinitely produced, which riſes above the Plane of the Baſe, and all other Points in VT below T or beyond z, repreſenting Points of its Original which lye under that Plane, and which can therefore project no Light into the Concave Surface of the Cone. Fig. 180. N°. 1.

If the Point T fall on the hither Side of the Baſe, then the Vaniſhing Point z of the Line VT muſt fall below the Plane of the Baſe, and the Point Σ can never appear between T and z, but may lye any where elſe in that Line, either above T or beyond z; thoſe Indefinite Parts of Tz repreſenting all that Part of the Original of TV indefinitely produced, which lies above the Plane of the Baſe, and Tz repreſenting the reſt of that Line indefinitely produced, which falls below that Plane; and if the Luminous Point be ſuppoſed Infinitely diſtant behind the Eye, the Point Σ will coincide with z. Fig. 180. N°. 4, 5.

If the Line VT be parallel to the Picture, the Point Σ may be any where in that Line produced beyond T, but cannot fall below T, ſeeing the Luminous Point muſt then be under the Plane of the Baſe. Fig. 180. N°. 2, 3.

Laſtly, if the Luminous Point be ſuppoſed either at a moderate or Infinite diſtance in the Directing Plane; the Directions of the Projecting Lines and of their Seats on the Plane of the Baſe being found, the Point T will be determined by the Interſection of VT drawn parallel to the Direction of the Projecting Lines, with A B drawn parallel to the Direction of their Seats on the Plane of the Baſe, through the Seat of V on that Plane[a]; and in either of theſe Caſes the Point Σ being infinitely diſtant, the Point V will biſect $T\sigma$[b].

Although when the Point T lies on the hither Side of the Baſe of the Cone (but in no other Situation of that Point) the Image of the Section produced may be a *Parabola* or *Hyperbola*, according as the Point Σ falls in the Line TV[c]; yet the Original of the Section is always an Ellipſis, except it ſhould happen to be a Circle, by cutting the Cone ſubcontrarily, which can never be unleſs the Cone be Scalene.

For although Σ ſhould be infinitely diſtant in the Line TV, yet bC muſt ſtill cut both Sides of the Cone, ſo long as VB makes an Angle with VT, that is, ſo long as T falls without the Baſe of the Cone; ſeeing VB muſt cut Aa parallel to VT in ſome Point a, through which Point bC alſo paſſes[d]. Fig. 180. N°. 3.

LEM. 11.

If any two Cones whoſe Baſes are in the ſame Plane, be ſimilar, and their Axes be parallel, then if any Diameter be drawn in the one Baſe, parallel to a Diameter in the other, the Sides of the Cones drawn to the Extremities of theſe Diameters will be reſpectively parallel.

Let ADBV and adb$v$ be the propoſed Cones, having their Baſes in the ſame Plane, and let their Axes SV and sv be parallel, and in the ſame Proportion to each other as the Diameters of their reſpective Baſes, by which means theſe Cones will be ſimilar. Fig. 181.

Draw any two Diameters AB, ab, parallel to each other, and the Sides VA, VB, and va, vb, of the given Cones; it muſt be ſhewn that VA is parallel to va, and VB to vb.

Dem. In the Triangles SVA, sva, the Sides SV, SA, being reſpectively parallel to sv, and sa, the Planes SVA, sva, are parallel[e], and the Angles VSA, vsa, are equal[f]; and SV being to SA as sv to sa, the Triangles SVA, sva, are ſimilar, and the Angles SAV, sav, equal, and conſequently va is parallel to VA[g].

The ſame may be ſhewn of the Sides vb, VB, or of any other two Sides of theſe Cones, terminated by the correſponding Extremities of parallel Diameters. *Q.E.D.*

P R O B. X.

The Center and Diſtance of the Picture, and the Image of a Cone, with the Vaniſhing Line of the Plane of its Baſe, and the Vaniſhing Point of its Axe, being given; thence to find the Place of the Vaniſhing Points of all the Sides of that Cone.

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[a] Caſe 3. and 4. Prob. 1. B. V.

[b] Cor. 5.

[c] Cor. 6, 7.

[d] Cor. 5.

[e] 15 El. 11.

[f] 10 El. 11.

[g] Lem. 1. B. I.

Fig. 182. Let O be the Center, and O I the Diſtance of the Picture, D A E V the given Cone, EF the Vaniſhing Line of the Plane of its Baſe, and x the Vaniſhing Point of its Axe V S.

Through S the apparent Center of the Baſe, draw A B, D E, repreſenting two Diameters of the Baſe, the one perpendicular and the other parallel to the Interſecting Line of its Plane; through O the Vaniſhing Point of A B draw O x, and through x draw e d Parallel to DE or EF; then draw the Sides VA, V B, of the Cone, cutting x O in a and b, and the Sides V D, V E, cutting e d in d and e: Conſider a b and d e as the Images of two Diameters of a Circle in the Plane EF parallel to the Originals of A B and D E, and by their help compleat the Image a d b e of that Circle[a], and the Curve thus formed will be the Place of the Vaniſhing Points of all the Sides of the given Cone D A E V.

Dem. For x and O being the Vaniſhing Points of S V and A B, x O is the Vaniſhing Line of the Plane AV B, and conſequently a and b are the Vaniſhing Points of VA and V B; and D E being parallel to EF, x e parallel to them is the Vaniſhing Line of the Plane D V E, wherefore d and e are the Vaniſhing Points of V D and V E: Now x being the Vaniſhing Point of V S, it is alſo the Image of the Interſection of the Plane EF with a Line drawn from the Eye parallel to the Original of V S[b], and for the ſame Reaſon, a, b, d, and e, are the Images of the Interſections of the Plane EF with Lines drawn from the Eye parallel to the Originals of VA, V B, V D, and V E, reſpectively; wherefore the Originals of a b and d e (taken as Lines in the Plane EF) are parallel to the Originals of A B and D E, and conſequently a b and d e repreſent two Diameters of a Circle in the Plane EF, the one perpendicular and the other parallel to its Interſecting Line, on which Circle a Cone being formed from the Eye as the Vertex[c], it will be ſimilar to the given Cone D A E V, and have its Axe parallel to the Original of V S; wherefore as all the Sides of this Cone will be parallel to the correſponding Sides of the Cone D A E V[d], every Point in the Image of the Circle a d b e formed by any Line drawn from the Eye, will be the Vaniſhing Point of the correſponding Side of the Cone D A E V, and conſequently the Curve produced by the help of a b and d e, will be the Place of the Vaniſhing Points of all the Sides of the Cone propoſed. *Q. E. I.*

S C H O L.

By the Poſition of the Points a and b with reſpect to x and to the Vaniſhing Line EF, it will be eaſy to determine which of the Conick Sections the Image of the Circle will be.

For if a and b lie both on the ſame Side of EF, the Image of the Circle will be an Ellipſis or a Circle: If the Point a or b be infinitely diſtant, that is, if VA or VB be parallel to x O, the Image will be a *Parabola*; and if a lye on the oppoſite Side of EF from b, the Section will be two oppoſite *Hyperbola's*[e], either of which Sections may be drawn from theſe *Data* by the Methods formerly propoſed[f]: And laſtly, if the Point x be infinitely diſtant, by which means d e can have no Repreſentation, the Center of the *Hyperbola's* then formed will be at O, whence their ſecond Diameter conjugate to the Diameter a b may be determined[g], whereby thoſe Sections may be deſcribed.

C O R. 1.

If through S any Diameter F G of the Baſe be drawn, and produced till it cut the Vaniſhing Line EF in w, draw wx, and that will cut the Curve of the Vaniſhing Points in f and g the Vaniſhing Points of the Sides V F and V G of the given Cone.

For F G and f g having the ſame Vaniſhing Point w, they repreſent parallel Diameters of the Baſe of the given Cone, and of the Circle in the Original Plane which produces the Vaniſhing Points of its Sides, wherefore Lines from the Eye to g and f being parallel to the Originals of the Sides V G and V F of the given Cone[h], g and f are the Vaniſhing Points of thoſe Sides.

And hence if wx be drawn, G V and F V will cut it in the ſame Points g and f, and thereby determine the Vaniſhing Points of thoſe Sides, without the Trouble of drawing the Curve a d b e.

If x be infinitely diſtant, wx muſt be drawn parallel to a b, which in this Caſe will be parallel to V S the Axe of the Cone, that Axe being then parallel to the Picture.

C O R. 2.

If from any Point y in the Vaniſhing Line EF any Vaniſhing Line $y\,p$ be drawn

neither

[a] Prob. 24. B. II.

[b] Th. 18. B. I.

[c] Prop. 9. B. IV.

[d] Lem. 11.

[e] Cor. 1, 2, 3. Caſe 2, Prop. 24. B. IV.

[f] Prop. 16, 17, 18. B. III.

[g] Caſe 2. Prob. 10. B. III.

[h] Lem. 11.

Fig. 181.

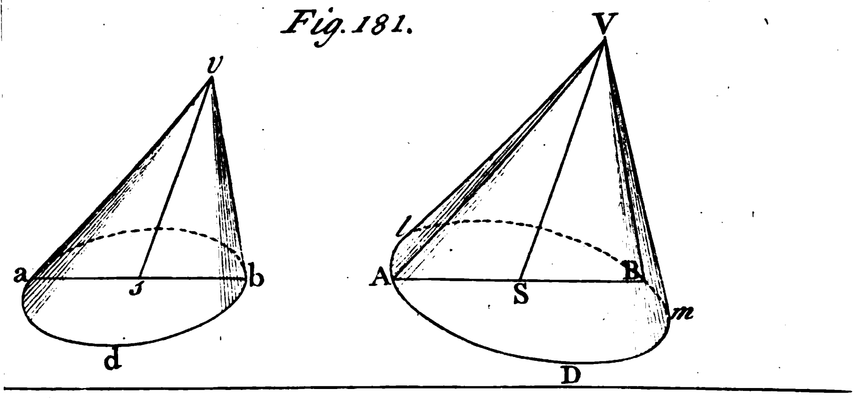
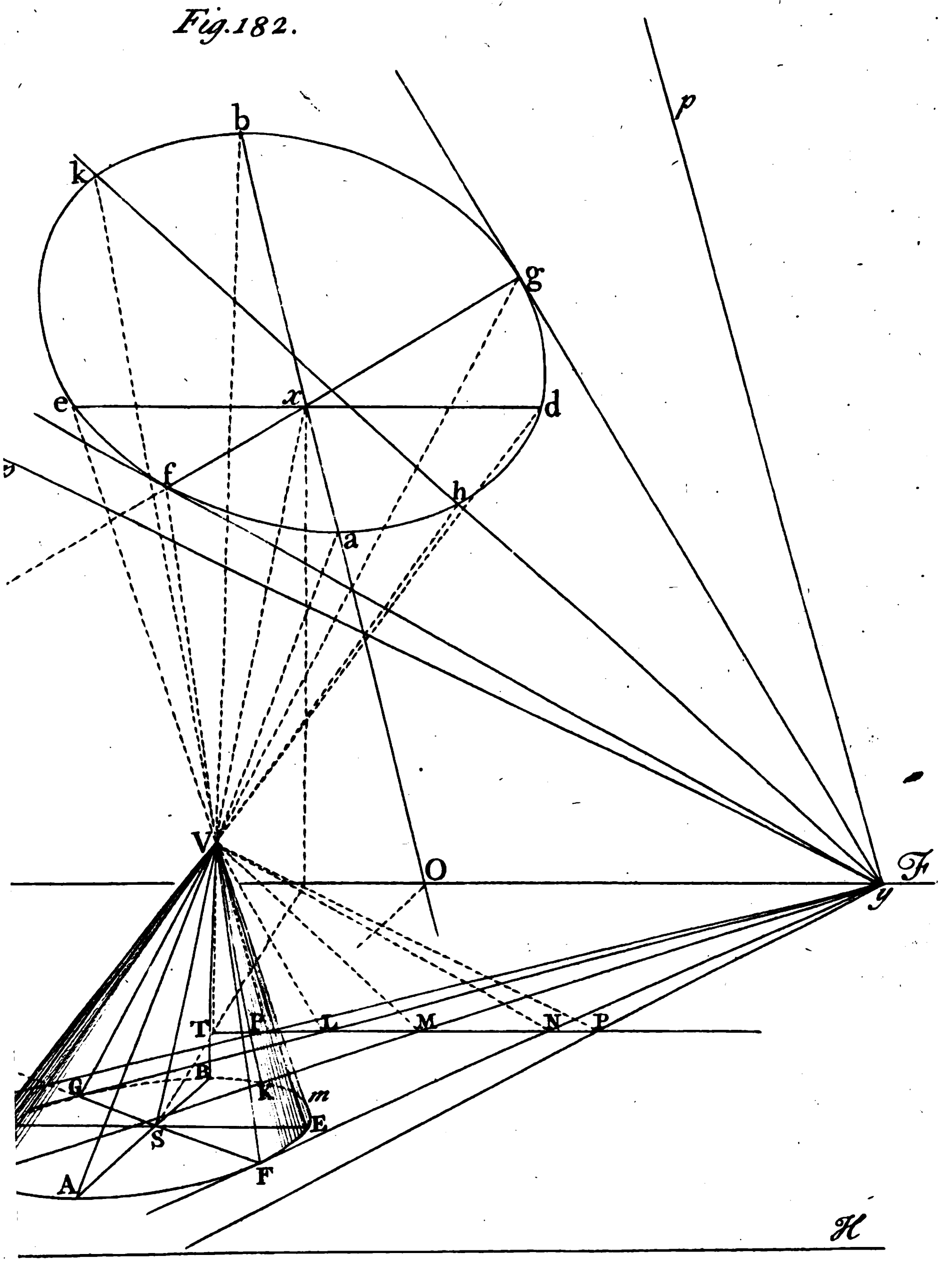


Fig. 182.



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neither touching nor cutting the Curve of the Vaniſhing Points, it will be a Vaniſhing Line of Planes, the Section of any of which with the Original of the given Cone D A E V, will be an Ellipſis or a Circle; and if from y through P the Parallel Seat of V on the Plane EF with reſpect to the cutting Plane yp, a Line y P be drawn, it will neither touch nor cut the Baſe A D B E.

For no Side of the given Cone being parallel to the cutting Plane, in regard none of them have their Vaniſhing Point in yp, the Section is therefore either an Ellipſis or a Circle [a]; and Py being the Interſection of the Plane EF with a Plane paſſing through V parallel to the cutting Plane [b], the Plane pyP can only touch the Cone in its Vertex V, and can therefore neither touch nor cut its Baſe.

C O R. 3.

If any Vaniſhing Line y f be drawn touching the Curve of the Vaniſhing Points in f, it will be a Vaniſhing Line of Planes, the Section of any of which with the Original of the given Cone will be a *Parabola*; and if through N the Parallel Seat of V on the Plane EF with reſpect to the cutting Plane y f, a Line y N be drawn, it will be a Tangent to the Baſe of the Cone.

For the cutting Plane y f will be parallel to the Original of the Side V F of the Cone whoſe Vaniſhing Point is f, and to none other of its Sides [c]; and the Plane f y F which paſſes through V parallel to the cutting Plane, muſt therefore touch the Cone in its Side V F, and conſequently F y the Interſection of that Plane with the Plane EF muſt touch the Baſe of the Cone in F.

The ſame is to be underſtood of the Vaniſhing Line y g, and of the Interſection y G of the plane EF with a Plane paſſing through V parallel to the cutting Plane y g.

C O R. 4.

If any Vaniſhing Line y k be drawn cutting the Curve of the Vaniſhing Points in h and k, it will be a Vaniſhing Line of Planes, the Section of any of which with the Original of the given Cone produced beyond its Vertex, will be two oppoſite *Hyperbola's*; and if through M the Parallel Seat of V on the Plane EF with reſpect to the cutting Plane y k, a Line yM be drawn, it will cut the Baſe of the Cone.

For in this Caſe, the cutting Plane muſt be parallel to the Originals of the Sides H V and K V of the Cone whoſe Vaniſhing Points are h and k, and muſt therefore cut both the given Cone and its oppoſite, and thereby form two oppoſite *Hyperbola's* [d], and the Plane which paſſes through V parallel to the cutting Plane, muſt therefore paſs through the Sides V H, V K, of the Cone, and conſequently cut its Baſe in H and K.

S C H O L.

If any ſtraight Line cut the Curve of the Vaniſhing Points in one Point, it will alſo cut it in another, except when that Curve is either a *Parabola*, and the propoſed Line is one of its Diameters, or elſe when the Curves are two oppoſite *Hyperbola's*, and the propoſed Line is parallel to one of their *Aſymptotes*, in which Caſes that Line can only cut the Curve or Curves in one Point. Nevertheleſs, in either of theſe Caſes, the Original of the Section produced will be two oppoſite *Hyperbola's* [e].

P R O B. XI.

The Center and Diſtance of the Picture, and the Image of a Cone, with the Vaniſhing Line of the Plane of its Baſe, being given; thence to find the Image of the Section of that Cone by any given Plane, whoſe Interſection with the Plane of the Baſe is given.

C A S E 1.

When the Vaniſhing Lines of the Plane of the Baſe and of the cutting Plane interſect.

Let O be the Center and O I the Diſtance of the Picture, F A G V the given Cone Fig. 183.
and EF the Vaniſhing Line of the Plane of its Baſe; and let it be propoſed to deſcribe N°. 1.
the Section of this Cone by a Plane whoſe Vaniſhing Line is yz, and its Interſection with the Plane EF is yP.

In the firſt Place, it is evident that if V be conſidered as a Projecting Point, or the Eye of a Spectator ſtanding on the Plane of the Baſe A D B E, the Projection or Image of this Baſe on the Plane zyP from the Point V, will repreſent the Section of the given Cone by that Plane [f].

[a] Con. Sec. Art. 8. B. III.

[b] Meth. 5. Prob. 6. B. V.

[c] Con. Sec. Art. 10. B. III.

[d] Con. Sec. Art. 13. B. III.

[e] Prop. 15. B. III. and Gen. Cor.

[f] Con. Sec. Art. 3. B. III.

How this Projection may be found, ſhall be ſhewn in the following Methods.

METHOD 1.

Having found T the Parallel Seat of the Vertex V on the Plane EF with reſpect to the cutting Plane zyP, draw Tw parallel to yP, and through S the apparent Center of the Baſe, draw any Diameter F G cutting Tw in t; find AB the Chord of the Tangents to the Baſe from t[a], cutting Py and Tw in p and q, and having found DE the Chord of the Tangents from q (which will alſo paſs through t[b],) cutting AB in C, draw the Sides AV, BV, DV, EV, of the Cone, and the Line CV; then having drawn Vt and Vq, through p draw pa parallel to Vq, cutting VA, VC, and VB, in a, c, and b; through a, c, and b, draw nr, de, and $\lambda\mu$, parallel to Vt, and through d and e the Interſections of de with VD and VE, draw $n\lambda$ and $r\mu$ parallel to Vq, and thereby a Parallelogram $nr\lambda\mu$ will be formed, within which a Curve being drawn in the uſual Manner[c], it will be the Projection of the Baſe on the Plane zyP from the Point V, and conſequently the Section of the given Cone by the Plane propoſed.

[a] Cor. 3. Prob. 3. B. III.
[b] Cor. 2. Lem. 14. B. III.
[c] Meth. 2. Prop. 16. B. III.

Dem. From t and q, draw the Tangents tA, tB, and qD, qE, to the given Baſe, forming by their mutual Interſections a *Trapezium* LMNR.

Then becauſe Tw is the Interſection of the Plane EF with a Plane paſſing through V and the Directing Line of the cutting Plane zyP, the Projections of all Lines in the Plane EF which meet in any Point t of the Line Tw are parallel to Vt[d], and the Projections of thoſe which meet in q are parallel to Vq; wherefore p being the Interſection of AB with the cutting Plane, and conſequently a Point of the Projection of that Line, pa parallel to Vq is its indefinite Projection, and a, c, and b, where pa cuts the Projecting Lines VA, VC, and VB, are therefore the Projections of A, C, and B; wherefore alſo nr, de, and $\lambda\mu$, drawn through a, c, and b, parallel to Vt, are the indefinite Projections of tA, tC, and tB; and the Points d and e where de cuts the Projecting Lines VD and VE, being therefore the Projections of D and E, $n\lambda$ and $r\mu$ drawn through d and e, parallel to Vq, are the Projections of qD and qE; and conſequently the Parallelogram $nr\lambda\mu$ is the Projection of the *Trapezium* NRLM, the Sides of which being Tangents to the Baſe in A, D, B, and E, the Sides of the Figure $nr\lambda\mu$ are alſo Tangents to the Curve formed by the Projection of the Baſe, in a, d, b, and e; wherefore the Curve inſcribed in the Figure $nr\lambda\mu$ as above directed, is the Projection of the Baſe ADBE on the Plane zyP from the Point V[e], and conſequently the Section of the given Cone by the Plane propoſed. *Q. E. I.*

[d] Meth. 7. Prob. 6. B. V.
[e] Con. Sec. Art. 19. B. III.

COR. 1.

The Projections ab and de of AB and DE, are two Conjugate Diameters of the Section.

For nr and $\lambda\mu$ which touch the Section in a and b, being parallel, ab which joins the Points of Contact is a Diameter; and for the like Reaſon de is alſo a Diameter of the Section, and being parallel to the Tangents nr and $\lambda\mu$, it is therefore a Diameter Conjugate to ab.

That ab and de biſect each other in c, is thus ſhewn.

Becauſe qA is Harmonically divided in q, B, C, and A, Vq, VB, VC, and VA, are Harmonical Lines, wherefore ab which is parallel to Vq, one of theſe Harmonicals, is biſected by the other three in a, c, and b.

Likewiſe becauſe tE is Harmonically divided in t, D, C, and E, Vt, VD, VC, and VE, are Harmonical Lines, wherefore de parallel to Vt is biſected by VD, VC, and VE, in d, c, and e.

COR. 2.

If from t any Line be drawn within the Angle BtA, it will cut the Baſe in two Points, and the Projection of that Line ſo terminated, will be a double Ordinate to the Diameter ab of the Section.

For every Line ſo drawn from t, being Harmonically divided by t and its Interſections with the Baſe and the Line AB[f], its Projection will be parallel to Vt, and conſequently to the Tangent nr, and biſected by the Diameter ab.

[f] Ellip. Art. 11. B. III.

The ſame is to be underſtood of all Lines drawn from q, within the Angle DqE, and terminated by the Baſe; the Projections of which Lines will be double Ordinates to the Diameter de of the Section.

Fig.183. N.º 1.

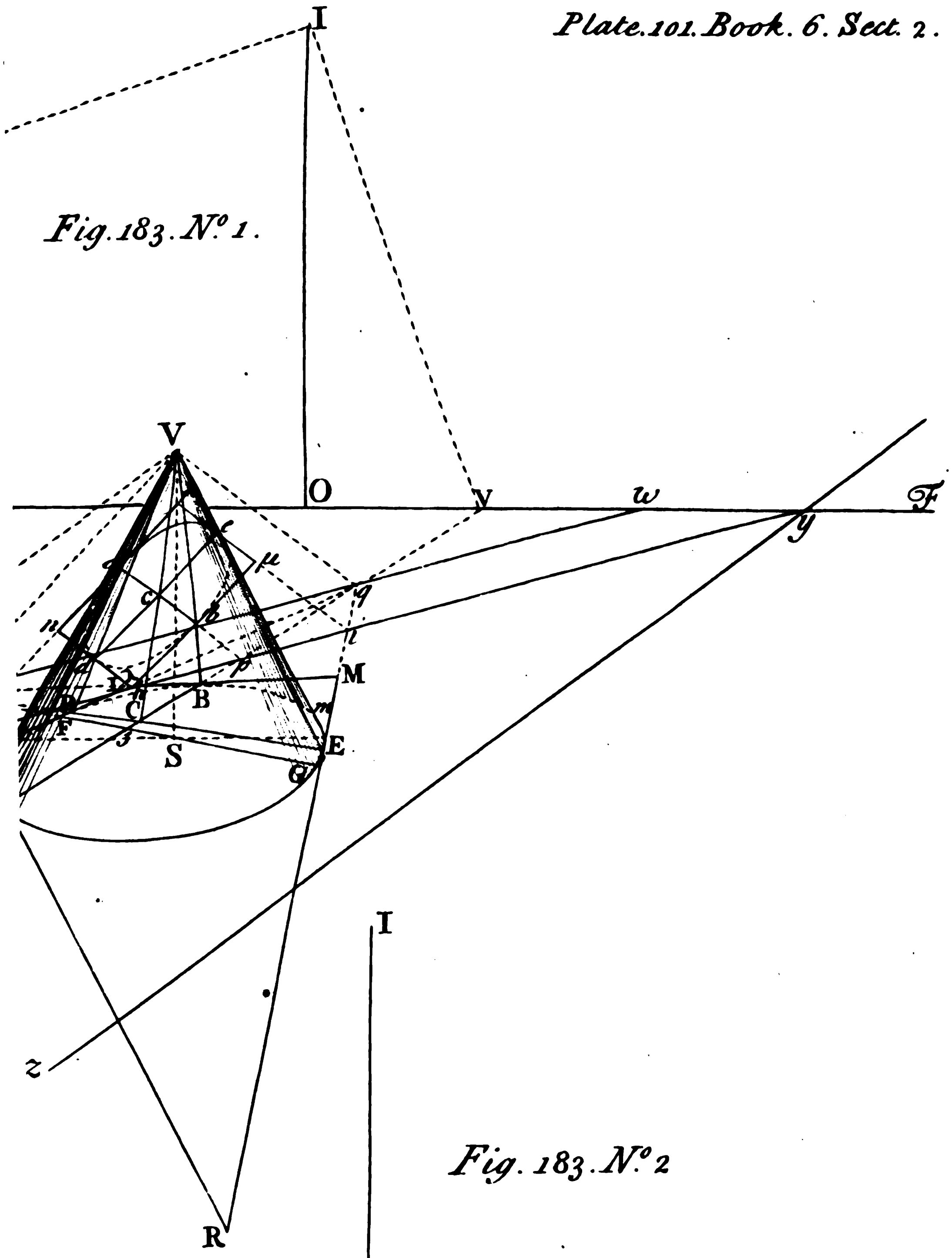
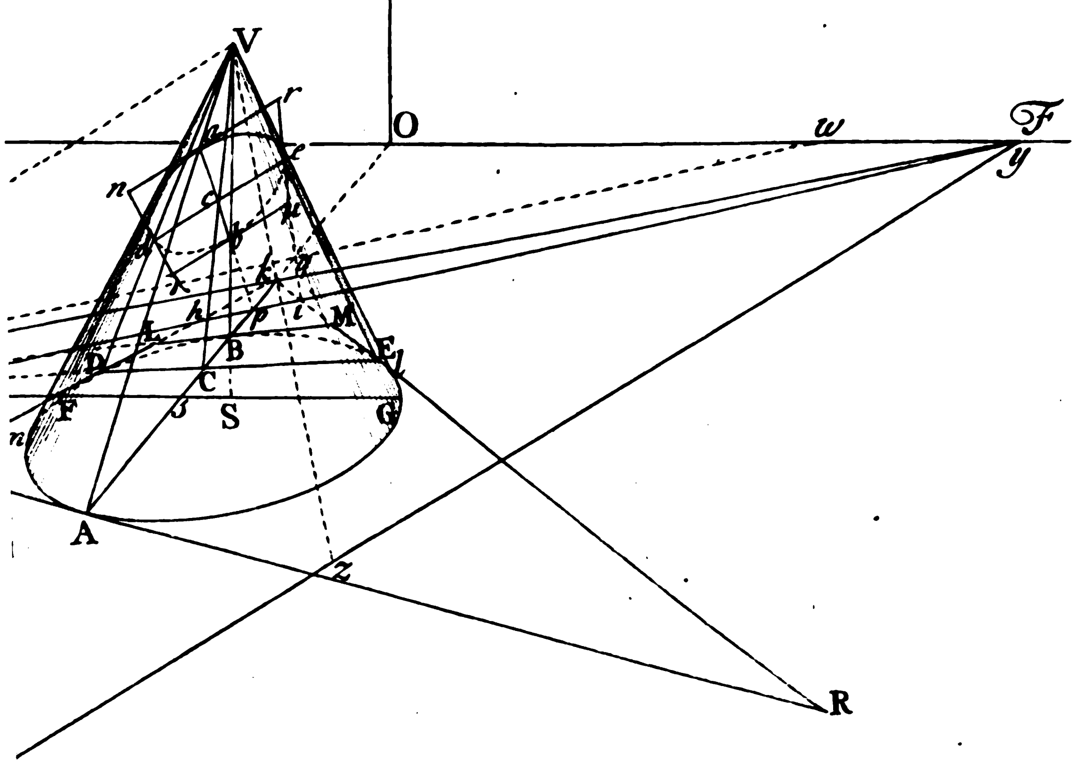


Fig. 183. N.º 2



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COR. 3.

The Tangents $n\lambda$ and μr of the Section may be found without the Trouble of determining the Diameter de, by drawing them parallel to Vq, through b and i the Interſections of the Tangents qD and qE with the Line Py.

For it is evident, b and i are Points of the Projections of qD and qE.

COR. 4.

Fig. 183. N°. 2.

If the Point t coincide with T, then TV being parallel to the Vaniſhing Line zy of the cutting Plane, the Tangents nr and $\lambda\mu$, at the Extremities of the Diameter ab, and conſequently all the Ordinates to that Diameter, will alſo be parallel to zy.

Here, the Diameter FG which paſſes through T, being parallel to the Picture, the Vaniſhing Point of AB the Chord of the Tangents from T, is at O the Center of the Vaniſhing Line EF.

COR. 5.

Fig. 183. N°. 3.

If the Point t be infinitely diſtant, that is, if the Diameter FG of the Baſe be drawn parallel to Py, and conſequently to Tw; the Chord AB of the Tangents from that infinitely diſtant Point, muſt biſect FG in s, and the Tangents to the Baſe in A and B will be parallel to Py[a], and the Projections of thoſe Tangents, and of all other Lines drawn through the Baſe parallel to Py, will alſo be parallel to that Line[b].

METHOD 2.

Fig. 183. N°. 2.

The ſame things being ſuppoſed as before; from T draw Ty cutting AB in k, and find DE the Chord of the Tangents from k paſſing through t[c], and cutting AB in C; and having drawn Vk till it cut the Vaniſhing Line zy in z, draw zp cutting VA, VC, and VB, in a, c, and b; and nr, de, and $\lambda\mu$, being drawn through a, c, and b, parallel to Vt, and de being terminated in d and e by the Sides VD and VE of the Cone as before, from z draw zd, ze, which by their Interſections with nr and $\lambda\mu$, will form a *Trapezium* $nr\lambda\mu$, within which a Curve being deſcribed in the uſual manner[d], it will be the Section required.

Dem. Becauſe Ty is the Interſection of the Plane EF with a Plane paſſing through V parallel to the cutting Plane zyP, the imaginary Projection of Ty coincides with the Vaniſhing Line zy[e], wherefore Vk cuts zy in z the Vaniſhing Point of the Projection of AB, and conſequently ab drawn from z through p, is the Projection of that Line; and a and b being the Projections of A and B, and nr and $\lambda\mu$ the Projections of the Tangents tA and tB as before, ab is therefore a Diameter of the Section, to which de the Projection of DE is a double Ordinate[f]: but becauſe DE is the Chord of the Tangents to the Baſe from k, the Projections of the Tangents kD and kE muſt paſs through d and e, and alſo through z the Projection of k; wherefore zd and ze are the Projections of kD and kE, and conſequently Tangents to the Section in the Extremities d and e of the double Ordinate de to the Diameter ab; wherefore the Figure $nr\lambda\mu$ being the Projection of the *Trapezium* NRLM formed by the Tangents to the Baſe from t and k, the Curve inſcribed in the Figure $nr\lambda\mu$ in the manner above directed, is the Section required. *Q. E. I.*

COR. 1.

The Tangents $n\lambda$ and $r\mu$ may be alſo found, by drawing them from z through b and i the Interſections of kD and kE with Py[g].

COR. 2.

If t coincide with T, then nr and $\lambda\mu$ being parallel to zy[h], and z being the Vaniſhing Point of $n\lambda$ and $r\mu$, the Figure $nr\lambda\mu$ will repreſent a Parallelogram in the Plane zyP, whoſe Sides nr and $\lambda\mu$ are parallel to the Picture.

COR. 3.

Fig. 183. N°. 3.

If the Point t be infinitely diſtant[i], and the Projections of the Tangents from the Point k of the Chord AB be uſed; the Figure $nr\lambda\mu$ will repreſent a *Trapezium* in the Plane zyP, whoſe Sides nr and $\lambda\mu$ have the ſame Directing Point with yP, and whoſe other Sides $n\lambda$ and $r\mu$ have z for their Vaniſhing Point; nevertheleſs in the Subdiviſion of that Figure, although the Sides nr and $\lambda\mu$ do not repreſent Lines parallel to the Picture, yet they muſt be divided Geometrically in the true Proportion

[a] Cor. 2. Prob. 3. B. III.

[b] Cor. 3. Meth. 7. Prob. 6. B. V.

[c] Cor. 2. Lem. 14. B. III.

[d] Meth. 3. Prop. 16. B. III.

[e] Cor. 2. Meth. 5. Prob. 6. B. V.

[f] Cor. 1. and 2. Meth. 1.

[g] Cor. 3. Meth. 1.

[h] Cor. 4. Meth. 1.

[i] Cor. 5. Meth. 1.

of the Sides of a Square circumſcribing a Circle, and not Stereographically from their Vaniſhing Points, as the other Sides $n\lambda$ and $r\mu$ muſt be, from their Vaniſhing Point z.

For the Tangents nr and $\lambda\mu$ being parallel to Py, and in the ſame Plane, their Originals muſt all have the ſame Directing Point[a], which is likewiſe the ſame with the Directing Point of Tw: and therefore, that the Subdiviſions of the Original of the Figure $nr\lambda\mu$ may paſs through the proper Points of the Original forming Circle of the Baſe, it is requiſite that the Originals of the Sides nr and $\lambda\mu$ be ſo divided, as that their Images or Projections nr and $\lambda\mu$ may be really divided in the true Proportion above-mentioned[b].

And for the ſame reaſon, if the Projections of the Tangents from q be uſed inſtead of thoſe from k[c], they will be parallel to Vq, and muſt be divided as the Tangents nr and $\lambda\mu$; the imaginary Projection of the Point q being the Directing Point of thoſe Tangents.

The ſame is to be underſtood of the Projections of Tangents to the Baſe from any Point whatſoever in the Line Tw, and ſhews the Coincidence of Method 1, with the Rules of Stereography.

COR. 4.

Fig. 183. N°. 4. If through S any apparent Diameter FG of the Baſe be drawn, cutting Tw in t, and AB the Chord of the Tangents from t be found, and thence its Indefinite Projection ab[d]; from v the Vaniſhing Point of AB, draw vV cutting ab in o, and having found fg the Projection of FG, of and og will be Tangents to the Section in f and g, and form with the Tangents $\lambda\mu$ and nr at the Extremities of the Diameter ab (if within reach) a *Trapezium* $\lambda\mu nr$, by the help of which the Section may be deſcribed.

For o being the Projection of the Vaniſhing Point v of the Line AB, of and og are the Projections of vF and vG the Tangents to the Baſe from v.

The Tangents of and og may be alſo found, by drawing them from o, through b and i the Interſections of vF and vG with Py.

COR. 5.

If from y through T the Parallel Seat of V on the Plane zyP with reſpect to the Plane of the Baſe, a Line yT be drawn, it will be the Line of the *Foci* of the Projections of all Lines in the Plane of the Baſe on the cutting Plane[e], in which Line the Projection o of any Vaniſhing Point v in the Line EF therefore lies, and may be found by the Interſection of Vv with yT, whence po may be determined, as well as by either of the former Methods.

COR. 6.

Fig. 183. N°. 5. If from t (here ſuppoſed to coincide with w the Vaniſhing Point of Tw) any Line DE be drawn within the Angle BtA, cutting the Baſe in D and E, and the Chord AB of the Tangents from t, in C; find the Point Γ in AB, where the Tangents in D and E meet that Line, by taking it in ſuch manner, that ΓA may be Harmonically divided in Γ, B, C, and A; then ΓV being drawn, it will cut ab the Projection of AB, in a Point γ through which the Projections λn and μr of the Tangents ΓD and ΓE muſt paſs; by the help of which, and of the Tangents $\mu\lambda$ and nr at the Extremities of ab, a Figure will be formed whereby the Section may be deſcribed.

For it is evident that γ is the Projection of Γ.

In this Figure, DE being Part of the Line Tw, it has no Projection; nevertheleſs the Projections $n\lambda$ and $r\mu$ are obtained, either by drawing them through γ parallel to VD and VE, or through b and i the Interſections of ΓD and ΓE with Py.

COR. 7.

Fig. 183. N°. 6. If Py cut the Baſe in D and E; find FG the Chord of the Tangents from y, which will be an apparent Diameter of the Baſe, and in it the Point Γ where the Tangents in D and E meet; then biſect DE in C, and draw ΓC cutting the Baſe in A and B, and having found ab the Projection of AB[f], draw ΓV cutting it in γ; laſtly, through a and b draw Parallels to Py, which by their Interſections with γD and γE, will form a Figure $nr\lambda\mu$, by the help of which the Section may be deſcribed[g].

Here, it is evident that AB is a real Diameter of the Baſe, as well as the Original of a Diameter of the Section, and that DE is a double Ordinate common to both theſe Diameters, DE being its own Projection.

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[a] Cor. 5. Theor. 12. B. I.

[b] Cor. 4. Meth. 2. Prop. 16. B. III.

[c] Meth. 1.

[d] Meth. 1. or 2.

[e] Cor. 1. Meth. 4. Prob. 6. B. V.

[f] Meth. 1.

[g] Cor. 5. Meth. 1. and Cor. 6. Meth. 2.

Fig. 183 No. 7.

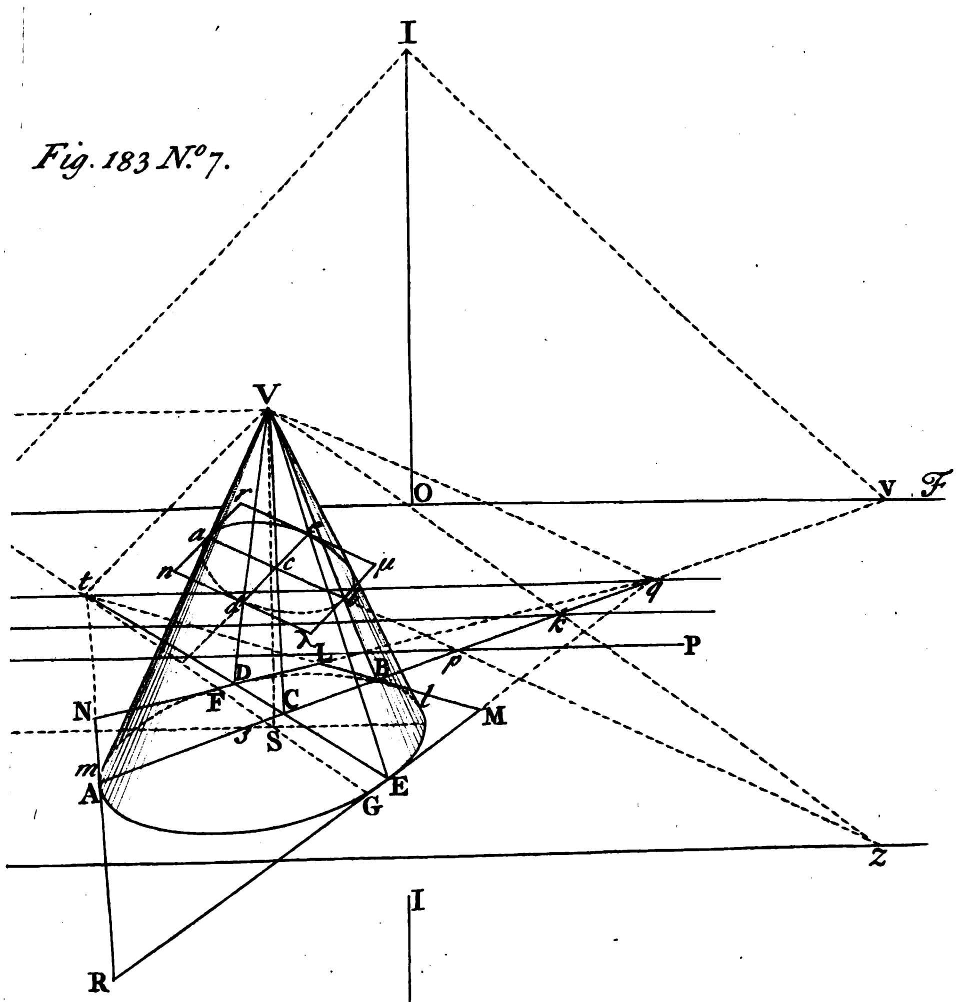
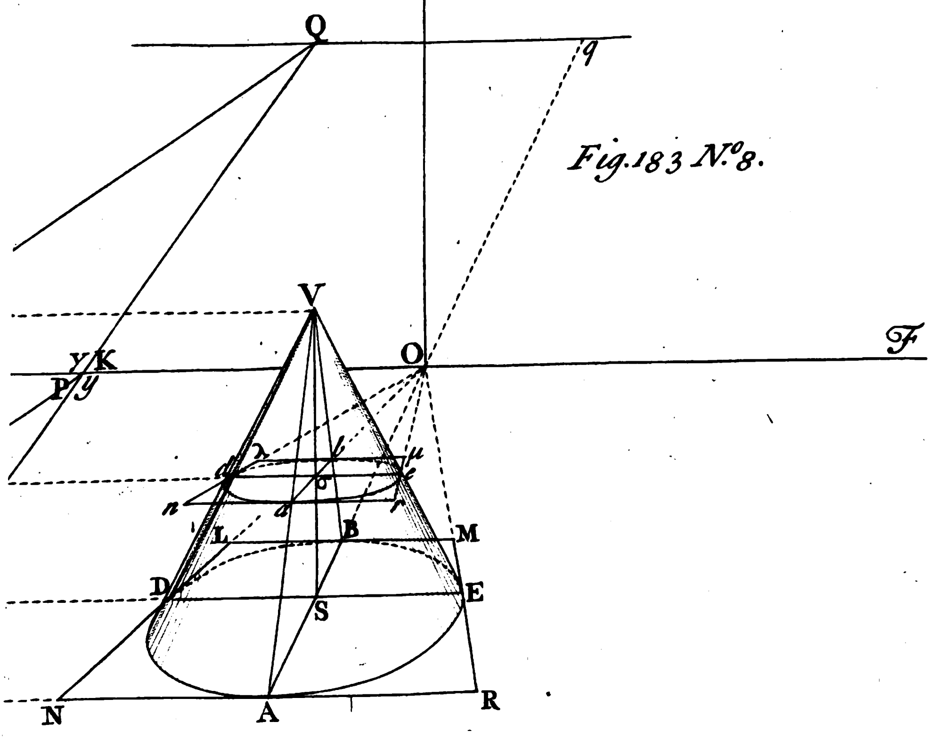


Fig. 183 No. 8.



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Here, the Figure nrDE incloſes all that Part of the Section which lies above the given Baſe, and is to be conſidered as repreſenting one Moiety of a Square circumſcribing a Circle, whoſe Diameter is repreſented by DE; but the other Moiety of that Square which lies below the Plane of the Baſe, cannot be expreſſed in this Figure, the Extremity b of the Diameter ab of the Section, through which the Side $\lambda\mu$ of the *Trapezium* oppoſite to nr, ought to paſs, being here out of reach.

GENERAL COROLLARY 1.

If the Line Tw neither touch nor cut the given Baſe, the Section produced will be an Ellipſis or a Circle.

Fig. 183. N°. 1, 2, 3, 6.

For in this Caſe Vq falling wholly without the Triangle AVB, and the Diameter ab of the Section being parallel to Vq[a], it muſt neceſſarily cut both Sides VA, VB, of the Cone on the ſame Side of its Vertex V, and thereby become a determinate Diameter of the Section; and as the Projections of all Lines drawn from t within the Angle BtA, and terminated by the Baſe, are double Ordinates to the Diameter ab[b], which they muſt cut between its Extremities a and b, the Section thus produced muſt be a Curve terminated every where, and conſequently an Ellipſis, if it ſhould not happen to be a Circle by ſubcontrary Section.

The produced Section may be alſo proved to be an Ellipſis or a Circle, by the Place of the Point of Concourſe of the Tangents at the Extremities of any double Ordinate, with its proper Diameter.

Fig. 183. N°. 2, 3.

For kA being Harmonically divided in k, B, C, and A, Vk, VB, VC, and VA, are Harmonical Lines, and the Diameter ab which is parallel to none of theſe, therefore cuts them all four, and is Harmonically divided by them in z, b, c, and a[c]; but z is the Point of Concourſe of the Tangents zd, ze, with the Diameter ab produced without the Section, de is therefore a double Ordinate to the Diameter ab of an Ellipſis[d]: the ſame may be ſhewn of all other Ordinates to the Diameter ab, and the Tangents at their Extremities; wherefore the Section thus produced is an Ellipſis.

Note, *The Section cannot be a Circle unleſs the Ordinate* de *be perpendicular to the Diameter* ab, *and then only when the Diameter* ab *is equal to its Conjugate.*

GENERAL COROLLARY 2.

If Tw touch the given Baſe, the Section produced will be a *Parabola*.

Fig. 183. N°. 4.

For if Tw be a Tangent to the Baſe in any Point A, the Chord AB of the Tangents to the Baſe from any Point t in the Line Tw, muſt paſs through A, and in this Caſe A and q coinciding, Vq coincides with the Side VA of the Cone, to which the Diameter ab of the produced Section muſt be therefore parallel[e]; wherefore the Extremity a of that Diameter which ought to be determined by its Interſection with VA, being infinitely diſtant, the Diameter ab is Indefinite towards a; and as all Diameters of the Section muſt in this Caſe be parallel to VA, in regard their Originals paſs through the ſame Point A of the Baſe, and muſt each of them be Indefinite towards that End which ought to be determined by VA, the Section thus produced is therefore a *Parabola*[f].

This Section may be alſo proved to be a *Parabola*, by the Place of the Point of Concourſe of the Tangents at the Extremities of any double Ordinate, with its proper Diameter.

Having found fg the Projection of FG, and drawn vV cutting the Indefinite Diameter ab in o, draw the Tangents of and og[g].

Then, becauſe vA is Harmonically divided in v, B, s, and A, Vv, VB, Vs, and VA, are Harmonical Lines, wherefore ab which is parallel to VA, one of theſe Harmonicals, is biſected by the other three in o, b, and σ; conſequently o being the Point where the Tangents of and og at the Extremities of the double Ordinate fg, meet its proper Diameter ab beyond its Vertex b, and ob and $b\sigma$ being equal, fg is a double Ordinate to a Diameter ba of a *Parabola*[h].

The ſame may be ſhewn of all other Ordinates to the Diameter ba, and of the Tangents at their Extremities; wherefore the Section thus produced is a *Parabola*, which by theſe *Data* may be deſcribed[i].

GENERAL COROLLARY 3.

If Tw cut the given Baſe, the Sections produced will be oppoſite *Hyperbola's*.

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[a] Meth. 1.

[b] Cor. 2. Meth. 1.

[c] Lem. 8. B. III.

[d] Ellip. Art. 11. B. III.

[e] Meth. 1.

[f] Parab. 3. B. III.

[g] Cor. 4. Meth. 2.

[h] Parab. Art. 6. B. III.

[i] Meth. 1. Prop. 17. B. III.

Fig. 183. N°. 5. For if Tw cut the Baſe in E and D, AB the Chord of the Tangents to the Baſe from any Point t in the Line Tw, muſt cut that Line in ſome Point q between E and D within the Baſe, and the Diameter ab formed by the Projection of AB, being parallel to Vq, muſt neceſſarily cut the Sides VA and VB of the Cone in a and b, the one below and the other beyond the Vertex V; and the Projection of q being infinitely diſtant, ab becomes the Complement of the Projection of AB, all Points in the Part qA being projected below a in ap indefinitely produced beyond p, and all Points of qB being projected beyond b; and as of Conſequence the Projections of all Lines drawn from t within the Angle BtA, and which form double Ordinates to the Diameter ab of the Section[a], muſt fall beyond its Extremities a and b, and cannot fall between thoſe Points, the Sections thus generated muſt be two oppoſite *Hyperbola's*, the one formed by the Projection of the Part EAD of the Baſe, and which will be the real Section of the given Cone by the propoſed Plane, and the other formed by the Part DBE of the Baſe which gives the Section of the oppoſite Cone.

If the Point t be placed at w the Vaniſhing Point of Tw, as in the preſent Figure; then AB the Chord of the Tangents from t, will paſs through S the apparent Center of the Baſe, and form a Diameter of the Section, to which the Projections of all Lines drawn from t within the Angle BtA and terminated by the Baſe, will be double Ordinates, except only the Line Tw, which hath no Projection; and the Tangents at the Extremities of every ſuch Line ſo terminated, will meet in ſome Point of AB without the Baſe, if they be not parallel to it[b]: Now if the Part DE of the Line Tw be uſed as the Original of one of theſe double Ordinates, and the Point Γ where the Tangents to the Baſe in E and D meet AB be found[c], the Projection γ of the Point Γ will be the Center of the *Hyperbola's*, and $\gamma\lambda$, $\gamma\mu$, drawn through γ parallel to the Sides VD and VE of the Cone, will be the *Aſymptotes*, ab will be a firſt Diameter, and nr and $\lambda\mu$ drawn through the Extremities a and b of that Diameter parallel to Vt and terminated by the *Aſymptotes*, will be Tangents to the Sections in a and b, and $\delta\delta$ drawn through γ parallel and equal to either of theſe Tangents, will be the ſecond Diameter Conjugate to the Diameter ab[d]; by which *Data* the Sections may be deſcribed[e].

For $V\Gamma$, VB, VC, and VA, being Harmonical Lines, the Diameter ab which is parallel to VC or Vq, is biſected by the other three in a, γ and b, wherefore γ is the Center of the Sections; and $\gamma\lambda$, $\gamma\mu$, drawn parallel to VD and VE, being the Projections of the Tangents ΓD, ΓE, and the Projections of D and E being infinitely diſtant, $\gamma\lambda$ and $\gamma\mu$ are therefore Tangents to the produced Sections at an infinite Diſtance, that is, they are the *Aſymptotes* of the *Hyperbola's*[f]: the reſt is evident.

The Sections thus formed may be alſo proved to be oppoſite *Hyperbola's*, by the Place of the Point of Concourſe of the Tangents at the Extremities of any double Ordinate, with its proper firſt Diameter.

From t through S draw the apparent Diameter FG of the Baſe, the Concourſe of the Tangents at the Extremities of which, is at v the Vaniſhing Point of AB, then vV will cut the Diameter ab in o the Point of Concourſe of the Tangents to the Section in f and g, the Extremities of the double Ordinate fg formed by FG[g].

Then becauſe Vv, VB, VS, and VA, are Harmonical Lines, ab which cuts them all Four, is Harmonically divided by them in σ, a, o, and b, and the Point of Concourſe o of the Tangents in f and g, falling between the Extremities a and b of the Diameter ab, that Diameter is therefore a firſt Diameter of oppoſite *Hyperbola's*, to which fg is a double Ordinate[h]; and as the ſame Thing may be ſhewn of any other double Ordinate to the Diameter ab, and the Tangents at its Extremities, the Point of Concourſe of which with ab muſt always fall between a and b, the Sections thus generated are therefore oppoſite *Hyperbola's*; which may likewiſe be deſcribed from theſe *Data*[i].

GENERAL COROLLARY 4.

If the Original of the produced Section be an Ellipſis or a Circle, its Projection or Image (whatever Conick Section it may form) will neither touch nor cut the vaniſhing Line of the cutting Plane; if the original Section be a *Parabola*, its Projection will touch that Vaniſhing Line, and if the Original of the Section be oppoſite *Hyperbola's*, their Projection will cut that Line. Fig. 183. N°. 2, 3.

For if Ty neither touch nor cut the given Baſe, in which Caſe the Original of the Section is an Ellipſis or a Circle[k], the Projection of Ty which coincides with the Vaniſhing Line zy[l], can neither touch nor cut the Projection of the Baſe: If Ty touch the

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[a] Cor. 2. Meth. 1.

[b] Lem. 13. B. III.

[c] Cor. 6. Meth. 2.

[d] Hyp. Art. 6. B. III.

[e] Meth. 1. Prop. 18. B. III.

[f] Hyp. Art. 5. B. III.

[g] Cor. 4. Meth. 2.

[h] Hyp. Art. 26. B. III.

[i] Meth. 2. Prop. 18. B. III.

[k] Cor. 2. Prob. 10.

[l] Meth. 2.

the Baſe, in which Caſe the Original Section is a *Parabola*[a], zy muſt alſo touch the Projection of the Baſe; and if Ty cut the given Baſe, in which Caſe the original Sections are oppoſite *Hyperbola's*[b], zy muſt alſo cut the Projection of the Baſe; which Projection in all Caſes is the Image of the Section required; the Species of which Image is determined by the Poſition of the Line Tw[c], independent of that of the Line Ty.

S C H O L.

This Problem may alſo be ſolved by finding the Projection of any *Trapezium* whatſoever formed by Tangents to the Baſe; but as no Chord of any Tangents to the Baſe, except of ſuch as meet in ſome Point of Tw, or are parallel to it, can form a Diameter of the Section required, ſeeing the Projections of the Tangents at the Extremities of ſuch Chord cannot in any other Caſes be parallel, the Methods before propoſed are the moſt convenient; in regard that by them, either two Conjugate Diameters of the required Section, or one Diameter with a double Ordinate to it are always found, by which the Species of the Section is determined; and the intire Section can be thence eaſily deſcribed, either by ſubdividing the Projection of the Figure which incloſes the Baſe, as here directed, or by any other of the Methods formerly propoſed for deſcribing the Conick Sections[d].

It will therefore be ſufficient to hint at ſome other Methods of this kind, and leave the putting them in Execution to the Practice of the Learner.

Thus if the Points of Concourſe of the Tangents which incloſe the Baſe be both taken in the Line Ty, the Projections of thoſe Points (other than of the Point T) will be Vaniſhing Points in zy, to which the Sides of the projected *Trapezium* will tend, and which will then repreſent a Parallelogram or Square in the cutting Plane having none of its Sides parallel to the Picture[e].

Or if the *Trapezium* incloſing the Baſe, have its Sides tending to two Vaniſhing Points in EF, in which Caſe the Lines which join the Points of Contact will be apparent Diameters of the Baſe; then the Projections of thoſe Vaniſhing Points will fall in a Line drawn from y through the Parallel Seat of V on the cutting Plane with reſpect to the Plane of the Baſe, which is the Line of the Foci of the Projections of all Lines in this laſt Plane on the cutting Plane[f]; and that Line being uſed as a Vaniſhing Line, the intire Projection may be thence found.

Or laſtly, if the Tangents to the Baſe be ſo drawn as to compoſe a real Parallelogram, their Projections may be found by the help of a Line drawn in the cutting Plane, through the Parallel Seat of V on that Plane, parallel to Py, which Line is the imaginary Projection of the Directing Line of the Plane of the Baſe[g], in which the Projections of the Tangents to the Baſe will meet; ſo that this Line being uſed as a Vaniſhing Line will likewiſe ſerve for the Deſcription of the Section.

But in either of theſe three laſt Ways, the Projections of the Chords of the Tangents to the Baſe will neither be Diameters of the Section, nor be Ordinately applied to each other, in regard that neither of them will biſect the other, nor will the Tangents at their reſpective Extremities be parallel.

C A S E 2.

When the Vaniſhing Lines of the Plane of the Baſe and of the cutting Plane are either parallel or coincide,

Let AGBFV be the given Cone, EF the Vaniſhing Line of its Baſe, and yz the Vaniſhing Line of the cutting Plane parallel to EF, and PP their common Interſection. Fig. 183. N°. 7.

This Caſe varies from the preceeding only in the Method of finding in the Plane of the Baſe, the Lines marked Tw and Ty in the former Figures, the Projections of which coincide with the Directing and Vaniſhing Lines of the cutting Plane.

This is done by transferring the Oblique Seat S of the Vertex V of the Cone, to T in any convenient ſubſtituted Plane yyP, and taking TΣ in that Plane, parallel and equal to VS; for then ΣQ parallel to yT cuts yT in Q, through which Qq being drawn parallel to EF, it will be the former of the Lines ſought[h]; and Σy being drawn cutting yT in K, Kk parallel to EF will be the other Line deſired[i]: And either of theſe Lines being uſed as before directed[k], the Section may be found as in the Figure.

And when the Vaniſhing Lines of the given Planes coincide, the Lines Kk and PP coincide with them; ſo that the ſubſtituted Plane $\Delta\delta$y being drawn, and VS being transferred to ΣT in that Plane, by the help of T the Seat of Σ on the cutting Plane, the Fig. 183. N°. 8.

[a] Cor 3. Prob. 10.

[b] Cor. 4. Prob. 10.

[c] Gen. Cor. 1, 2, 3.

[d] Prob. 16, 17, 18. B. III.

[e] Meth. 5. Prob. 6. B. V.

[f] Meth. 4. Prob. 6. B. V.

[g] Meth. 6. Prob. 6. B. V.

[h] Meth. 7. Prob. 7. B. V.

[i] Meth. 5. Prob. 7. B. V.

[k] Meth. 1, or 2.

the double Ordinate de to the Diameter ab of the Section is found, from whence the whole may be compleated as in the Figure[a].

Or if the Chord of the Tangents to the Baſe from q the Interſection of AB with Qq (found as before) be uſed, a Parallelogram may be thereby formed which will incloſe the propoſed Section[b].

COR.

The ſeveral Corollaries of Caſe 1, may be applied to this; and when the given Vaniſhing Lines are parallel, the produced Section may be either of the Conick Sections: but when the cutting Plane is parallel to the Baſe, the Original of the Section is always a Circle, and its Image muſt be either an Ellipſis or a Circle.

GENERAL COROLLARY.

This Problem ſerves to find the Shadow of any Straight Line, or of the Straight Edge of any Opake Body, on a given Cone, from any Luminous Point; the Plane of that Shadow, and its Interſection with the Plane of the Baſe, being given.

For in whatever Point of the Plane of the Shadow the Luminous Point be placed, the Section of the Cone by that Plane will be the ſame.

But according to the various Places of the Luminous Point in that Plane, the true Shadow will fall on different Parts of the Section; the Shadow being only ſuch Part of the Section as falls upon that Part of the Cone which is expoſed to the Light, and of this ſo much only will be viſible to the Eye as falls on the viſible Part of the Cone.

This Problem likewiſe furniſhes a ready Method to find the like Shadow on the Surface of any given Pyramid whoſe Baſe is any Rectilinear Figure, regular or irregular.

Fig. 183. N°. 9. Thus, let VCABD be a Pyramid, and EF the Vaniſhing Line of its Baſe, yz the Vaniſhing Line of the Plane of the Shadow, and Py its Interſection with the Plane EF.

Having found T the Parallel Seat of V on the Plane EF with reſpect to the Plane zyP, and drawn Tw parallel to Py, produce any Side CA of the Baſe till it cut Py and Tw in p and q; then draw Vq, and through p draw pa parallel to it, cutting VA, and VC, in a, and c, and thereby ac the Projection of AC is had; the Projection bd of the Side BD of the Baſe is obtained after the ſame manner, and the Points a and b give the Projection of AB[c]; and thus the propoſed Shadow on the three viſible Faces of the Pyramid is determined; and by the like Method the Projections of the other Sides of the Baſe, and thereby the compleat Section of the given Pyramid with the propoſed Plane, may be found if required; and for this Purpoſe it is only neceſſary to find the Projection of every alternate Side of the Baſe by the Method propoſed, the Projections of the intermediate Sides being thereby determined without farther Trouble.

SECTION III.

Of the Image of the Cylinder and its Sections.

LEM. 12.

Fig. 184. N°. 1. IF from the Eye at Σ, a Line be drawn Parallel to the Axe Ss of a Cylinder AEBaeb till it cut the Plane of its Baſe AEBD in T, and from T there be drawn two Tangents to the Baſe, meeting it in l and m; then, if from theſe Points of Contact, two Sides ll, mm, of the Cylinder be drawn, they will terminate its viſible Part from Σ.

Dem. For it is evident the Planes ΣTll, and ΣTmm, touch the Cylinder in ll and mm. *Q. E. D.*

COR. 1.

Where-ever the Eye is placed in the Line ΣT, the ſame Part of the Cylinder will remain viſible.

For all Lines drawn from any Point in ΣT to any Point of ll, or mm, will be in the Plane ΣTll, or ΣTmm, and conſequently Tangents to the Cylinder in ſome Point.

COR. 2.

The Line ΣT is the common Interſection of all Planes whatſoever, which paſs through

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[a] Meth. 2.

[b] Meth. 1.

[c] Meth. 1.

Plate 105. Book 6. Sec. 2.3.

183. No. 9.

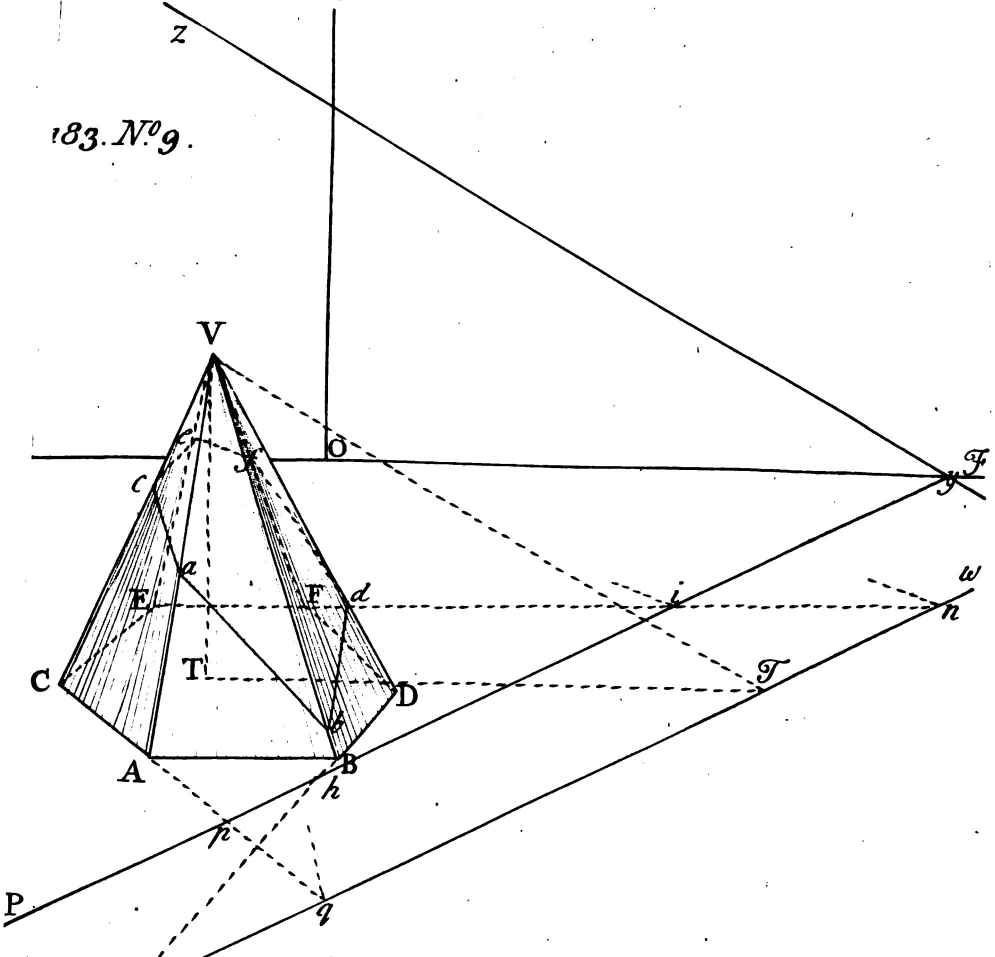
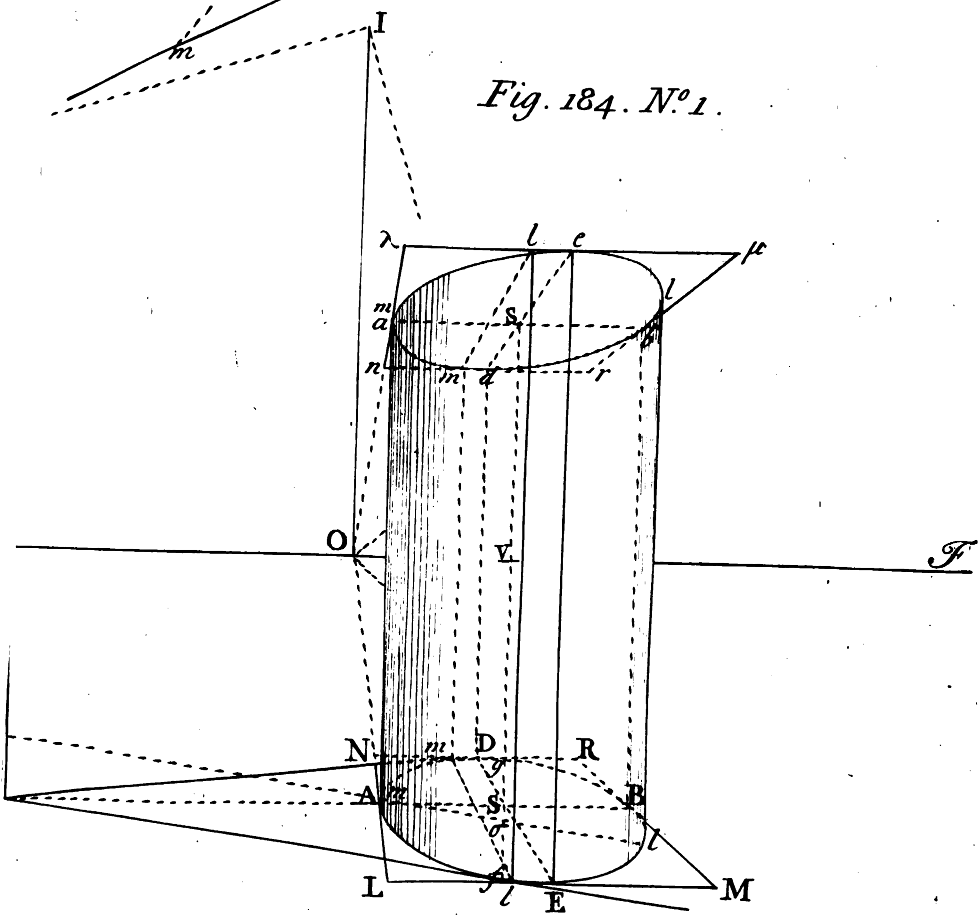


Fig. 184. No. 1.



through any Side of the Cylinder and the Point Σ.

For all the Sides of the Cylinder are parallel to ΣT.

SCHOL.

If a Cylinder be conſidered as a Cone whoſe Vertex is infinitely diſtant, the Method of finding its viſible Part will appear to be the ſame with that for finding the viſible Part of a Cone; for the Extremity of the Axe Ss being ſuppoſed infinitely diſtant, the Line ΣT parallel to Ss may be imagined to meet it in that infinitely diſtant Point[a]. [a] Lem. 10.

And here it may not be improper to add a few Articles relating to a Cylinder, and its Sections by Planes.

1. Every Section of a Cylinder by a Plane not parallel to its Axe, is either a Circle or an Ellipſis, the Center of which Section is always a Point of the Axe; but if the cutting Plane be parallel to the Axe, or paſs through it, the Section is a Parallelogram.

2. If a Cylinder be cut by a Plane not parallel to its Axe, another Cylinder may be fitted on that Section, whoſe Axe ſhall make any propoſed Angle with the Axe of the firſt Cylinder; that is to ſay, innumerable Cylinders may be fitted on any Circular or Elliptick Baſe, by drawing Lines from all Points of the Circumference of the Baſe, parallel to any Line propoſed, all which Lines will compoſe a Cylindrical Surface.

3. Every Cylinder may be ſo cut by a Plane, that the Section ſhall be a Circle.

4 When the Axe is perpendicular to the Plane of the Circular Section, the Cylinder is called a *Right* Cylinder; if the Axe inclines to the Plane of the Circular Section, then the Cylinder is *Scalene.*

5. If any Line be a Tangent to a Cylinder, it will alſo be a Tangent to the Sections of that Cylinder by all Planes paſſing through that Line, and likewiſe a Tangent to all Cylinders formed on any ſuch Section as a Baſe.

6. All Circular Arches or Vaults may be conſidered as Portions of Cylinders; if the Arch be Semicircular, it forms one Moiety of a Cylinder cut by a Plane paſſing through its Axe; if the Arch be but a Quadrant, it repreſents a like Portion of a Cylinder cut by two Planes perpendicular to each other, and paſſing through its Axe; and either of theſe may be either Concave or Convex, according as the Concave or Convex Surface of the Arch is conſidered.

The ſame is to be underſtood of Elliptick Arches; which may be taken as Portions of Scalene Cylinders, whoſe Sections by Planes perpendicular to their Axes, form Ellipſes, of which the Curvatures of the propoſed Arches are Portions.

PROB. XII.

The Center and Diſtance of the Picture, and the Image of any Diameter of the Circular Baſe of a Cylinder, and the Vaniſhing Line of its Plane, together with the Length of the Axe, and its Inclination to the Plane of the Baſe, being given; thence to deſcribe the Image of the Cylinder, and to determine its viſible Part.

CASE I.

When the Axe of the Cylinder is parallel to the Picture.

Let O be the Center, and IO the Diſtance of the Picture, AB the given Diameter of the Baſe, and *EF* the Vaniſhing Line of its Plane; and let the Axe of the Cylinder be ſuppoſed Perpendicular to its Baſe, and equal to a known Line. Fig. 184. N°. 1.

Having by the given Diameter AB drawn the Image LMNR of a Square circumſcribing the Baſe, and by its help deſcribed the Baſe ADBE[b], from the Center S draw Ss Perpendicular to the Plane of the Baſe, repreſenting a Line equal to the propoſed Axe, and through s draw *ab* and *ed*, repreſenting Parallels to AB and ED in the Baſe; from A, B, E, and D, draw the Sides A*a*, B*b*, E*e*, and D*d*, of the Cylinder, parallel to Ss, cutting *ab* and *ed* in *a*, *b*, *e*, and *d*, and by the help of theſe Points compleat the *Trapezium* *λμnr*, within which a Curve being deſcribed as uſual, it will repreſent the Face of the Cylinder oppoſite and parallel to the given Baſe; laſtly, draw *ll* and *mm* touching the oppoſite Faces of the Cylinder, and ADBE*adbe* will be the intire Image of the Cylinder, and *l*E*mlem* its viſible Part. [b] Prob. 24. B. II.

Dem For Ss repreſenting the Axe of the Cylinder, and *aebd* repreſenting a Circle parallel and equal to the Original of AEBD, having s for its Center, it therefore repreſents the oppoſite Face of the Cylinder; and Ss being parallel to the Picture, a Line drawn from the Eye parallel to Ss, muſt lye wholly in the Directing Plane, and will therefore cut the Plane of the Baſe in the Directing Point of the Tangents by which the viſible Part of the Cylinder is determined[a]; the Images of which Tangents being therefore parallel to that Line, and conſequently to Ss, mm and ll drawn parallel to Ss, and touching the Baſe in m and l, are the Images of thoſe Tangents; and in regard thoſe Tangents are in the ſame Planes reſpectively with the Sides of the Cylinder which terminate its viſible Part, and both theſe Planes paſſing through the Eye, the Images of theſe Tangents muſt therefore coincide with the Images of the terminating Sides of the Cylinder[b], and conſequently ll and mm alſo repreſent thoſe Sides, and are therefore alſo Tangents to the upper Face[c]; wherefore $mElmel$ is the viſible Part of the Cylinder. *Q. E. I.*

The ſame Method ſerves, although the Axe of the Cylinder be not perpendicular to its Baſe, ſo as it be parallel to the Picture.

COR.

The Points m and l, where the required Tangents meet the Baſe, may be determined in this manner.

Produce Ss till it cut the Baſe in f and g, and the Vaniſhing Line EF in v; conſider fg as an apparent Diameter of the Baſe, having v for its Vaniſhing Point, and having biſected fg in σ, from w a Point in EF perpendicular to the Vaniſhing Point v, draw wσ, which will cut the Baſe in l and m the Points of Contact required.

For it is evident, ml thus found, is the Chord of the Tangents to the Baſe from the Directing Point of the Diameter fg[d], which Directing Point is that, where a Line drawn from the Eye parallel to Ss cuts the Directing Line of the Plane EF.

CASE 2.

When the Axe of the propoſed Cylinder is not parallel to the Picture.

Fig. 184. N°. 2. This Caſe differs from the laſt, only in that the Axe Ss of the Cylinder not being parallel to the Picture, it muſt have ſome Point x for its Vaniſhing Point, to which the Sides Aa, Bb, Ee, and Dd, of the Cylinder alſo tend: and as x is the Vaniſhing Point of Ss, it alſo repreſents a Point in the Plane of the Baſe AEBD, where a Line from the Eye parallel to the Original of Ss, meets that Plane[e]; wherefore the Tangents to the Baſe by which the viſible Part of the Cylinder is determined, muſt be drawn from x, the Images of which Tangents, for the Reaſon before-mentioned[f], will coincide with the Images of the terminating Sides of the Cylinder, and will therefore alſo be Tangents to its oppoſite Face $aebd$: and the Points of Contact l and m with either of the Faces from x, may be found as formerly ſhewn[g], conſidering x as a Point in the Plane of either Face propoſed, both which Planes have ef for their Vaniſhing Line. *Q. E. I.*

If the Plane of the Baſe be parallel to the Picture, the Images of both Faces of the Cylinder will be true Circles, as being ſimilar to their Originals; but this will make no other Difference in the Practice, ſave in the Facility of drawing the Curves.

COR. 1.

Fig. 184. N°. 1. The Shadow of the Cylinder, N°. 1. on the Plane of its Baſe from any Luminous Point Σ, is terminated on each Side by the Tangents Tl, Tm, from the Point T, which Tangents are the Projections of the terminating Sides ll and mm of the Cylinder from Σ; and the Extremity of the Shadow, if the Point Σ be higher than the upper Face $aebd$ of the Cylinder, will be determined by the Projection of the Part lbm of that Face on the Plane EF, to which Projection Tl and Tm will be Tangents; but if Σ be lower than that Face, the Shadow will be Indefinite.

COR. 2.

Fig. 184. N°. 2. The Shadow of the Cylinder, N°. 2. on the Plane EFL on which it reſts with its Side Aa, from any Luminous Point Σ, is found after the ſame manner as that of a Cone[h], uſing the Point x as its imaginary Vertex.

Or it may be done, by finding the Projections of the oppoſite Faces of the Cylinder on the Plane EFL from the Point Σ, in the following manner.

Having through T the Parallel Seat of Σ on the Plane of the Face AEBD with reſpect to the Plane EFL, drawn Tt parallel to their common Interſection yL, through S the apparent Center of the Face AEBD, draw AB parallel to the Vaniſhing Line ef, cutting Tt in t; find FG the Chord of the Tangents from t, which will

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[a] Lem. 12.

[b] Theor. 17. and Cor. 1. B. I.

[c] Lem. 12. and Cor. 1.

[d] Cor. 2. and 3. Prob. 3. B. III.

[e] Theor. 18. B. I.

[f] Caſe 1.

[g] Cor. 3. Prob. 3. B. III.

[h] Cor. 4. Caſe 2. Prob. 8.

g. 184. N^o. 2.

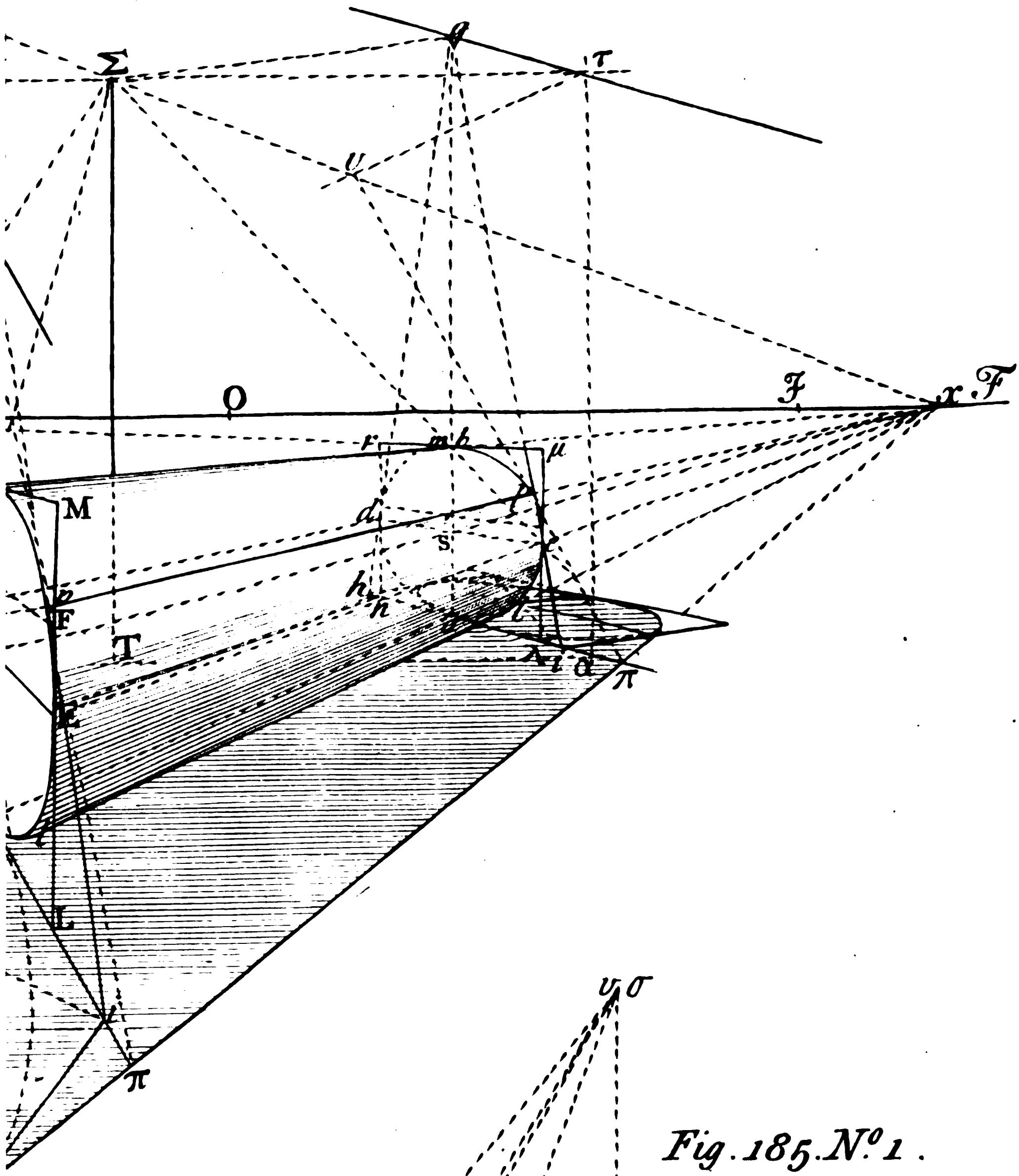
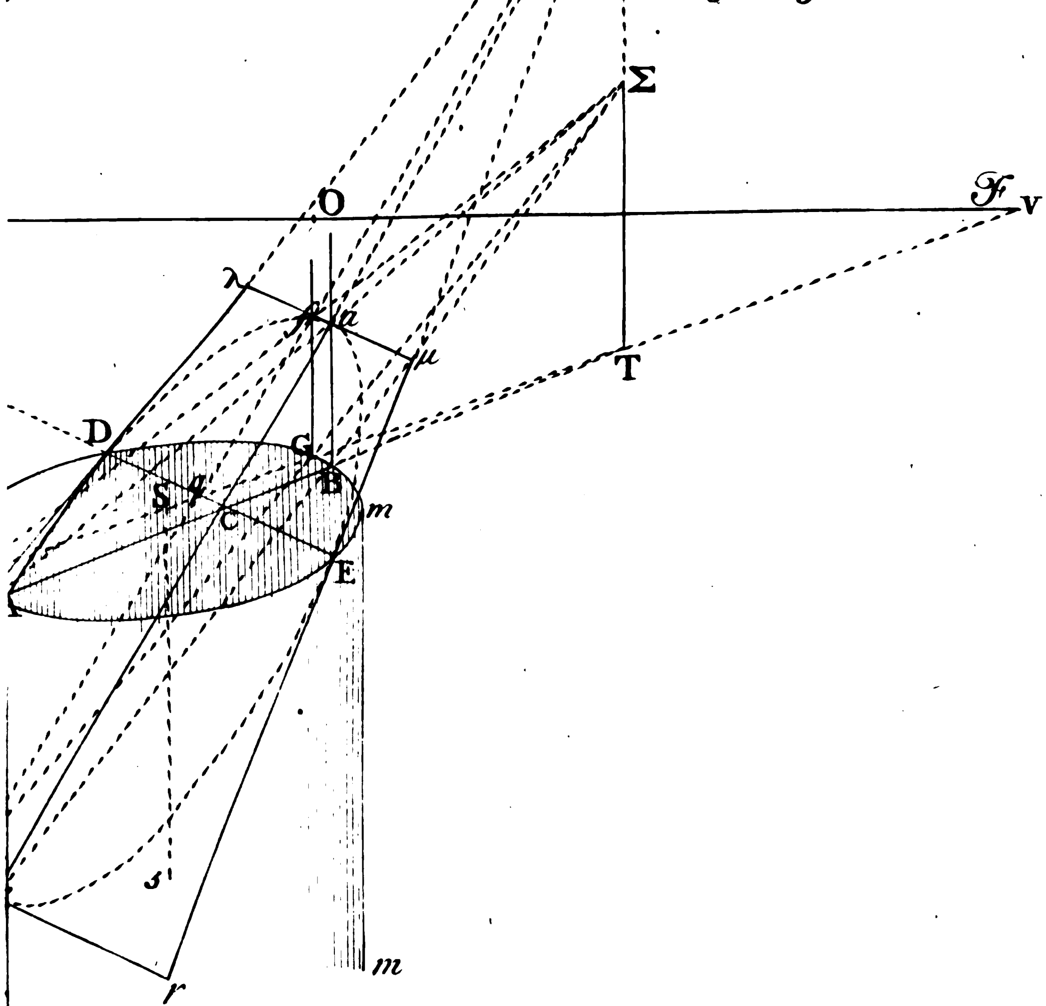


Fig. 185. N^o. 1.



J. Mynde sc.

will have y the Center of the Vaniſhing Line ef for its Vaniſhing Point, in which Point the Tangents in A and B alſo meet; having therefore drawn the Tangents to the Face AEBD from t and y, the Projections of tG, tA, and tF, will be parallel to Σt[a], and the Projections of the Tangents from y will alſo have y for their Vaniſhing Point; wherefore Ab the Projection of AB being found, and in it the Points b and σ the Projections of B and s, by the help of Ab and fg a *Trapezium* is formed, within which the Projection of the propoſed Face is deſcribed as uſual, of which Projection fg is a Diameter, and Ab a double Ordinate to it.

After the ſame manner, the Projection of the Face $aebd$ of the Cylinder is obtained by the help of τ the Parallel Seat of Σ on the Plane of that Face, whereby τq, parallel to ya the Interſection of that Face with the Plane EF, is found, and thence the Point q, where ab parallel to ef cuts τq; the Projections of the Tangents from which Point q are parallel to Σq, and the Projections of the Tangents from y have y for their Vaniſhing Point: the Projections of the two Faces being thus found, Tangents to theſe Projections from x will terminate the Shadow both ways, of which it is eaſy to perceive how much will be viſible.

The only thing remaining, is to determine the Extent of the Light on the Body of the Cylinder; which is done, either by finding the Line pp which produces the Projection $\pi\pi$ which terminates the Shadow on the hither Side[b], or by finding o the Interſection of $x\Sigma$ with the Plane of the Face AEBD, the Chord gp of the Tangents from which Point o, will give the Points g and p of that Circumference, from whence the terminating Sides of the enlightned Part of the Cylinder are to be drawn to x[c]; Or if v the Interſection of Σx with the Plane of the Face $aebd$ be found, the Chord of the Tangents to that Face from v will give the correſponding Points in the Circumference of that Face, through which the terminating Sides of the Cylinder paſs.

SCHOL.

The Points o and v are found by the Interſections of $x\Sigma$ with yT and $y\tau$, which are the Lines in the Planes of the two Faces of the Cylinder, the Projections of which coincide with the Vaniſhing Line EF[d].

Note, the Point y being here the Center of the Vaniſhing Line ef, and alſo a Point in EF, the Projections of the Tangents to both the Faces from y, have the ſame Point y for their Vaniſhing Point[e], and the Tangents from y in A and a are the ſame with the Interſections of thoſe two Planes with the Plane EF; but if the Center of the Vaniſhing Line ef had been out of the Line EF, the Projection of that Center muſt have been found in the Line yT, drawn through y the Interſection of the Vaniſhing Lines of the propoſed Planes, and T the Parallel Seat of Σ on the Plane EF with reſpect to the Planes ef; which is the Line of the *Foci* of the Projections of all Lines in the Planes ef on the Plane EF[f].

The Method here propoſed for finding the Shadow of the Cylinder by the Projections of its oppoſite Faces, is equally applicable to the finding the Shadow of a Cone on a Plane paſſing through one of its Sides, by the Projection of its Baſe on that Plane; for Tangents to that Projection drawn from the Vertex of the Cone, will terminate the Shadow on both Sides; and the Sides of the Cone of which thoſe Tangents are the Projections, will determine the enlightned Part of its Convex Surface by the Luminous Point.

COR. 3.

If the Axe of the Cylinder be parallel to the Picture, then the Point x being Infinitely diſtant, the Line Σx and all others which ſhould tend to x, muſt be drawn parallel to the Axe; but all the reſt of the Operation will be the ſame as before.

PROB. XIII.

The Center and Diſtance of the Picture, and the Image of a hollow Cylinder, with the Vaniſhing Line of the Plane of one of its Faces, being given; thence to find the Boundary of the Light on its Concave Surface, which can enter it from a given Luminous Point, whoſe Seat on the Plane of that Face is given, and to determine the Species of the Curve thereby produced.

[a] Meth. 7. Prob. 6. B.V.

[b] Gen. Cor. Prob. 10. B.V.

[c] Lem. 12.

[d] Meth. 5. Prob. 6. B.V.

[e] Gen. Cor. 1. Prob. 3. B.V.

[f] Meth. 4. Prob. 6. B.V.

CASE 1.

When the Axe of the Cylinder is parallel to the Picture.

Fig. 185. N°. 1. Let O be the Center, and JO the Diſtance of the Picture, $ADBEllmm$ the given hollow Cylinder, $ADBE$ one of its Faces, and EF the Vaniſhing Line of its Plane; and let Σ be the given Luminous Point, and T its Seat on that Plane.

Having found the Parallel Seat of Σ on the given Plane with reſpect to the Axe of the Cylinder (which is here the ſame with T, the Axe Ss being ſuppoſed perpendicular to its Faces) find DE the Chord of the Tangents to the given Face from T; then having from T drawn any Line TA cutting the given Face in A and B, draw the Side Ab of the Cylinder, and the Interſection b of ΣB with Ab will be a Point of the Boundary of the Light required. And thus as many Points of that Boundary may be found as are requiſite to deſcribe the whole, which muſt terminate at the Points D and E.

This is demonſtrated in the ſame Manner as Prob. IX. *Q.E.I.*

COR. 1.

The Curve EbD thus formed, is Part of a Conick Section, lying in a Plane paſſing through the Chord of the Tangents DE, and the Diagonal ab of the *Trapezium* $BbAa$, formed by the mutual Interſections of ΣA and ΣB with the Sides Ba and Ab of the Cylinder.

Having biſected DE in C, draw TC cutting the given Face in A and B, and from b the Projection of B, draw bC cutting the Side Ba of the Cylinder produced, in a, and the Line ΣT in ſome Point σ, which it muſt do, in regard that bC and ΣT cannot be parallel. In like Manner, from g the Projection of G, through q the Interſection of FG with DE. draw gq cutting the Side Gf of the Cylinder, in f, and the Line ΣT in ſome Point v.

Then becauſe of the Harmonical Diviſion of TA, the Lines $T\Sigma$, Ba, Ab, and a Line drawn through C parallel to them, are Harmonical Parallels, wherefore $b\sigma$ is Harmonically divided by them in b, C, a and σ; and becauſe AT and $b\sigma$ which are both Harmonically divided, have their Point C in common, and $T\sigma$ joins their ſecond Points of Diviſion from C, bB and Aa which join their other Points, muſt meet in the ſame Point of $T\sigma$, and conſequently in Σ where bB cuts that Line, wherefore ba is the Diagonal of the *Trapezium* $bAaB$; and by reaſon of the Harmonical Diviſion of $b\sigma$, Ab, AC, Aa, and $A\sigma$, are Harmonical Lines, wherefore $T\sigma$ which is parallel to Ab, is biſected by the other three in T, Σ, and σ.

In like Manner it may be proved, that gv is Harmonically divided in g, q, f, and v; that the Point f is in the Line $F\Sigma$, and that the Point Σ biſects Tv, and conſequently that v and σ coincide; and that therefore the Curve $DgbEaf$ is in a Plane paſſing through DE and the Diagonals ba and gf of the *Trapezia* $bAaB$ and $gFfG$: And the Concave Cylinder being thus cut by a Plane inclining to the Plane of its Face, the Original of the Section produced is an Ellipſis, the Image of which may be either an Ellipſis or a Circle.

SCHOL.

This Demonſtration is in Effect the ſame with that of Cor. 1. Prob. IX. which relates to a Cone, where the Correſponding Points are marked with the ſame Letters; ſave that the Sides of the Cylinder being here ſuppoſed parallel to the Picture, their Vaniſhing Point which ſhould repreſent the Vertex V of the Cone, is infinitely diſtant; for which reaſon the Line $T\sigma$ which is there harmonically divided in V, T, Σ, and σ, is here biſected in T, Σ, and σ.

The ſecond and third Corollaries of that Propoſition are likewiſe applicable here.

COR. 2.

If the Luminous Point Σ be infinitely diſtant before or behind the directing Plane; then ΣT parallel to Ss becomes the Vaniſhing Line of all Planes which paſs through the Luminous Point and any Side of the Cylinder[a], and the Point T will be where ΣT cuts EF; but the Line $T\sigma$ will ſtill be biſected in Σ, and the reſt of the Operation will be the ſame as before.

COR. 3.

Fig. 185. N°. 2. If the Luminous Point be at a moderate Diſtance in the Directing Plane, the Line ΣT will fall wholly in that Plane, and the Points Σ, T, and σ, become Directing Points.

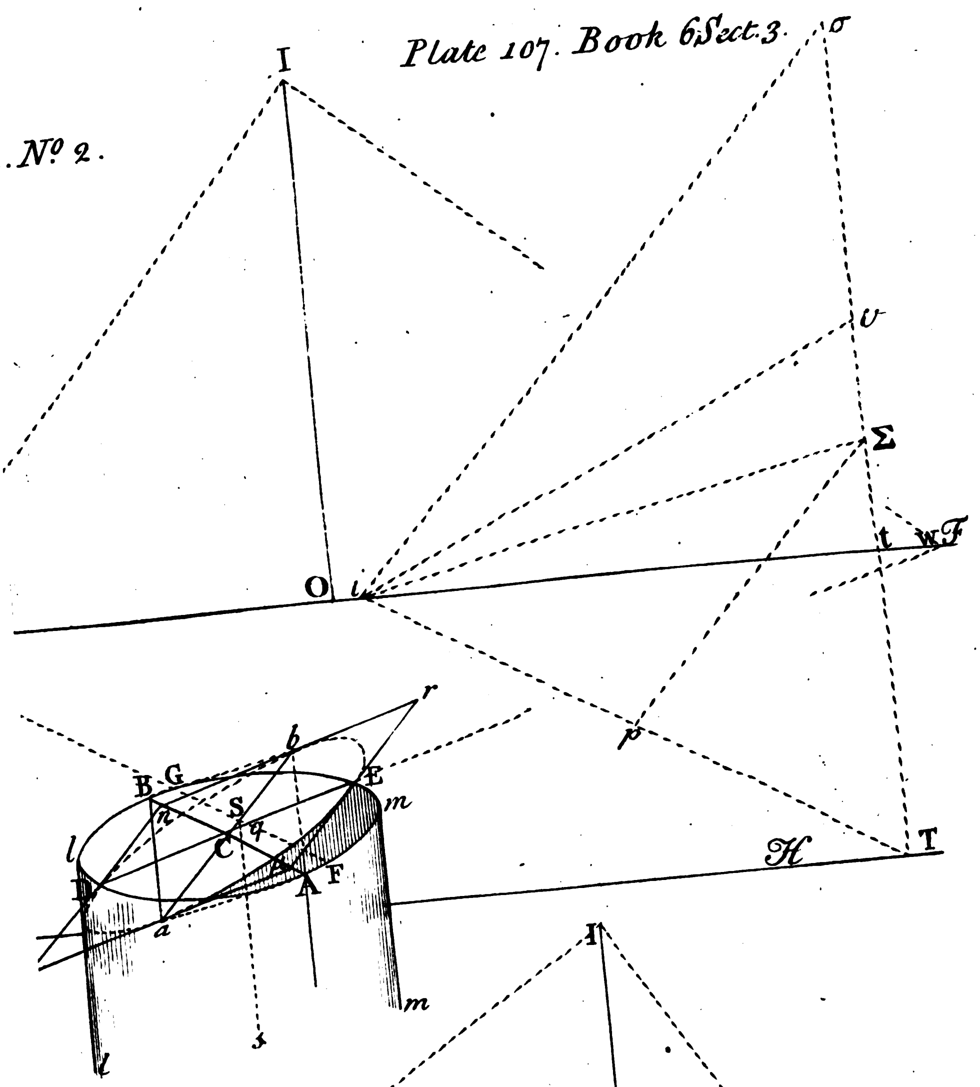
Having therefore found the Directions Σi and Ti of the Projecting Lines and their

Parallel

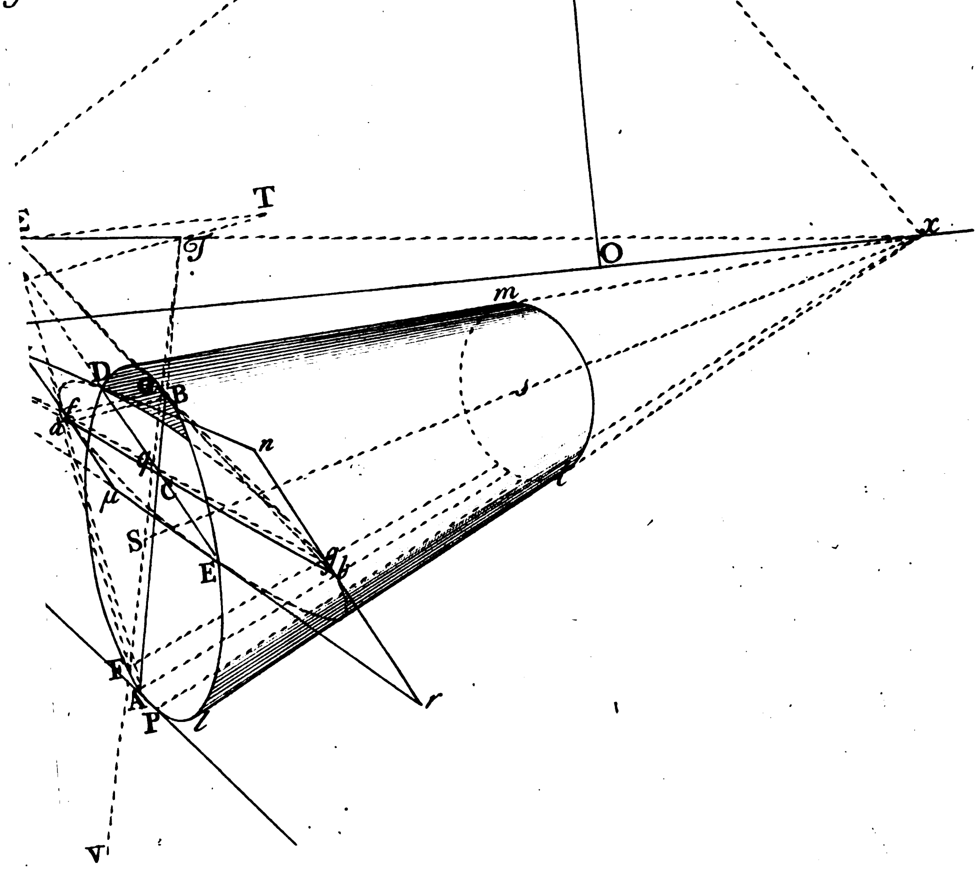
[a] Cor. 2. Lem. 12.

Plate 107. Book 6 Sect. 3.

5. Nº 2.



ig. 185. Nº 3.



Parallel Seats on the Plane of the given Face ADBE with reſpect to the Axe of the Cylinder[a], and taken $\Sigma\sigma$ equal to ΣT, and drawn σi; the apparent Diameter FG muſt be drawn parallel to the Direction Ti, and DE the Chord of the Tangents to the given Face from the Directing Point of FG, repreſented by T, will biſect FG in q[b]; and DE being biſected in C, AB muſt be drawn through C Parallel to FG or Ti, and conſequently tending to the ſame Directing Point[c]: Then the Sides Ba and Ab of the Cylinder being drawn, Bb and Aa drawn parallel to the Direction Σi, will by their Interſections with thoſe Sides, give ab a Diameter of the Section, to which DE will be the Conjugate Diameter; which Diameter ab, and conſequently the Tangents to the Section in D and E, will be parallel to the Direction σi, and the Tangents in a and b will be parallel to DE, which Tangents will by their mutual Interſections form the Parallelogram $\lambda\mu rn$ which incloſes the Section.

For FG being parallel to the Tangents to the given Face in D and E[d], and being biſected in q by DE, the Line DE is a Real Diameter of the Face ADBE, to which FG is a double Ordinate; and DE being biſected in C by AB parallel to FG, AB is a Diameter of that Face Conjugate to the Diameter DE, and is therefore biſected in C; and BaAb being by Conſtruction a Parallelogram, ab is likewiſe biſected in C: wherefore the Diameter ab of the Section, biſecting the Line DE Ordinately applied to it[e], and being biſected by it in C, DE is a Diameter of the Section Conjugate to the Diameter ab. Laſtly, the Triangle BAb having its Sides BA, Ab, Bb, reſpectively parallel to the Sides iT, TΣ, $i\Sigma$, of the Triangle iTΣ, ab which biſects the Side BA of the Triangle BAb, is parallel to a Line Σp drawn from Σ biſecting the Side iT of the Triangle iTΣ; but Tσ being biſected in Σ, and Ti in p, Σp is parallel to σi, wherefore ab is alſo parallel to σi: The reſt is evident.

And here, the Tangents λn and μr in D and E being parallel to ab, σ repreſents their common Directing Point.

COR. 4.

When the Luminous Point is at an infinite Diſtance in the Directing Plane; the only Difference is, that FG and AB muſt be drawn parallel to *EF*, that being the Direction of the Seats of the Projecting Lines[f], and the Seat of the Luminous Point on the Plane *EFGH* being at an infinite Diſtance in the Line FG; the Chord DE of the Tangents from that Point muſt paſs through O and biſect FG and AB: And in this Caſe, if Σi be the Direction of the Projecting Lines, the Direction vi of the Diameter ab of the Section, will be found by drawing through any Point Σ in the Direction Σi, a Line Σt parallel to the Axe of the Cylinder cutting *EF* in t, and taking Σv equal to Σt and drawing vi.

CASE 2.

When the Axe of the Cylinder inclines to the Picture.

The ſame Things being ſuppoſed as before; let x be the Vaniſhing Point of the Axe Ss of the Cylinder, Σ the Luminous Point, and $\mathcal{T}$ the Interſection of Σx with *EFP* the Plane of the Face ADBE. Fig. 185. N°. 3.

In this Caſe, the Method of finding the Section DbEa is exactly the ſame as for a Cone at Prob. IX. the Point x being conſidered as its Vertex; and the correſponding Points of this Figure being marked with the ſame Letters as thoſe of that Propoſition, the Conſtruction and Demonſtration there, will ſerve in this Place, only applying to a Cylinder what was there ſaid of a Cone, and putting the Point x every where inſtead of V. *Q. E. I.*

Likewiſe the firſt Four Corollaries of that Propoſition are equally true here, to which therefore the Reader is referred.

COR. 1.

If the Luminous Point Σ be infinitely diſtant before or behind the Directing Plane; Σx becomes a Vaniſhing Line as before[g], the Point $\mathcal{T}$ is at the Interſection of Σx with *EF*, and the Line Σx will be Harmonically divided in x, $\mathcal{T}$, Σ, and σ; but every Thing elſe is found as before.

COR. 2.

If the Luminous Point be at a moderate Diſtance in the Directing Plane; although that Point is then a Directing Point, yet in regard that $\mathcal{T}$ and σ lye in a Line drawn from the Luminous Point to the Vaniſhing Point x, the Points $\mathcal{T}$ and σ will have Real Images, which are found in this Manner. Fig. 185. N°. 4.

Having drawn the Directions Σi and Ti of the Projecting Lines and their Oblique Seats on the Plane of the given Face, draw xy perpendicular to *EF*, cutting it in y the Oblique Seat of x on that Plane; then draw $x\mathcal{T}$ and $y\mathcal{T}$ parallel to the Directi-

[a] Cor. 2. Caſe 3. Prob. 1. B. V.

[b] Cor. 2. and 3. Prob. 3. B. III.

[c] Cor. 5. Theor. 12. B I.

[d] Cor. 2. Prob. 3. B. III.

[e] Cor. 2. Prob. 9.

[f] Caſe 4. Prob. 1. B. V.

[g] Cor. 2. Caſe 1.

ons Σi and $\mathrm{T}i$, and their Interſection T will be one of the Points required[a]; and becauſe the Image of the Luminous Point Σ is here at an infinite Diſtance in the Line xT, take $x\sigma$ equal to xT, and σ will be the other Point ſought[b]; and the Points T and σ being thus found, every Thing elſe is done as before, only obſerving that A*a* and B*b* muſt be drawn parallel to the Direction Σi of the Projecting Lines.

COR. 3.

When the Luminous Point is at an infinite Diſtance in the Directing Plane; xT drawn Parallel to the Direction of the Projecting Lines, becomes a Vaniſhing Line[c], and cuts *EF* in the Point T; and Σ being here alſo infinitely diſtant in the Line xT, the Points T and σ will be equally diſtant from *x*; but this makes no Difference in what remains to be done.

CASE 3.

When the Faces of the Cylinder are Parallel to the Picture.

In this Poſition of the Cylinder, the Luminous Point may be at a moderate Diſtance, either between the Eye and the propoſed Face of the Cylinder, or in the Directing Plane, or behind it, or it may be at an infinite Diſtance behind that Plane; but it cannot be at a moderate Diſtance beyond the Plane of the given Face, nor at an infinite Diſtance either before or in the Directing Plane; ſeeing that in either of theſe laſt Poſitions, no Light could fall into the Concavity of the Cylinder viſible to the Eye.

It will therefore be ſufficient to ſhew, in either of the firſt mentioned Situations of the Light, after what manner the Points Σ, T, σ and *x* fall; which being determined, the reſt of the Operation will be compleated as in the former Caſes.

Fig. 185. N°. 5, 6. 1. In the firſt Place, let Σ be the Luminous Point at a moderate Diſtance between the Eye and the Plane of the Face ADBE of the Cylinder; and let T be the Perpendicular Seat of Σ on that Plane, which muſt be the ſame with the Interſection of that Plane with the Line Σx, when the Axe S*s* of the Cylinder is perpendicular to its Faces, the Point *x* being then the ſame with O the Center of the Picture.

In this Caſe, if $\Sigma\mathrm{T}$ be leſs than $\mathrm{T}x$, the Point σ will fall in $\mathrm{T}\Sigma$ produced beyond Σ[1], and its Original will be a Point between the Eye and the given Face; but if $\Sigma\mathrm{T}$ be greater than $\mathrm{T}x$, σ will fall in $\Sigma\mathrm{T}$ produced beyond *x*, in the Tranſprojective Part of that Line[2], and will repreſent a Point behind the Directing Plane; the Line Σx, in both Caſes, being Harmonically divided in *x*, T, Σ, and σ.

Having therefore from T drawn the Diameter AB of the given Face, and found the Chord DE of the Tangents from T, which muſt be perpendicular to, and biſected by AB, the Face ADBE in this Poſition of the Cylinder being a true Circle; ΣB and ΣA will cut the Sides A*x* and B*x* in *b* and *a*, and give the Diameter *ab* of the Section, which will paſs through σ where the Tangents to the Section in D and E alſo meet.

Fig. 185. N°. 7. 2. If the Luminous Point be at a moderate Diſtance behind the Directing Plane; its Image Σ will be in the Tranſprojective Part of $\mathrm{T}x$, and the Point σ will always fall between Σ and *x*.

For the Diagonal AB of the *Trapezium* A*a*B*b* meeting Σx in T beyond *x*, it is evident that *ab* the other Diagonal of that *Trapezium*, muſt meet Σx in σ between Σ and *x*.

Fig. 185. N°. 8. 3. If the Luminous Point be at a moderate Diſtance in the Directing Plane; its Image Σ being then infinitely diſtant in the Line $\mathrm{T}x$, the Point σ will fall at an equal Diſtance from *x* with the Point T.

And here, A*a* and B*b*, which determine the Extremities *a* and *b* of the Diameter *ab* of the Section, muſt be drawn parallel to $\mathrm{T}x$, which in this Caſe is parallel to the Direction of the Projecting Lines[d].

Note. *If the Cylinder be Scalene, whereby the Vaniſhing Point x of its Axe will be different from* O *the Center of the Picture, the Interſection of* Σx *with the Plane* ADBE *is found by the help of* T *the Perpendicular Seat of* Σ *on that Plane, by drawing from* T, *a Parallel to* O*x*, *which will cut* Σx *in the Point required*[e].

Fig. 185. N°. 9. 4. Laſtly, If the Luminous Point Σ be at an infinite Diſtance behind the Directing Plane; Σx becomes the Vaniſhing Line of all Planes which paſs through the Luminous Point and any Side of the Cylinder, the Interſections of all which Planes with the Face ADBE are parallel to Σx[f]; ſo that the Point T becomes infinitely diſtant in that Line, and conſequently the Point σ biſects Σx.

In this Caſe, the Diameter AB of the Circular Face, drawn parallel to Σx, is that which

[a] Caſe 3. Prob. 1. B. V.

[b] Cor. 5. Prob. 9.

[c] Caſe 4. Prob. 3. B. V.

[1] Fig. 185. N°. 5.

[2] Fig. 185. N°. 6.

[d] Caſe 3. Prob. 4. B. V.

[e] Prob. 4. B. V.

[f] Cor. Theor. 3. B. I.

Fig. 185. No. 4.

5.

Fig. 185. No. 6.

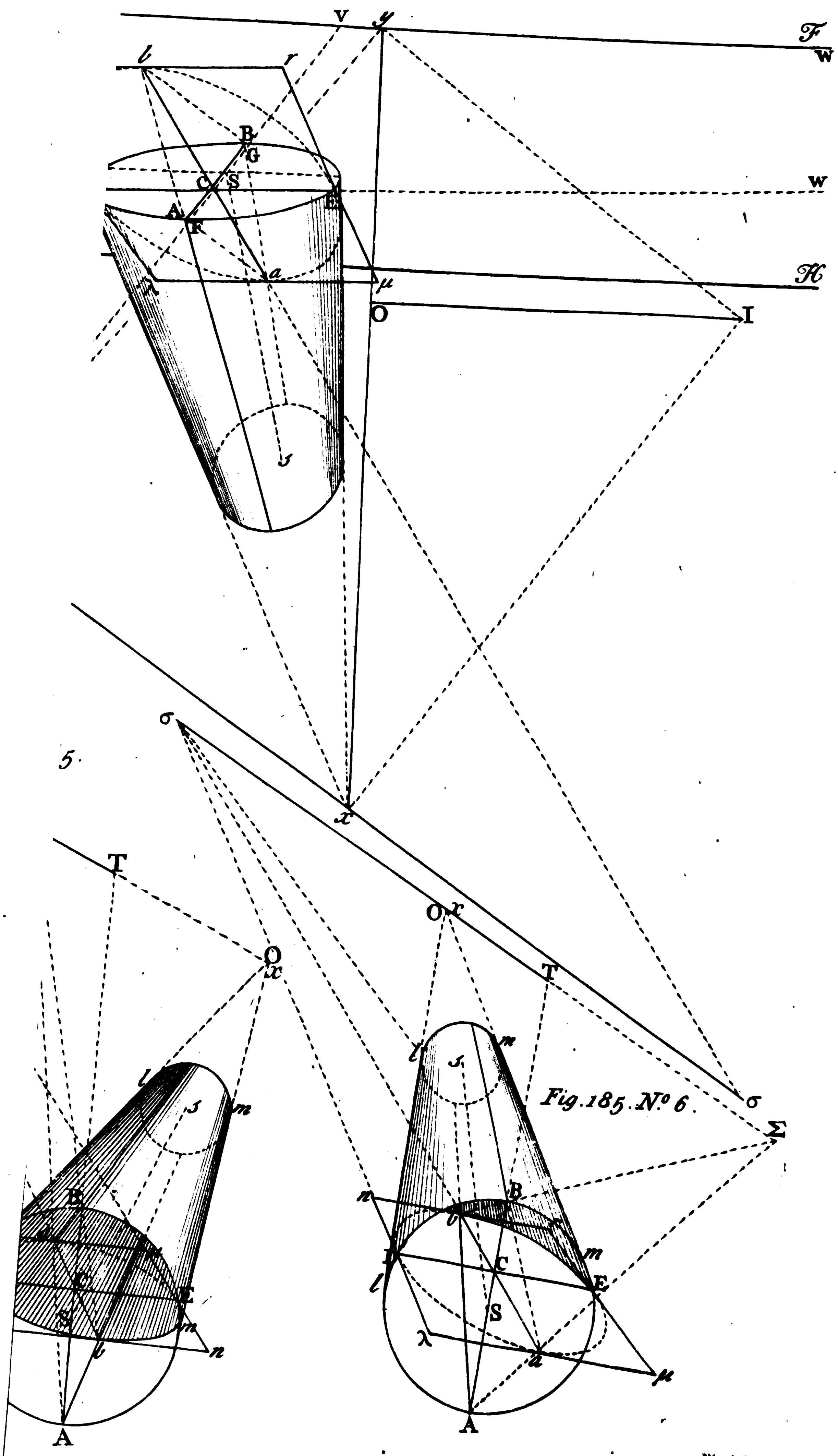


Plate 109. Book 6. Sect. 3.

Fig. 185. N.º 8.

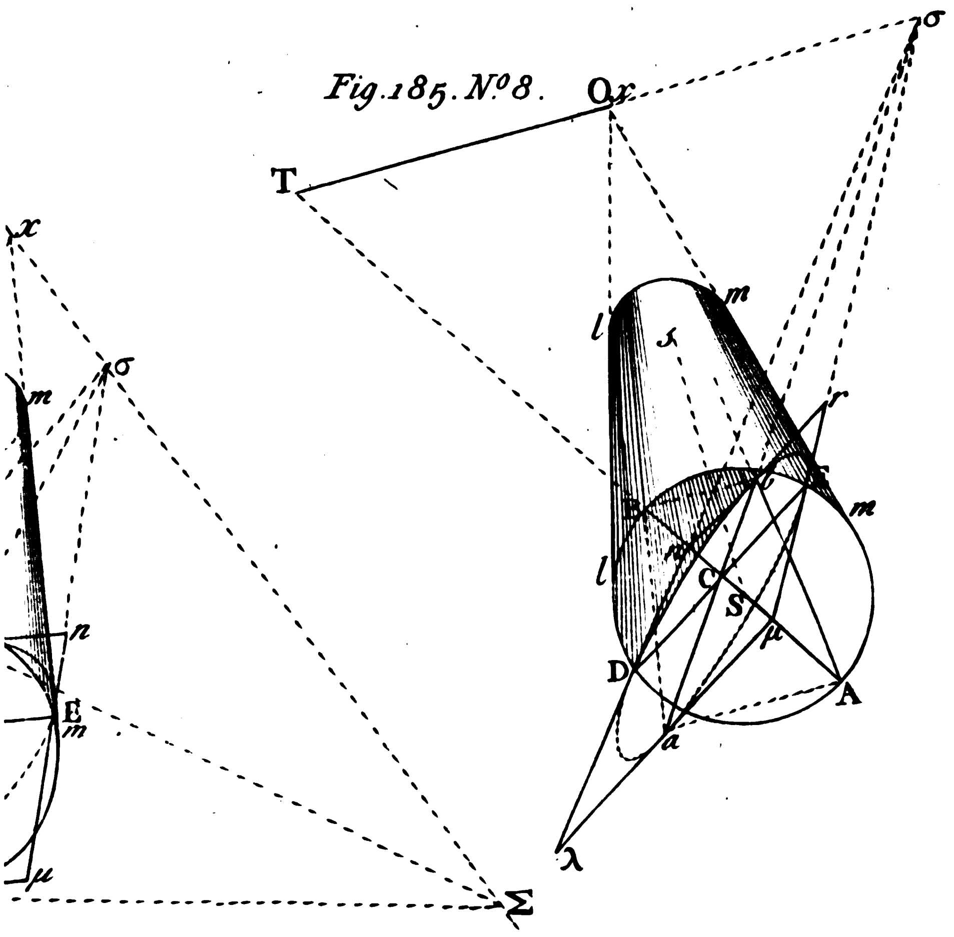
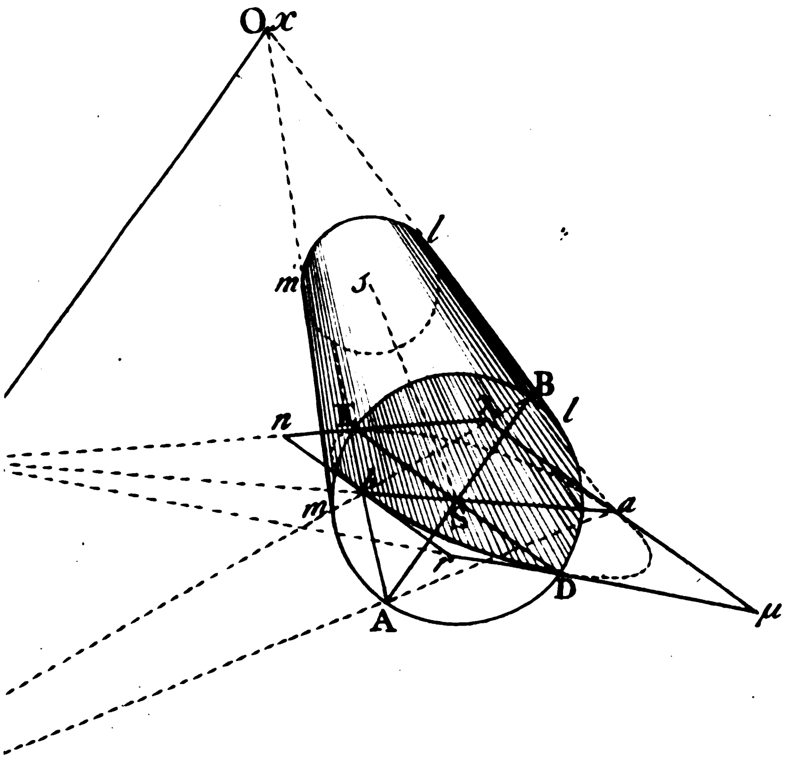


Fig. 185. N.º 9.



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which forms the Diameter *ab* of the Section; and the Diameter DE of the given Face, drawn perpendicular to AB or Σx, and which is the Chord of the Tangents to that Face from the Infinitely diſtant Point T, is a double Ordinate to the Diameter *ab*; the Tangents to the Section in *a* and *b* are therefore parallel to DE, and the Tangents in E and D meet in σ the Vaniſhing Point of *ab*.

GENERAL COROLLARY.

In all Caſes of this Problem, the Diameter *ab* of the Section being a Diagonal of the *Trapezium* A*b*B*a* whoſe Sides tend to Σ and *x*, of which *Trapezium* AB is the other Diagonal; it is evident, that when the Point T lies without Σx [1], the Point σ muſt fall between *x* and Σ; and on the contrary, that when T falls between *x* and Σ [2], the Point σ muſt fall on the outſide of them; and that when neither of thoſe Points is Infinitely diſtant, the Line Σx is always Harmonically divided in Σ, *x*, T, and σ [a]: But if *x* be Infinitely diſtant [3], Tσ will be biſected in Σ; if Σ be Infinitely diſtant [4], Tσ will be biſected in *x*; if σ be Infinitely diſtant, Σx will be biſected in T; and if T be Infinitely diſtant [5], Σx will be biſected in σ [b].

Note, *The Point* T *here meant, is marked T in Fig. No.* 3, 4.

SCHOL.

The ſeveral Corollaries of this Propoſition relating to the different Poſitions of the Luminous Point, are equally applicable to the correſponding Problem concerning a Cone; only obſerving that with reſpect to a Cone, the Point T can never be a Vaniſhing Point, nor can it be Infinitely diſtant, unleſs the Situation of the Luminous Point be ſuch, that a Line drawn from thence to the Vertex of the Cone, may cut the Plane of its Baſe in its Directing Line; which will happen, if the Luminous Point be as far diſtant from the Directing Plane behind the Eye, as the Vertex of the Cone is before it, the Oblique Supports of the Luminous Point and of the Vertex of the Cone on the Plane of its Baſe being at the ſame time equal.

For the Vertex of the Cone being a Point at a moderate Diſtance, a Line drawn from thence to the Luminous Point, where-ever it be ſituated, muſt cut the Plane of the Baſe in ſome Point likewiſe at a moderate Diſtance, which Point will therefore always have a Real Image, unleſs it happen to fall in the Directing Plane.

[1] Fig. 185. N°. 7.

[2] Fig. 185. N°. 3, 5, 6.

[a] Lem. 22. B. III.

[3] Fig. 185. N°. 1. 2.

[4] Fig. 185. N°. 4. 8.

[5] Fig. 185. N°. 9.

[b] Cor. 1, 2. Lem. 22. B. III.

PROB. XIV.

The Center and Diſtance of the Picture, and the Image of a Cylinder, with the Vaniſhing Line of the Planes of its Faces, being given; thence to find the Image of the Section of that Cylinder by any given Plane, whoſe Interſection with the Plane of either Face is given.

CASE 1.

When the Axe of the Cylinder is parallel to the Picture.

Let O be the Center, and OI the Diſtance of the Picture, ADBE*lm* the given Cylinder, and *EF* the Vaniſhing Line of its Faces; and let it be required to deſcribe the Section of this Cylinder by a Plane, whoſe Vaniſhing Line is *yz*, and its Interſection with the Plane of the Face ADBE is P*y*. Fig. 186. N°. 1.

In this Caſe, if the given Cylinder be conſidered as a Cone whoſe Vertex is Infinitely diſtant in the Directing Plane, the Projection or Image of the Face ADBE on the Plane *zy*P, from that Infinitely diſtant Vertex taken as a Projecting Point, will be the Section of the given Cylinder by that Plane.

For the Sides of the Cylinder, which are all parallel to its Axe S*s*, may be taken as the Projecting Lines of the Face ADBE from a Point at an Infinite diſtance in the Directing Plane, having a Line parallel to S*s* for their Direction; and it is evident that the Interſections of the Sides of the Cylinder with the Plane *zy*P form the Section required.

This Section may be therefore found by the following Methods.

METHOD 1.

Having through any Point P in *y*P, drawn Pπ parallel to *EF*, and P*q* parallel to *yz*, from O draw the Diameter AB of the Face ADBE cutting P*y* in *p*, and having biſected AB in C, through C draw DE parallel to *EF*, and the Sides A*a*, B*b*, D*d*,

Ee, of the Cylinder, parallel to Ss, and the Line Cc parallel to them; then having drawn $y\pi$ parallel to AB cutting $P\pi$ in π, draw πq parallel to Ss cutting Pq in q, and draw yq.

This being done; from p draw pb parallel to yq, cutting Aa, Cc, and Bb, in a, c, and b; through a, c, and b, draw rn, ed, and $\mu\lambda$, parallel to yz, and through e and d the Interſections of ed with Ee and Dd, draw $r\mu$ and $n\lambda$ parallel to ab, and there- by a Parallelogram $nr\mu\lambda$ will be formed, within which a Curve being deſcribed as uſual, it will be the Section deſired.

Dem. Through A and B draw the Tangents NR, LM, parallel to EF, and through D and E draw NL, RM, parallel to AB, which will be Tangents in D and E, AB and DE being two Real Conjugate Diameters of the Face ADBE.

Then, if $EFP\pi$ and $zyPq$ be conſidered as the Directing Planes of the given Planes brought into the Picture[a], Σy parallel to Ss as the Direction of the Projecting Lines, and $y\pi$ parallel to AB as the Director of that Line; πq parallel to the Direction Σy cutting Pq in q, gives yq the Direction of the Projections of AB and its parallels NL and RM[b]; wherefore pb parallel to yq is the Projection of AB, and the Points a, c, and b, are the Projections of A, C, and B; and in regard that DE is here parallel to $P\pi$, a Line from y parallel to DE coincides with EF and can never cut $P\pi$ to determine its Directing Point, which Point being therefore at an Infinite Diſtance in $P\pi$, its Projection is at an Infinite Diſtance in Pq, wherefore the Direction of the Projections of DE and its parallels, is parallel to Pq, and coincides with yz; and conſequently rn, ed, and $\mu\lambda$, drawn through a, c, and b, parallel to yz, are the Projections of RN, ED, and ML, and the Parallelogram $rn\lambda\mu$ is therefore the Projection of the Parallelogram RNLM: the reſt needs no farther Demonſtration. *Q. E. I.*

COR.

If p the Interſection of AB with yP ſhould be at an inconvenient Diſtance, δ the Interſection of DE with that Line may be uſed; for δd drawn parallel to yz will give the Points e, c, and d, by the help of which ab parallel to yq, and thence the Parallelogram $rn\lambda\mu$ may be found.

SCHOL.

The Section may be alſo determined by the help of any two Real Conjugate Diameters of the Face ADBE[c], as well as by thoſe here uſed; the Directions of both thoſe Diameters being found after the ſame manner as the Direction yq of the Diameter AB.

METHOD 2.

The ſame things being ſuppoſed as before; from O the Vaniſhing Point of AB draw Oz parallel to Ss till it cut yz in z, and find FG the Chord of the Tangents to the given Face from O, cutting AB in S; then pz cutting Aa, Ss, and Bb, in a, σ and b, will give the Diameter ab of the Section, and fg drawn through σ parallel to yz, and terminated in f and g by the Sides Ff, Gg, of the Cylinder, will be a double Ordinate to it; the Lines zf and zg will be Tangents to the Section in f and g, which, together with the Tangents $\lambda\mu$ and nr, will form a *Trapezium*, within which the Section may be deſcribed as uſual.

Dem. For z being the Projection of O[d], pz is the Indefinite Projection of AB; and fg being the Projection of FG, zf and zg are the Projections of the Tangents OF, and OG: the reſt is evident. *Q. E. I.*

SCHOL.

Here alſo, it is not neceſſary that the Diameter AB which tends to O, ſhould be uſed, but any other Real Diameter of the given Face will ſerve, in regard that the Tangents at the Extremities of any Real Diameter being parallel, their Projections will alſo be parallel[e]; but then the Direction of the Projections of theſe Tangents muſt be found as before ſhewn[f], which Trouble is ſaved by uſing the Diameter AB, the Tangents at the Extremities of which being parallel to EF, their Projections are parallel to zy.

CASE 2.

When the Axe of the Cylinder is not parallel to the Picture.

Fig. 186. N°. 2. The ſame things being ſuppoſed as before, ſave that the Axe Ss hath x for its Vaniſhing Point; then if this Point be conſidered as a Projecting Point at an Infinite diſtance before or behind the Directing Plane, the Section of the given Cylinder $ADBElm$ by the Plane zyP is found in this manner,

Having

[a] Cor. Caſe 4. Prob. 6. B. V.

[b] Meth. 2. Caſe 4. Prob. 6. B. V.

[c] Ellip. Art. 1. and 2. B. III.

[d] Meth. 1. Caſe 4. Prob. 6. B. V.

[e] Schol. Caſe 3. Prob. 6. B. V.

[f] Meth. 1.

Plate 110. Book 6. Sect. 3.

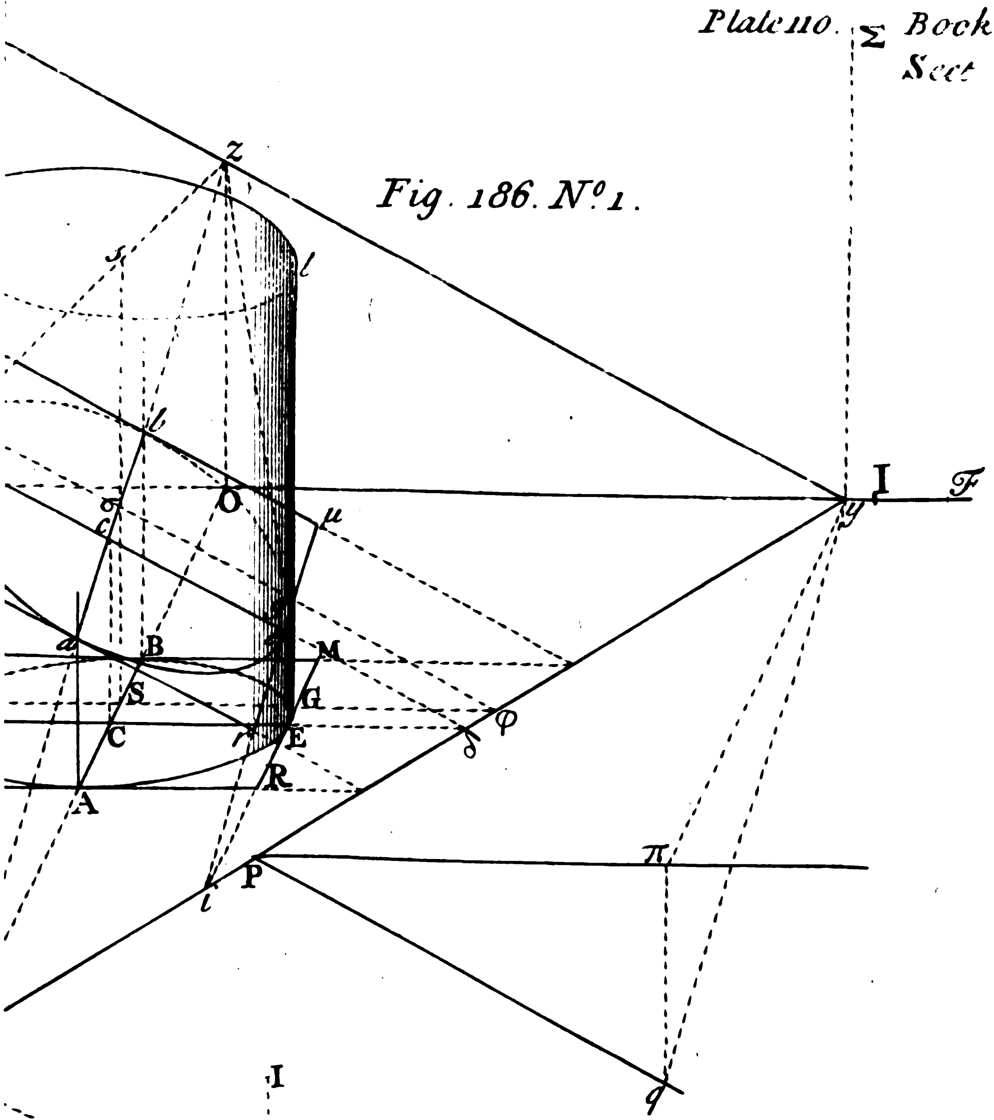

Fig. 186. No. 1.

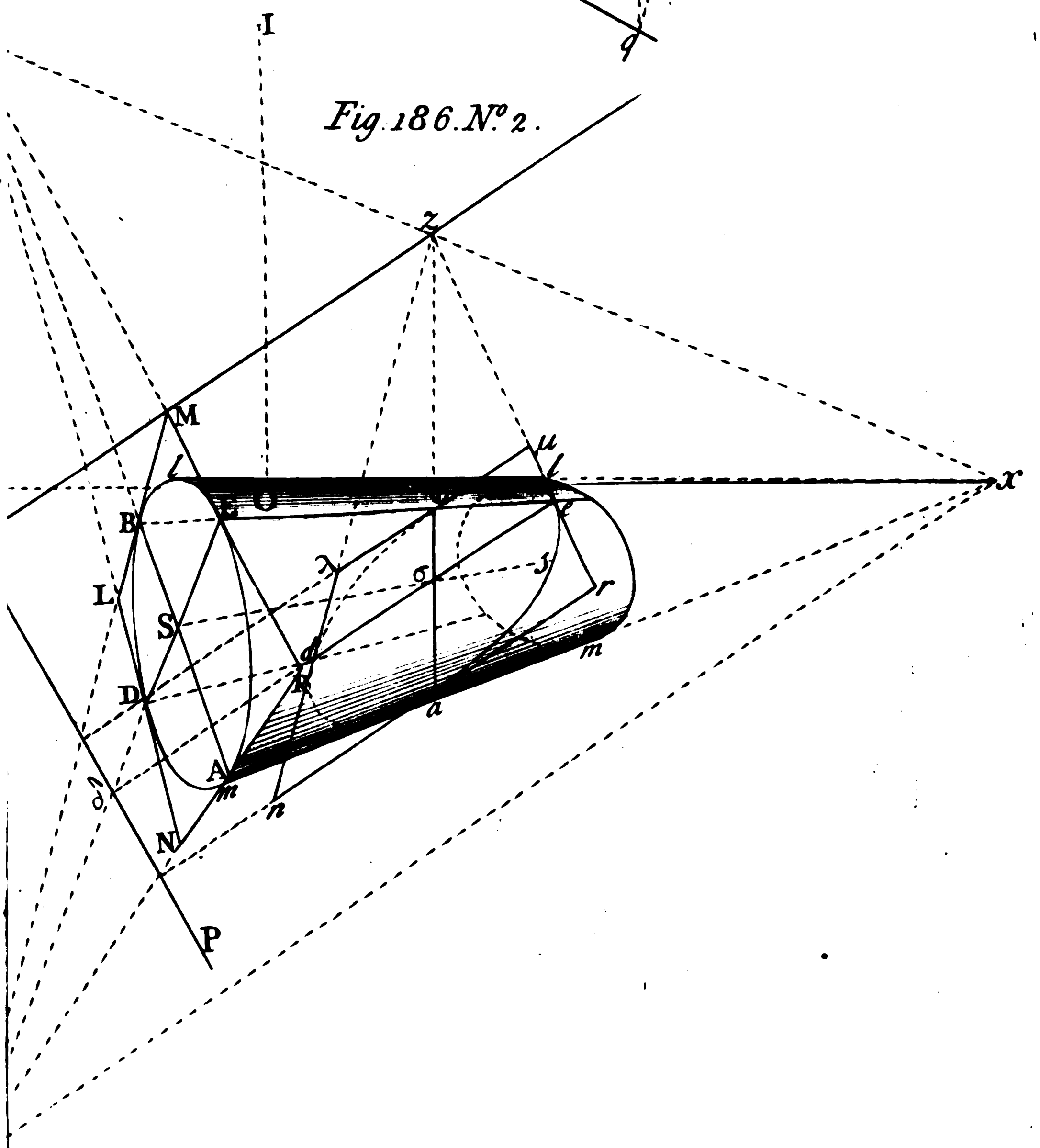

Fig. 186. No. 2.

J. Mynde sc.

Having drawn xT parallel to yz cutting EF in T, from T through S draw an apparent Diameter DE of the given Face ADBE, and find AB the Chord of the Tangents from T, which will likewiſe be an apparent Diameter, and from v the Vaniſhing Point of AB draw vx cutting yz in z; then having drawn the Sides Aa, Bb, Dd, and Ee, of the Cylinder, and its Axe Ss, from δ the Interſection of DE with yP, draw δe parallel to yz cutting Dd, Ss, and Ee, in d, σ and e; from z through d, σ and e, draw $n\lambda$, ab, and $r\mu$, and through a and b the Interſections of ab with Aa and Bb, draw nr, $\lambda\mu$, parallel to de, and thereby the *Trapezium* $nr\lambda\mu$ will be obtained, which incloſes the Section.

Dem. For T being the Parallel Seat of x on the Plane EF with reſpect to the Plane zyP, the Projections of DE and of the Tangents LM and NR which meet in T, are parallel to Tx or yz[a]; and δ being the Interſection of DE with the cutting Plane, δe parallel to yz is therefore the Indefinite Projection of DE, and d, σ, and e are the Projections of D, S, and E; and z being the Vaniſhing Point of the Projections of AB, and of the Tangents LN and MR whoſe Vaniſhing Point is v[b], zd, $z\sigma$, and ze, are thereſore the Indefinite Projections of thoſe Lines; and conſequently $\lambda nr\mu$ is the Projection of the *Trapezium* LNRM on the Plane zyP from the Point x; wherefore the Curve $daeb$ is the Section required. *Q. E. I.*

SCHOL.

This correſponds to Meth 2. Caſe 1. Prob. XI. the Line there marked Ty, being here the ſame with EF[c]; and if through T in this Figure, a Line Tt be drawn parallel to Py, it will correſpond to Tw in that Problem, the Projections of all Lines in the Plane EFP which paſs through any Point t of Tt, being parallel to a Line drawn from x to that Point[d]; and by the help of Tt, the Section might be found according to the firſt Method there propoſed: and here alſo, by the Poſition of the Line Tt with reſpect to the given Face ADBE, the Species of the produced Section is determined[e], but the Original of that Section muſt always be an Ellipſis or a Circle, unleſs the cutting Plane be parallel to the Axe of the Cylinder, in which Caſe the Section is a Parallelogram[f].

The General Corollary at the End of Prob. XI. is likewiſe applicable here; and the ſeveral Methods of this Problem may be applied to the finding the Section of a Priſm by any propoſed Plane, in the ſame manner as the Methods of that Problem were applied to the finding the Section of a Pyramid; ſeeing a Priſm may be conſidered as a Pyramid, whoſe Vertex is infinitely diſtant.

And in this manner may be found the Section of any ſtraight Corniſh or Entablature in Pieces of Architecture, by any propoſed Plane, the *Profil*, or any other Section of ſuch Corniſh or Entablature being given; ſeeing they may be conſidered as Priſms, having the given *Profil* or Section for their Baſe.

PROB. XV.

The Center and Diſtance of the Picture, and the Image of a Cylinder, being given; thence to find the Image of the Section of that Cylinder by any given Plane, whoſe Interſection with a Plane paſſing through one of the Sides of the Cylinder, is given.

CASE 1.

When the Axe of the Cylinder is parallel to the Picture.

Fig. 187. N°. 1. Let O be the Center, JO the Diſtance of the Picture, and ADBEADBE the given Cylinder, reſting with its Side EE on the Plane EFEE perpendicular to the Picture; and let Oo be the Vaniſhing Line of the Faces of the Cylinder, yz the Vaniſhing Line of the cutting Plane, and yP its Interſection with the Plane EFEE.

Through E draw the apparent Diameter ED of either of the Faces, perpendicular to the Plane EFP, and conſequently parallel to Oo, and another Diameter AB parallel to the Plane EFEE; and therefore tending to O the Interſection of EF with Oo; then having drawn the Sides AA, DD, BB, of the Cylinder, and its Axe SS, through e the Interſection of EE with Py, draw ed parallel to zy, cutting SS and DD in σ and d; from y, through d and σ draw $n\lambda$ and ab, and through a and b the Interſections of ab with AA and BB, draw nr and $\lambda\mu$ parallel to de, which will give the *Trapezium* $rn\lambda\mu$ which incloſes the Section.

[a] Meth. 2. Caſe 2. Prob. 6. B.V.

[b] Meth. 3. Caſe 2. Prob. 6. B.V.

[c] Meth. 3. Caſe 2. Prob. 6. B.V.

[d] Meth. 5. Caſe 2. Prob. 6. B.V.

[e] Gen. Cor. 1, 2, 3. Caſe 1. Prob. 11.

[f] Schol. Lem. 12. Art. 1.

Dem. For the Diameter DE and the Tangents NR and LM being parallel to the Vaniſhing Line Oo, their Projections de, nr, $\lambda\mu$, are parallel to the Vaniſhing Line yz; and O being the Vaniſhing Point of AB, NL, and RM, Oy parallel to SS cuts yz in y the Vaniſhing Point of their Projections ab, $n\lambda$ and $r\mu$[a]. The reſt is evident. *Q. E. I.*

[a] Meth. 2. Caſe 1. Prob. 14.

COR. 1.

If the given Plane which touches the Cylinder be not perpendicular to the Picture, another Plane perpendicular to the Picture may be thence found, which ſhall touch the given Cylinder in one of its Sides, whereby the Problem may be ſolved as before.

Thus, let $\varepsilon\varphi$ be the Vaniſhing Line of the given Plane touching the Cylinder in its Side ee, the Interſection eo of that Plane with the Plane of the Face adbe being a Tangent to it in e; and let vπ be the Interſection of the Plane $\varepsilon\varphi$ee, with the cutting Plane zy.

Draw the apparent Diameter DE of the Face adbe parallel to Oo the Vaniſhing Line of its Plane, and through E draw the Tangent EO which muſt tend to O the Center of the Vaniſhing Line Oo, the ſame with the Center of the Picture, the Axe of the Cylinder being here ſuppoſed perpendicular to its Faces; through O draw EF parallel to $\varepsilon\varphi$ cutting zy in y, and through t the Interſection of the Tangents eo and EO, draw $t\pi$ parallel to EF cutting vπ in π, and from y through π draw yP.

Then EFP will be a Plane perpendicular to the Picture touching the Side EE of the Cylinder, and yP will be the Interſection of that Plane with the cutting Plane; which reduces the Problem to that which was at firſt propoſed.

For it is evident that t is a Point of the Interſection of the Planes $\varepsilon\varphi$e and EFE, and therefore that $t\pi$ parallel to $\varepsilon\varphi$ and EF, is their common Interſection, and π is therefore a Point in the Interſection of the cutting Plane zyP with both thoſe Planes; wherefore as vπ is its Interſection with the Plane $\varepsilon\varphi$, $y\pi$ is its Interſection with the Plane EF.

COR. 2.

If the Touching Plane $\varepsilon\varphi$ee be given, the Interſection of the Cylinder with the Plane $zy\pi$ may be found without ſubſtituting another Plane EFP perpendicular to the Picture; by drawing the apparent Diameters de and ab, the one tending to the Vaniſhing Point of Perpendiculars to the Plane $\varepsilon\varphi$ee in the Line Oo[b], and the other to o the Center of the Vaniſhing Line $\varepsilon\varphi$; for then v will be the Vaniſhing Point of the Projections of ab and of the Tangents in e and d, and a Line drawn from the Vaniſhing Point of de parallel to $\varepsilon\varphi$, will cut zy in the Vaniſhing Point of the Projections of de, and of the Tangents in a and b.

[b] Cor. 3. Prop. 20. B. IV.

For although the Projection of ab thus found, be not a Real Diameter of the Section, the Tangents at its Extremities tending to a Vaniſhing Point in zy, yet it may ſometimes be more expeditious to deſcribe the Section by the Help of the Oblique *Trapezium* thus determined, than to find the Direction of the Projection of de, and conſequently of the Tangents at the Extremities of the Real Diameter of the Face adbe to which de is a double Ordinate[c]; it being neceſſary for that End, firſt to find the Interſection of the Cutting Plane with the Plane of that Face. However, this laſt Method may be imployed in caſe the Vaniſhing Point of the Projection of de happen to be out of Reach, as it is in the preſent Figure.

[c] Meth. 1. Caſe 1. Prob. 14.

COR. 3.

If the Plane EF paſs through the Axe of the Cylinder ſo as to cut it in AABB, and ya be the Interſection of that Plane with the cutting Plane zy, meeting the Sides AA and BB of the Cylinder in a and b, and its Axe in σ; it is evident the Section *adbe* may be found as before, only uſing the Point σ inſtead of e the Interſection of EE with yP.

CASE 2.

When the Axe of the Cylinder is not parallel to the Picture.

Fig. 187. N°. 2. The ſame Things being ſuppoſed as before; from x the Vaniſhing Point of the Axe Ss, draw xT parallel to the Vaniſhing Line yz of the cutting Plane, till it meet the Vaniſhing Line oT of the Faces of the Cylinder in T, through which draw Tt parallel to Py the Interſection of the cutting Plane zyP with the Plane EFP on which the Cylinder reſts with its Side EE, which Plane EFP is here ſuppoſed perpendicular to the Picture; then having drawn the apparent Diameter DE of the Face ADBE parallel to oT, produce it till it cut Tt in t, and draw xt, and having found AB the

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Plate 111. Book 6 Sect. 3.

Fig. 187. No. 1.

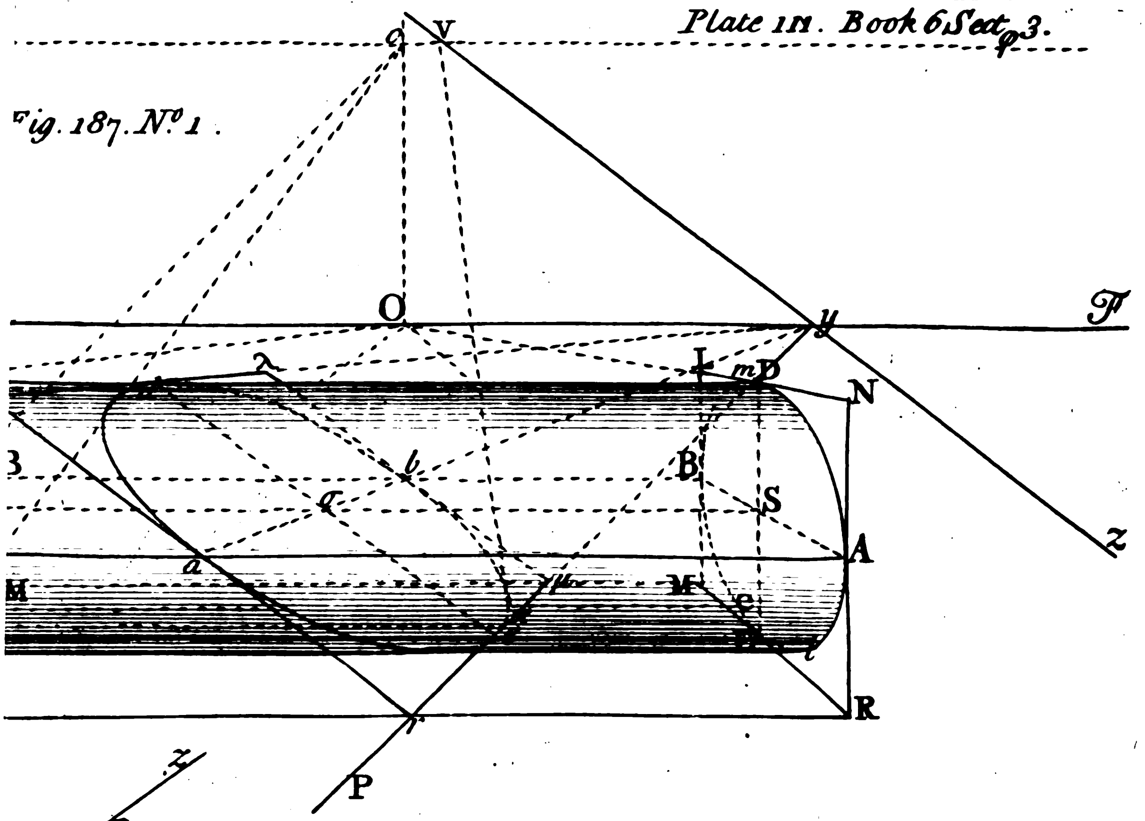


Fig. 187. No. 2.

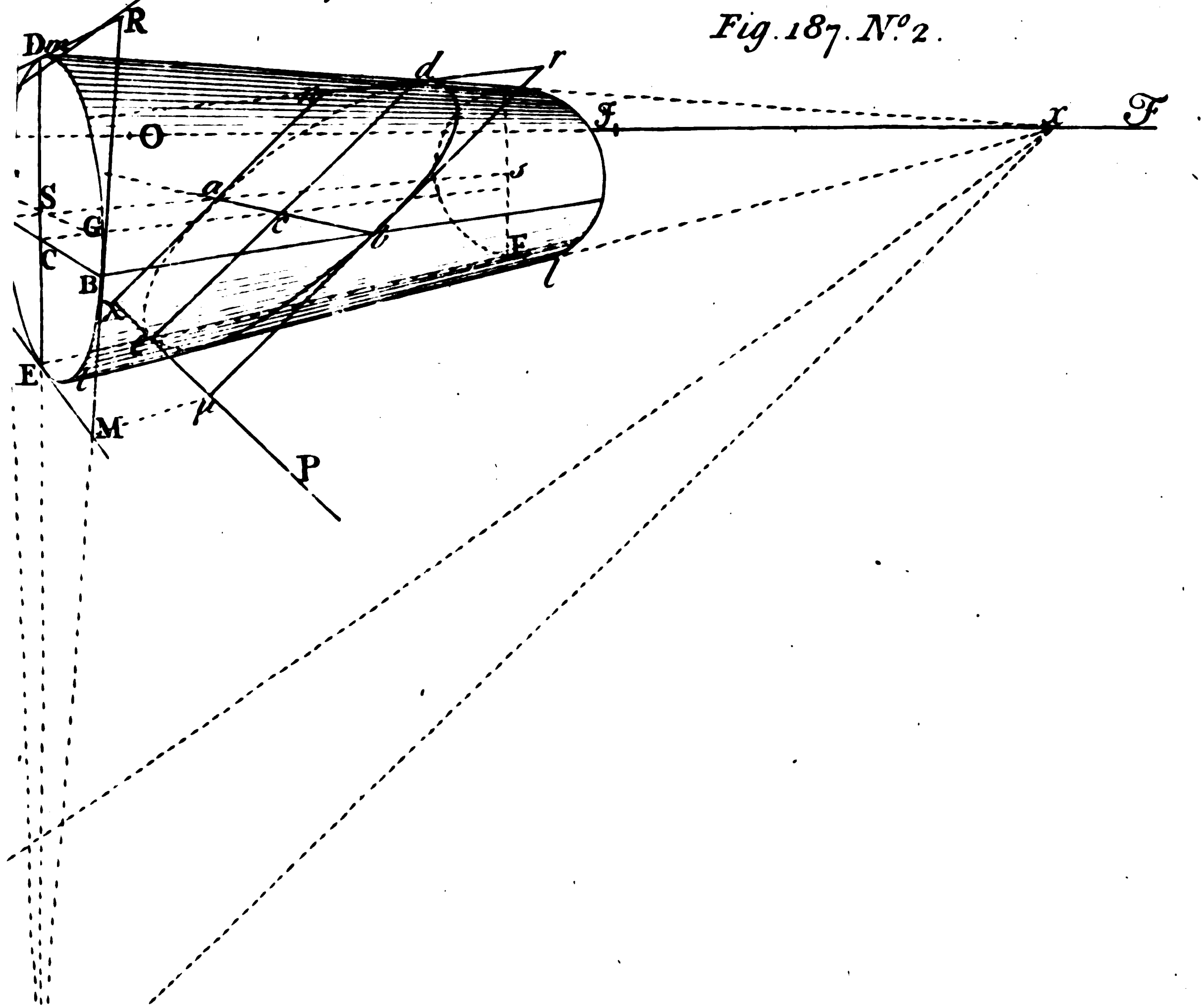
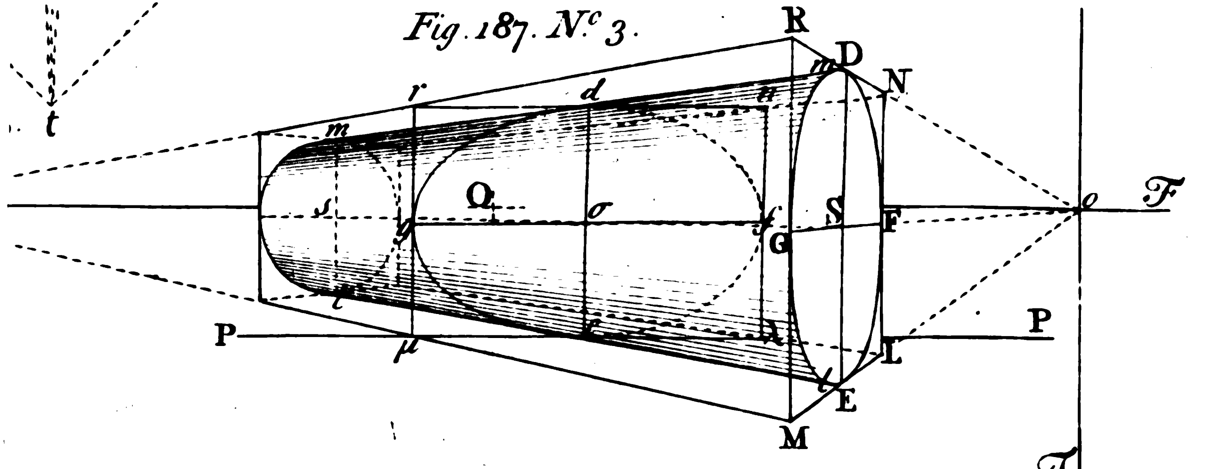


Fig. 187. No. 3.



J. Mynde sc.

Chord of the Tangents from t, which muſt paſs through o the Interſection of EF with oT, cutting DE in C, and drawn the Sides Aa, Bb, Dd, of the Cylinder, and a Line Cc all tending to x, through e the Interſection of E E with Py, draw ed parallel to xt cutting Cc and Dd in c and d; then from y through c and d draw ab and nr, and through a and b the Interſections of ab with Aa and Bb, draw $n\lambda$ and μr parallel to ed, and thereby the *Trapezium* $\lambda\mu r n$ will be obtained which incloſes the Section.

Dem. For the Projections of E D, L N, and M R, which meet in t, are parallel to xt[a], and the Projections of A B, N R, and L M, which meet in o, have y for their Vaniſhing Point as before. *Q. E. I.*

[a] Schol. Caſe 2. Prob. 14.

C O R. 1.

If the Plane EFP be not perpendicular to the Picture, the Diameter D E muſt tend to the Vaniſhing Point of Perpendiculars to the Plane EFP in the Line To[b], which in this Caſe paſſes through that Vaniſhing Point[c]; but this makes no Difference in the Remainder of the Work, ſeeing D E will ſtill cut Tt in ſome Point t to be uſed as before.

[b] Cor. 2. Caſe 1.

[c] Cor. Prop. 8. B. IV.

C O R. 2.

If the Vaniſhing Line zy of the cutting Plane be parallel to the Vaniſhing Line oT of the Faces of the Cylinder; the Point T, and conſequently the Line Tt and the Point t become infinitely diſtant, and the apparent Diameter F G will be the Chord of the Tangents from the infinitely diſtant Point t; wherefore the Projections of D E and of the Tangents in F and G will be parallel to oT and zy, but y will ſtill be the Vaniſhing Point of the Projections of L M, F G, and N R.

C O R. 3.

Fig. 187. N°. 3.

If the cutting Plane be parallel to the Picture, the Projections of DE and of the Tangents in F and G will be parallel to oT, and the Projections of LM, F G, and N R, will be parallel to EF.

For the Originals of D E and of the Tangents in F and G being parallel to the Picture, a Line xv drawn through x parallel to them, and conſequently to oT, is the Vaniſhing Line of the Planes which paſs through x and thoſe three Lines, the Interſections of which Planes with the cutting Plane (which are the ſame with their Projections from x) are therefore alſo parallel to them; and becauſe EF is the Vaniſhing Line of the Planes which paſs through F G, N R, L M, and the Point x, the Interſections of thoſe three Planes with the cutting Plane are therefore parallel to EF[d].

[d] Cor. Theor. 3. B. I.

P R O B. XVI.

The Center and Diſtance of the Picture, and the Images of two Cylinders of equal Diameters, whoſe Axes cut each other, being given; thence to deſcribe the common Interſections of thoſe Cylinders.

Fig. 188. N°. 1.

Let O be the Center, and I O the Diſtance of the Picture, X X and Y Y the given Cylinders, and Ss and Cc their Axes interſecting in σ; and let EF be the Vaniſhing Line of a Plane paſſing through the given Axes, and cutting the two Cylinders in their Sides A a, B b, and A B, a b, thereby forming a *Trapezium* A a b B whoſe Angles A, a, b, and B, are common to the Surfaces of both Cylinders.

Draw the Diagonals A b, a B, of the *Trapezium* A a b B interſecting in σ, and cutting the Vaniſhing Line EF in y and z, through which draw two Vaniſhing Lines yv, zw, of Planes perpendicular to the Plane EFA a; then find the Sections Adbe, adBe, of either of the Cylinders X X, with the Planes vyL N and w$z\lambda n$ which paſs through A b and a B[e]; and the Curves thus found will be the common Interſections of the Cylinders propoſed.

[e] Cor. 3. Caſe 1. Prob. 15.

Dem. For the Originals of the Sides L N and $r\mu$ of the *Trapezium* L Nμr which incloſes the Section Adbe by the Plane vyL N, being perpendicular to the Plane EFA a which paſſes through the given Axes[f], and the Points A and b being common to the Surfaces of both the Cylinders, L N and $r\mu$ which are Tangents to the Cylinder X X in A and b, are alſo Tangents to the Cylinder Y Y in the ſame two Points[g]; and becauſe σ is a Point in the Axes of both the Cylinders, de drawn through σ perpendicular to the Plane EFA a, repreſents a Diameter common to them both, wherefore

[f] Prob. 15.

[g] Schol. Lem. 12. Art. 5.

fore its Extremities d and e are alſo Points common to the Surfaces of both Cylinders; and the Sides $\mathrm{L}r$ and $\mathrm{N}\mu$ of the *Trapezium* $\mathrm{LN}\mu r$ which are Tangents in d and e to the Cylinder XX, are therefore alſo Tangents to the Cylinder YY in the ſame two Points; and conſequently the Curve $\mathrm{A}d\mathrm{b}e$ incloſed within the *Trapezium* $\mathrm{LN}\mu r$, which is the Section of the Cylinder XX by the Plane $\mathrm{v}y\mathrm{LN}$, is likewiſe the Section of the other Cylinder by the ſame Plane: In like manner it may be ſhewn that the Curve $\mathrm{a}d\mathrm{B}e$ incloſed within the *Trapezium* $n\lambda\mathrm{RM}$ is the common Section of both the Cylinders by the Plane $\mathrm{w}z\lambda n$; wherefore theſe two Sections agreeing every where with the Surfaces of both the Cylinders, they are the Interſections of the given Cylinders with each other. *Q. E. I.*

COR. I.

If a Cylinder XX be cut by a Plane EF paſſing through its Axe $\mathrm{S}s$ and its Sides Aa and Bb, and from any Point O in EF two Lines OA, Oa, be drawn in the Plane EF, cutting the Sides Aa and Bb, in A, B, a, and b, thereby forming a *Trapezium* AabB repreſenting a Parallelogram, and the Sections $\mathrm{A}d\mathrm{b}e$, [illegible]$d\mathrm{B}e$, of that Cylinder by two Planes $\mathrm{v}y\mathrm{LN}$, $\mathrm{w}z\lambda n$, paſſing through the Diagonals Ab, aB, of that *Trapezium* perpendicular to the Plane EF, be found[a]; then, if on either of theſe Sections $\mathrm{A}d\mathrm{b}e$, a Cylinder YY be fitted, having its Axe OC parallel to the Originals of AB and ab[b], that Cylinder will alſo agree with the other Section $\mathrm{a}d\mathrm{B}e$; and theſe two Sections will be the common Interſections of the given Cylinder XX with the Cylinder YY thus formed.

For A and b being by ſuppoſition two Points in the Surface of the Cylinder YY, OA and Ob are two Sides of that Cylinder, wherefore B and a are alſo Points in its Surface; now LN being a Tangent to the Section $\mathrm{A}d\mathrm{b}e$ in A, it is therefore alſo a Tangent in A to the Cylinder YY formed on that Section[c]; wherefore a Plane IOLN paſſing through LN and the Side AB of the Cylinder YY touches it in that Side, and RM which is a Line in that Plane (the Originals of LN and RM being parallel) is therefore a Tangent to the Cylinder YY in B; in like manner $r\mu$ being a Tangent to the Section $\mathrm{A}d\mathrm{b}e$ in b, the Plane $\mathrm{IO}r\mu$ touches the Cylinder YY in its Side ab, and λn is therefore a Tangent to that Cylinder in a: Again, the Points d and e where the two Sections croſs, being Points common to the Surfaces of both the Cylinders, dy or $\mathrm{L}r$ which is a Tangent to the Section $\mathrm{A}d\mathrm{b}e$ in d, is alſo a Tangent to the Cylinder YY in the ſame Point; wherefore the Plane $EF\mathrm{L}\lambda$, which paſſes through $\mathrm{L}r$ and the Side $d\mathrm{O}$ or $\mathrm{G}g$ of the Cylinder YY, touches that Cylinder in its Side $\mathrm{G}g$, and conſequently zd or $\lambda\mathrm{R}$, which is a Line in the ſame Plane, is alſo a Tangent to the Cylinder YY in d, and for the like reaſon ze or $n\mathrm{M}$ is a Tangent to the Cylinder YY in e; thus the Sides of the *Trapezium* $n\lambda\mathrm{RM}$ being Tangents to the Cylinder YY in a, d, B, and e, the Curve $\mathrm{a}d\mathrm{B}e$ is the Section of that Cylinder by the Plane $\mathrm{w}z\lambda n$, which Cylinder therefore agreeing with both the Sections, theſe Sections are the common Interſections of the Cylinders XX and YY.

COR. 2.

If the Originals of the Axes $\mathrm{S}s$ and $\mathrm{C}c$ be perpendicular, and the Sides Aa and AB of the *Trapezium* AabB repreſent equal Lines, then AabB will repreſent a Square, and $\mathrm{C}c$ and $\mathrm{S}s$ will repreſent Diameters of both the Cylinders equal to each other, and if the Original of their common Diameter de be alſo equal to them, both Cylinders will be Right, as is ſuppoſed in the Problem; but if the Original of de be bigger or leſs than thoſe of the Diameters $\mathrm{S}s$ and $\mathrm{C}c$, the Cylinders will both be Scalene, that is, the Section ADBE of the Cylinder XX by the Plane IOLN perpendicular to its Axe $\mathrm{S}s$ will be an Ellipſis, of which AB and DE will repreſent the Axes; and the Section AGaF of the Cylinder YY by the Plane $\mathrm{L}\lambda n\mathrm{N}$ perpendicular to its Axe $\mathrm{C}c$ will alſo be an Ellipſis, of which Aa and FG will repreſent the Axes; which Ellipſes will be equal and ſimilar, the Originals of their Axes AB, Aa, and DE, FG, being reſpectively equal, theſe laſt being equal to the Original of de.

COR. 3.

If the Originals of the Sides AB and Aa of the *Trapezium* AabB be unequal, and either of the Cylinders XX be Right, the other Cylinder YY will be Scalene; and then if AB be the greater Side, FG will repreſent the longer, and Aa the ſhorter Axe of the Ellipſis formed by the Section AGaF of the Cylinder YY; but if AB be the ſhorter Side, then Aa will be the longer, and FG the ſhorter Axe of that Ellipſis; FG

in

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[a] Prob. 15.

[b] Schol. Lem. 12. Art. 2.

[c] Schol. Lem. 12. Art. 5.

Plate 112 Book 6. Sect. 3.

Fig. 188. No. 1.

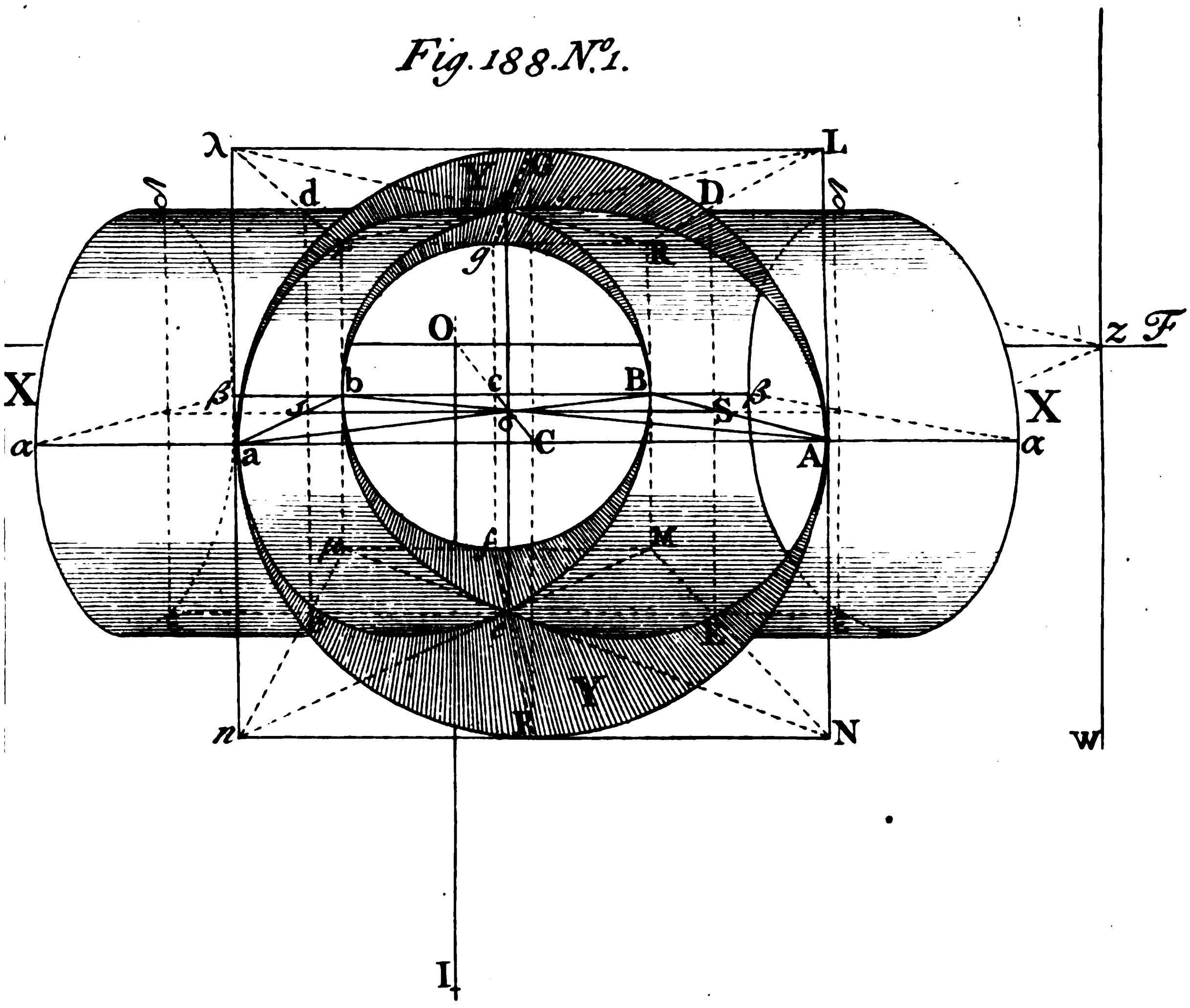
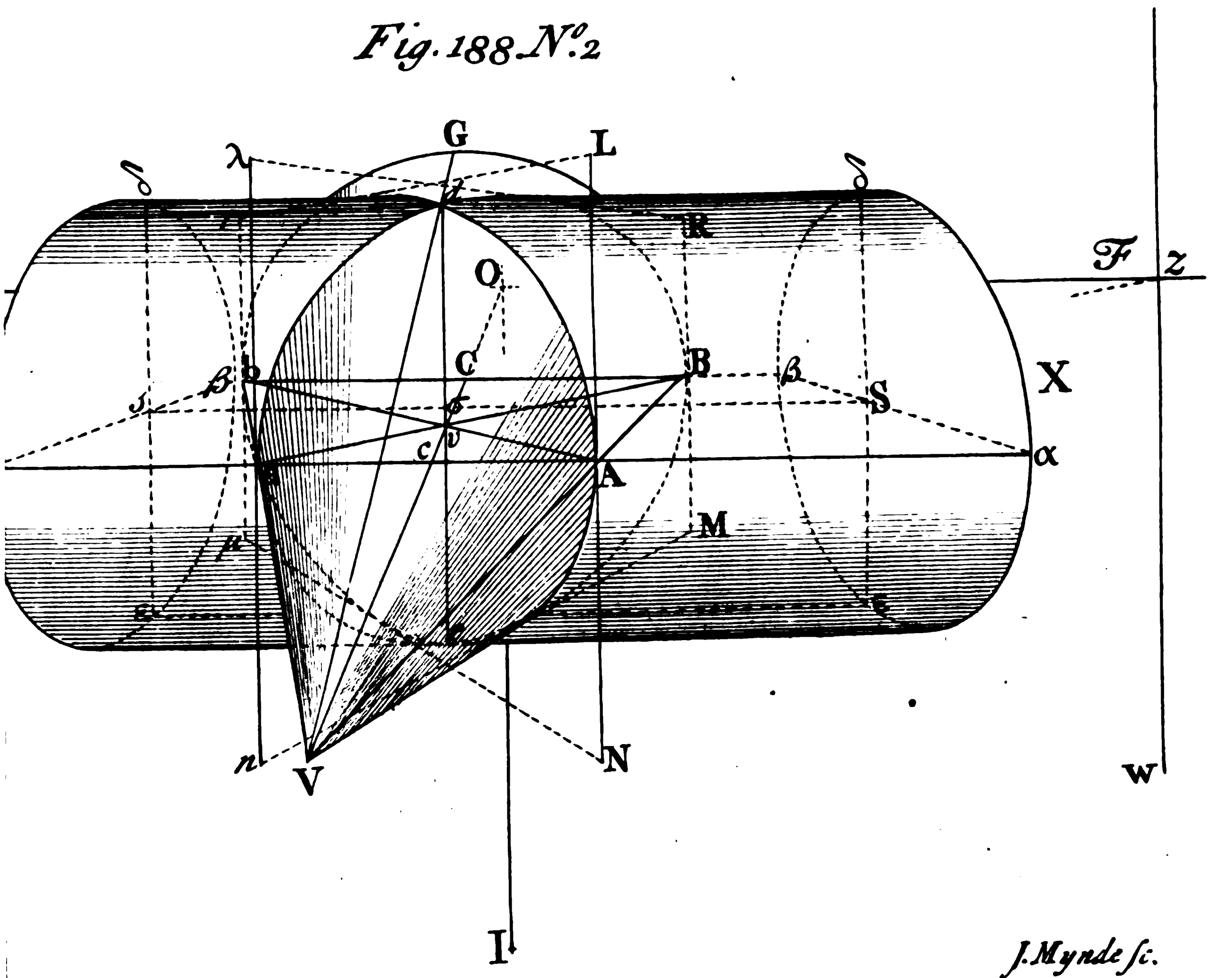


Fig. 188. No. 2



J. Mynde sc.

in either Caſe, repreſenting a Line equal to the Original of *de* or DE, which is here ſuppoſed equal to that of AB, the Cylinder XX being Right.

COR. 4.

If the Sides AB, ab, of the *Trapezium* AabB, do not repreſent Parallel Lines, Fig. 188. but be drawn ſo as to meet in any Point V not their Vaniſhing Point, and the Secti- N°. 2. ons A*d*b*e*, a*d*B*e*, of the given Cylinder XX by two Planes v*y*LN, w*zλn*, paſſing through the Diagonals Ab, aB, of that *Trapezium*, perpendicular to the Plane *EF*Aa, be found as before; then if on either of theſe Sections A*d*b*e* a Cone be fitted having V for its Vertex, that Cone will agree with the other Section a*d*B*e*; and theſe two Sections will be the common Interſection of the Cylinder XX with the Cone thus formed.

This evidently follows from the firſt Corollary, there being no Difference in the Demonſtration, whether the Sides AB, ab, of the *Trapezium* AabB meet in a Vaniſhing Point O, or in any other Point V at a moderate Diſtance in the Plane *EF*Aa.

SCHOL.

But here it muſt be obſerved, that the Interſection *v* of the Diagonals Ab and aB, is not a Point in the Axe S*s* of the Cylinder XX, and conſequently that *de* is not an apparent Diameter of that Cylinder, as it is when the Originals of AB and ab are parallel; but *de* is the Chord of the Tangents to the Cylinder XX in *d* and *e* from the Point V, and likewiſe a Diameter of the Cone BGbV, the Sides V*d*, V*e*, of that Cone being Tangents to the Cylinder XX in *d* and *e*.

This Corollary is alſo applicable to the Interſections of two Cones, having one common Diameter perpendicular to the Plane which paſſes through their Axes; which Diameter will be the common Chord of the Tangents to each Cone from the Vertex of the other.

If what has been ſaid here, be compared with Prob. IX. and XIII, it will appear that the whole is founded on the ſame Principles; the Methods there propoſed for finding the Boundary of the Light which can enter a hollow Cone or Cylinder from a Luminous Point, being only Ways to find one of the Sections of ſuch Cone or Cylinder by another Cone or Cylinder formed by the Rays of Light, the open Baſe of the given Cone or Cylinder being the other of thoſe Sections; and what was ſhewn at thoſe Problems, will ſerve to extend this to all Varieties of Caſes.

COR. 5.

The ſame things being ſuppoſed as in the laſt Corollary; if on the Sections A*d*b*e*, a*d*B*e*, two Cylinders be fitted, having their Sides parallel to the Original of the Axe VC of the Cone which paſſes through both Sections; thoſe two Cylinders will cut each other in their common Sides O*d* and O*e*, and the common Portion of theſe two Cylinders will be two Curvilinear Surfaces (either Circular or Elliptick) meeting in two Angles at their mutual Interſections in O*d* and O*e*, the Diſtance of which Angles from each other is meaſured by the Line *de*, which paſſes through *v* the Interſection of the Diagonals Ab and aB, perpendicular to the Plane *EF*Aa.

This is evident, it being impoſſible that the two Cylinders thus formed ſhould coincide.

GENERAL COROLLARY I.

It having been already obſerved that all Circular or Elliptick Arches or Vaults may be conſidered as Portions of Cylinders[a]; this Problem ſerves to find the Interſections of all Cylindrical Arches which meet or croſs each other in Vaulted Roofs, *&c.* and likewiſe furniſhes ſome Remarks touching the different Curvatures of Arches of different Breadths or Heights, which cut each other in ſuch manner that their common Interſections (which are ſometimes called the *Miter Groins)* may hang perpendicularly over their Baſes, as is generally required in Building that they ſhould do.

Thus, if the Semi-Cylinder XX be conſidered as an Arch ſpringing from its Sides Fig. 188. Aa and Bb, repreſenting the Tops of the upright Walls on which it reſts, lying in a N°. I. Plane parallel to the Horizon, and the Semi-Cylinder YY be another Arch ſpringing from its Sides AB and ab, theſe two Arches having the ſame Height σd; then

I. It is evident that the Curves A*d*b, a*d*B, which lye in Planes perpendicular to the Plane AabB, are the Interſections of theſe two Arches, or the *Miter Groins* required, and that each of them forms a compleat Semi-Ellipſis, having the Diagonals Ab, aB, for one of their Axes, and their common Height σd for their Semi-Conjugate

[a] Schol. Lem. 12. Art. 6.

[a] Cor. 1. gate Axe; the Original of *σd* biſecting Ab and aB[a], and being perpendicular to their common Interſection.

2. If the Breadths AB and Aa of theſe Arches be equal, their Curvatures will be
[b] Cor. 2. equal and ſimilar[b]; but if their Breadths be unequal, their Curvatures will alſo be unequal, and diſſimilar, that is, if the Curvature of the one be Circular, that of the other
[c] Cor. 3. will be Elliptick[c], the wider Arch having always the flatter Curvature.

Fig. 188. N°. 2. 3. If two Arches have not the ſame Height, as when a lower Arch is inſerted into the Side of a higher for the Reception of a Door or Window, and their Interſections or *Miter Groins* be required to be perpendicular over their Baſes; if the higher Arch be Circular or Elliptick, the other muſt be Gothick; that is, it will not form one continued Curvature, but two diſtinct Curvatures, one on each Side, meeting in an An-
[d] Cor. 5. gle at the Top[d]: and the *Miter Groins* A*d* and a*d* will be Portions of the two Ellipſes A*d*b*e*, a*d*B*e*, which have Ab and aB for one of their Axes, and the Line *vd* a common Ordinate to both of them.

4. If the ſmaller Arch were for the Reception of a Door or Window in an upright Circular Wall, and it were required to have a ſingle Curvature, and that its Interſection with the Circular Wall ſhould lye in a Plane; the Curvature of the inſerted Arch muſt be Part of a Cylinder fitted on a Section of the upright Circular Wall by a ſloping Plane, and muſt always be leſs than one half of that Cylinder.

Thus, ſuppoſe the Axe S*s* of the Cylinder XX were perpendicular to the Horizon, and an Arch of the Breadth of *de*, leſs than the Diameter of that Cylinder, were to be inſerted in it; *dae* would then be the Section of the Surface of that Cylinder by the Arch propoſed, which Arch would be a Portion of a Cylinder fitted on the Section a*d*B*e*, but terminated at its Sides O*d* and O*e*, and therefore leſs than a Semi-Cylinder.

5. But if it were required to inſert a Semi-Circular or Semi-Elliptick Arch into a larger; the Interſection of thoſe Arches will not be a plain Figure, but will form a Sort of a Curve Line, lying in a Curvilinear Surface, with which no Plane can agree; nevertheleſs the Image of ſuch a Section may be found in this manner.

Fig. 188. N°. 3. Let O be the Center, and IO the Diſtance of the Picture, XX an upright Cylinder, S*s* its Axe, and *EF* the Vaniſhing Line of the Planes of its Faces; and let it be required to deſcribe the Section of this Cylinder by a ſmaller Cylinder, having DE for its Diameter, and C*c* for its Axe, both in a Plane parallel to the Faces of the Cylinder XX, and meeting its Axe S*s* in σ.

On the Diameter DE deſcribe the Section DaEa of the ſmaller Cylinder by a Plane paſſing through S*s* and DE, and having found a Section AFBG of the Cylinder XX parallel to its Faces, at a convenient Diſtance from the ſmaller Cylinder, draw its Diameter FG, repreſenting a parallel to the Diameter DE, and transfer the Points D and E to *d* and *e* in the Diameter FG by Lines parallel to S*s*; then from O the Vaniſhing Point of the Axe C*c*, draw O*d*, O*e*, cutting the Section AFBG in d and e, and from d and e draw the Sides dδ, eε, of the Cylinder XX, till they be cut in δ and ε by OD and OE, and δ and ε will be two Points of the Section required; in like manner, draw any Line bb in the Section DaEa parallel to S*s*, and produce it till it cut FG in *p*, and having drawn O*p* meeting the Section AFBG in P, draw the Side Pβ of the Cylinder XX, which will be cut by Ob, Ob, in β and β, two more Points of the required Section: And thus, as many more Points α, α, γ, γ, &c. in that Section may be obtained as are neceſſary for the Deſcription of the whole.

For it is evident that aα, bβ, cγ, Dδ, &c. which all tend to O, and repreſent parallels to the Axe C*c*, are Sides of the ſmaller Cylinder, and are in the ſame Planes reſpectively with the Sides Aα, Pβ, Qγ, dδ, of the Cylinder XX; and that therefore their Interſections α, β, γ, δ, are Points common to the Surfaces of both the Cylinders, and conſequently that the Curve αδαε thus found is the Section required.

And here it is manifeſt, that the Sides dδ, eε, of the Cylinder XX, determined by the help of the Extremities D and E of the Diameter DE of the ſmaller Cylinder, are Tangents to the Section αδαε in δ and ε.

After the ſame manner, the Section of the Cylinder XX, on its hinder Part, by the ſmaller Cylinder, might be obtained if deſired.

GENERAL COROLLARY 2.

The Arches hitherto mentioned, whether Circular or Elliptick, are called *Right Arches*, as taking their riſe from Sides lying in a Plane parallel to the Horizon; but beſides theſe, there are alſo ***Rampant*** Arches which ſpring from Sides lying in a Plane inclining to the Horizon.

Now this Inclination may be either lengthwiſe of the Arch, its Breadth remaining parallel to the Horizon; Or the Inclination may be in the Breadth, the Axe remaining Horizontal; Or laſtly, the Arch may incline to the Horizon, both in its Breadth and Length.

1. Thus, let O be the Center of the Picture, and *EF* the Vaniſhing Line of the Horizon, or the Horizontal Line, *zy* the Vaniſhing Line of the Plane $\alpha\alpha\beta\beta$ on which the Semi-Cylinder or Arch $\alpha\delta\beta\alpha\delta\beta$ reſts with its Sides $\alpha\alpha$ and $\beta\beta$, its Axe S*s* being parallel to *zy*, and conſequently to the Picture; then this Arch will be a Rampart Arch of the firſt Sort, its Axe S*s* inclining to the Horizon, but the Baſe Lines $\alpha\beta$, $\alpha\beta$, of its Sections right a-croſs being parallel to it, as having O for their Vaniſhing Point. Fig. 188. N°. 4.

In this Caſe, the Arch may be either Circular or Elliptick, but more uſually Elliptick, and of ſuch a Curvature, that its Section $\alpha\delta\beta$ right a-croſs by a Plane perpendicular to the Horizon, may be a Semicircle; to the End it may agree, or make a true Joint with a Right Circular Arch having its Axe perpendicular to that Plane, and conſequently parallel to the Horizon; for if the Rampant Arch were Circular, then its Section $\alpha\delta\beta$ by a Plane perpendicular to the Horizon, muſt be a Semi-Ellipſis, and could therefore only agree with a Right Arch of the ſame Elliptick Curvature.

2. If from O any two Lines AB, ab, be drawn, forming with the Sides $\alpha\alpha$ and $\beta\beta$ of the given Arch, a Parallelogram AabB, and the Sections A*d*b, a*d*B, of that Arch by the Planes *y*v, *z*w, paſſing through the Diagonals Ab, aB, perpendicular to the Horizon be found; the Semi-Cylinder AGaB*g*b which paſſes through theſe Sections, will be a Rampant Arch of the ſecond Sort, having its Axe C*c* parallel to the Horizon, and the Baſe Lines Aa, Bb, of its Sections right a-croſs, inclining to it.

In this Caſe, the Arch AGaB*g*b muſt be Elliptick, and of ſuch a Form, that its Section right a-croſs by a Plane LλAa perpendicular to the Horizon, may be a Semi-Ellipſis; but then the Diameter Aa on which that Section reſts, is not one of its Axes (as it is in the Caſe of Right Elliptick Arches) but only ſuch a Diameter as has its Semi-Conjugate CG, and conſequently the Tangents AL, aλ, at its Extremities, perpendicular to the Horizon, to the End theſe Tangents may agree with the upright Walls from which the Arch ſprings; which they could not do were the Arch Circular, in regard the Tangents would in that Caſe be perpendicular to Aa which inclines to the Horizon.

3. If *o* were the Center of the Picture, $\epsilon\phi$ the Horizontal Line, and O the Center of the Vaniſhing Line *zy*, the other things remaining as before; the Arch $\alpha\delta\beta\alpha\delta\beta$, would repreſent an Arch of the third Sort, inclining to the Horizon both in its Breadth $\alpha\beta$, and its Length S*s*; which Arch muſt alſo be Elliptick, the Section $\alpha\delta\beta$ reſting on a Diameter $\alpha\beta$ inclining to the Horizon, and having Sδ perpendicular to the Horizon for its Semi-Conjugate; and the Croſs Arch AGaB*g*b would alſo be of the ſame Kind, its Axe C*c*, and the Baſe Line Aa of its Section AGa, both inclining to the Horizon, to which the Tangents LA, λa, are perpendicular.

Now theſe being all the Varieties of Circular and Elliptick Arches which can ſpring from two Parallel Sides; this Propoſition ſerves equally to find the Interſections of any of them with the other, having the Vaniſhing Line *zy* of the Plane which paſſes through their Axes, and the Horizontal Line *EF* given; the Planes v*y*AL and w*z*aλ which paſs through the Diagonals Ab and aB of the Parallelogram AabB formed by the Sides from whence the Arches ſpring, being always made to repreſent Planes perpendicular to the Horizon, whatever inclination the Plane *zy* may have to it: all which is ſufficiently evident.

GENERAL COROLLARY 3.

This Problem may likewiſe be applied to the finding the Interſections of ſimilar Gothick Arches or Vaults with each other.

For if a Figure be drawn ſimilar to a Perpendicular Section of the propoſed Gothick Vault, and that Figure be incloſed in a Parallelogram, having the Baſe Line of the Arch for one of its Sides, and the two Curve Sides of the Arch be each divided into three or more equal Parts at Diſcretion, and the Parallelogram be ſubdivided by Lines paſſing through thoſe Diviſions, after the ſame manner as is done for a Circle or an Ellipſis; a Model for the Deſcription of that Arch will be got, which will likewiſe ſerve for the Deſcription of the Section of that Arch by any Plane whatſoever; the Fig. 189. N°. 1, 2, 3, 4.

Sides

Sides of the *Trapezia* which incloſe ſuch Sections, being ſo divided, as to repreſent Lines divided in the ſame Proportion as the correſponding Sides of the Model.

But as Gothick Arches are of many different Forms, ſome having only one Portion of an Arch for each Side, and others having each Side compoſed of two or more Arches of different Curvatures, ſo joined together, as that Tangents at each Juncture may be Tangents to both the Contiguous Arches; a different Model muſt be made for every different kind of Gothick Arch; whereas, for Circular or Elliptick Arches, the ſame Model ſerves them all: Gothick Arches have nevertheleſs this in common with the other Sort, that all Sections of the ſame Gothick Vault by any Planes however differently inclined to each other, are to be conſidered as Gothick Arches of the ſame kind, and may be deſcribed by the help of one and the ſame Model.

Gothick Arches, as well as the Circular or Elliptick, may be Rampant in all the different Manners already explained[a]; and in that Caſe are always ſo formed, that the Line which Meaſures their Height, and alſo the Tangents at the Extremities of the Baſe of every Perpendicular Section, may be perpendicular to the Horizon, to the End the Sides of the Arch may reſt perpendicularly on the Walls from whence they ſpring; and what has been ſaid of the Deſcription of the Interſections of Rampant Arches of the other Sort, is alſo applicable to theſe.

[a] Gen. Cor. 2.

The Figures here referred to, repreſent Perpendicular Sections of Gothick Arches of ſeveral kinds fitted with Models.

Fig. 189. N°. 1. In the firſt, the Curve Sides AC, BC, are Arches of Circles deſcribed from B and A as Centers with the *Radius* AB, their Interſection C is the Crown of the Arch, SC its Height, and AB its Span or Breadth.

Fig. 189. N°. 2. In the ſecond, the Breadth AB is divided into three equal Parts in D and E, and the Sides AC, BC, are Arches of Circles deſcribed from E and D as Centers with the *Radius* EA or DB.

Fig. 189. N°. 3. In the third, the Breadth AB is alſo divided into three equal Parts in D and E, AF and BG are perpendicular to AB, and equal to AE or DB; the Part A*d* of the Curve Side AC is an Arch of a Circle deſcribed from the Center D with the *Radius* AD, and terminated at *d* by its Interſection with GD, and the Remainder *d*C of that Side is an Arch of a Circle deſcribed with the Center G and *Radius* G*d*; the other Side BC is in like manner formed of two Arches B*e* and *e*C, the firſt from the Center E with the *Radius* EB, and the other from the Center F with the *Radius* F*e*: and here 'tis evident, that if through *d*, a Perpendicular to G*d* be drawn, it will be a Tangent to both the Contiguous Arches A*d* and *d*C in their Juncture *d*, the Centers of theſe two Arches being at D and G in the Line G*d*; and for the ſame Reaſon a Perpendicular to F*e* drawn through *e*, is a Tangent in that Point to both the Arches B*e* and *e*C.

Fig. 189. N°. 4. In the fourth, the Breadth AB is likewiſe divided into three equal Parts; the Points F and G are the Vertices of two Equilateral Triangles formed on AE and DB as their Baſes, D and E are the Centers of the Arches A*d* and B*e*, which are terminated at *d* and *e* by GD and FE, and G and F are the Centers of the Arches *d*C and *e*C.

In all theſe Arches, AL and BM perpendicular to AB, are Tangents to the Curve Sides in A and B, the Centers of the Arches which ſpring from A and B being always in the Line AB.

Theſe Arches are all Right Arches, their Baſe Line AB being ſuppoſed parallel to the Horizon, and the Curves of which they are compoſed being all Portions of Circles; but in the Rampant Gothick Arches, all thoſe Curves become Elliptick, and are to be deſcribed by varying the Angles of the Models, preſerving the Proportion of the Diviſions of their Sides.

Thus, if it were required to deſcribe a Rampant Arch of the ſame Species with the Right Arch, N°. 1. whoſe Baſe Line aB may incline to the Horizon in any Angle aBA; a Model for it is made by drawing the Baſe Line AB with the Inclination propoſed[b], the Sides LA and MB of the Parallelogram LMAB continuing perpendicular to the Horizon; and dividing the Sides LA and AB of this Model in the ſame Proportion as the correſponding Sides of the Model, N°. 1, and drawing the Curve of the Arch through the proper Subdiviſions, as in the Figure.

[b] Fig. 189. N°. 5.

Beſides the Gothick Right Arches here deſcribed, which are all formed of Portions of Circles, there are many other different Sorts compoſed as well of Circular as Elliptick Segments; but whatever their Variety be, the Method of conſtructing Models for the Deſcription of them is the ſame.

S C H O L.

In the ſeveral Figures of this Propoſition, the Axes of the given Cylinders are ſuppoſed to interſect at Right Angles, but the Method is the ſame whatever Angle they make, the Quantity of that Angle being nowiſe concerned in the Demonſtrations. Fig. 188. N°. 1, 2, 3, 4.

SECTION IV.

Of the Image of the Sphere or Globe and its Sections.

D E F.

IF a Globe or Sphere be anywiſe cut by a Plane, the Section will be a Circle; if the cutting Plane paſs through the Center of the Sphere, the Circle thereby formed is called a *great Circle of the Sphere*, it being that by the Revolution of which round any of its Diameters, the Spherical Surface is generated; ſo that all great Circles of the ſame Sphere are equal and biſect it, and conſequently each other; and therefore every Circle of the Sphere, whoſe Plane is perpendicular to the Plane of a great Circle, is biſected by it.

If the cutting Plane do not paſs through the Center of the Sphere, the Circle thereby formed is called a *ſmaller Circle of the Sphere*; every ſuch Circle, whoſe Plane is nearer the Center of the Sphere, is larger than one more remote, and all ſmaller Circles of the Sphere divide it into two unequal Parts.

L E M. 13.

If from the Eye at Σ a Line be drawn to the Center S of a Globe or Sphere, cutting it in A and B, and on AB as a Diameter a Circle AFBG be deſcribed, and the Chord DE of the Tangents to that Circle from Σ be found, cutting AB in C; then if from C as a Center with the Diameter DE, a Circle $DfEg$ be deſcribed in a Plane perpendicular to the Line ΣC, that Circle will terminate the viſible Part of the Sphere from Σ. Fig. 190.

Dem. For ΣD and ΣE being Tangents to the Circle AFBG in D and E, and that Circle being a great Circle of the Sphere[a], ΣD and ΣE are therefore alſo Tangents to the Sphere in the ſame two Points; now if the Circle AFBG be imagined to revolve round its Diameter AB, it will thereby deſcribe the intire Spherical Surface, and at the ſame time the Line DE which is perpendicular to AB, will by its Extremities D and E deſcribe the ſmaller Circle $DfEg$ in a Plane perpendicular to AB; but ΣD and ΣE will ſtill continue Tangents to the Circle AFBG in the ſame Points D and E, in what manner ſoever that Circle be turned on its Diameter AB, the Line ΣB remaining unaltered by that Motion; and conſequently theſe Tangents by their Motion round with the Circle AFBG, will become ſucceſſively Tangents to the Spherical Surface in every Point of the Circle $DfEG$, which Circle will therefore terminate its viſible Part from the Point Σ. *Q. E. D.*

[a] Def.

S C H O L.

Thus the Rays by which a Sphere is ſeen, form a Right Cone whoſe Vertex is at the Eye, and its Baſe is a ſmaller Circle of the Sphere, terminated by the Contact of the Sphere with the Viſual Rays which form the Conick Surface, the Axe of which Cone is a Line drawn from the Eye to the Center of the Sphere.

C O R.

If the Sphere AFBG be cut by any Plane paſſing through Σ, that Plane will likewiſe cut the Plane of the Circle $DfEg$ in a ſtraight Line, which Line will be the Chord of the Tangents from Σ to the Circle formed by the Section of the Sphere with the cutting Plane.

For it is evident that a Line Σf drawn from Σ to any Point f of the Circle $DfEg$, is a Tangent to the Sphere in that Point, and conſequently a Tangent to any Circle of the Sphere whoſe Plane paſſes through Σf.

L E M. 14.

The Center and Diſtance of the Picture, and the Image of the Center, and of any Diameter of a Sphere, with the Vaniſhing Point of that Diameter, being given; thence to find the Image of any other Diameter of the Sphere whoſe Vaniſhing Point is alſo given.

Fig. 191. Let O be the Center, and I O the Diſtance of the Picture, and let S be the Center, and A B the given Diameter of the Sphere, and y its Vaniſhing Point, and let z be the Vaniſhing Point of the Diameter ſought.

Having through the given Vaniſhing Points y and z drawn a Vaniſhing Line yz, conſider A B as the Image of a Diameter of a Circle in the Plane yzA B, and having from z through the Center S drawn an indefinite Diameter D E of that Circle, find its Extremities D and E[a], and D E will be the Diameter of the Sphere required.

Dem. For it is manifeſt that any two Diameters of a Sphere muſt lye in a Plane whoſe Vaniſhing Line paſſes through the Vaniſhing Points of thoſe Diameters, which Plane paſſing through the Center of the Sphere, cuts it in a great Circle, whoſe Diameter is the ſame with the given Diameter A B of the Sphere. *Q. E. I.*

C O R.

If the given Diameter be parallel to the Picture, its Vaniſhing Point being then infinitely diſtant, the Vaniſhing Line of the Plane of the Circle muſt be drawn through the Vaniſhing Point of the required Diameter, parallel to that which is given; but if the required Diameter be alſo parallel to the Picture, then theſe two Diameters lying in a Plane parallel to the Picture, their Images will be equal.

P R O B. XVII.

The Center and Diſtance of the Picture, and the Image of any Diameter of a Sphere parallel to the Picture, being given; thence to find the Image of the ſmaller Circle of the Sphere which terminates its viſible Part.

Fig. 192. Let O be the Center, and I O the Diſtance of the Picture, and let ab be the given Diameter of the Sphere, and S its Center.

Here, S being the Indefinite Image of a Line from the Eye to the Center of the Sphere, it repreſents not only the Vaniſhing Point of that Line, but alſo the Diameter of the Sphere which paſſes through the Eye, and likewiſe the Point in that Diameter through which the Chord of the Tangents paſſes, which forms the Diameter of the terminating Circle[b]; wherefore it becomes neceſſary to make uſe of a ſubſtituted Plane in the following manner.

Having found the Vaniſhing Line xy of Planes perpendicular to the Vaniſhing Point S[c], through S draw a Diameter a b of the Sphere parallel to xy, repreſenting a Diameter parallel to the Picture, and conſequently equal to the given Diameter ab[d], and produce a b at pleaſure to E and F; then EF will be the Vaniſhing Line of a Plane paſſing through the Eye and the Diameter a b of the Sphere, in which Line the intire Image of the great Circle of the Sphere formed by its Section with that Plane, and of all Lines in that Circle therefore lye[e]: then having drawn any Line $\alpha\beta$ parallel to EF, and at a convenient Diſtance from it, from a, S, and b draw Perpendiculars to EF cutting $\alpha\beta$ in α, s, and β, and $\alpha\beta$ will be the Oblique Seat of a b on a ſubſtituted Plane $EF\alpha\beta$ parallel to the Plane EF; on $\alpha\beta$ as a Diameter deſcribe the Image of a Circle $\alpha f\beta g$ in this ſubſtituted Plane, cutting Ss in f and g, and having biſected fg in c, through c draw de parallel to $\alpha\beta$ meeting the Circle $\alpha f\beta g$ in d and e, and transfer de to D E in the Line EF by the Perpendiculars dD and eE; laſtly, on D E as a Diameter deſcribe the Image A D B E of a Circle in the Plane xyD E, and that will repreſent the ſmaller Circle of the Sphere which terminates its viſible Part, or the Outline of the Image of the Sphere propoſed.

Dem. For the ſubſtituted Plane $EF\alpha\beta$ being parallel to the Plane EF, the Oblique Seat of the great Circle of the Sphere which lies in the Plane EF, is alſo a Circle in the ſubſtituted Plane, equal to it; and $\alpha\beta$ being the Seat of the Diameter ab of that great Circle, the Circle $\alpha f\beta g$ is its intire Seat on the ſubſtituted Plane, and conſequently fg is the Seat of the Diameter S of that Circle[f]; and de being the Chord of the Tangents to the Circle $\alpha f\beta g$ from the Directing Point of its Diameter fg[g], which is

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[a] Cor. 2. Meth. 2. Prob. 24. B. II.

[b] Cor. Theor. 8. B. I. and Lem. 13.

[c] Prop. 21. B. IV.

[d] Cor. Lem. 14.

[e] Cor. 1. Theor. 17. B. I.

[f] Prop. 42. B. IV.

[g] Cor. 2. and 3. Prob. 3. B. III.

Fig. 190.

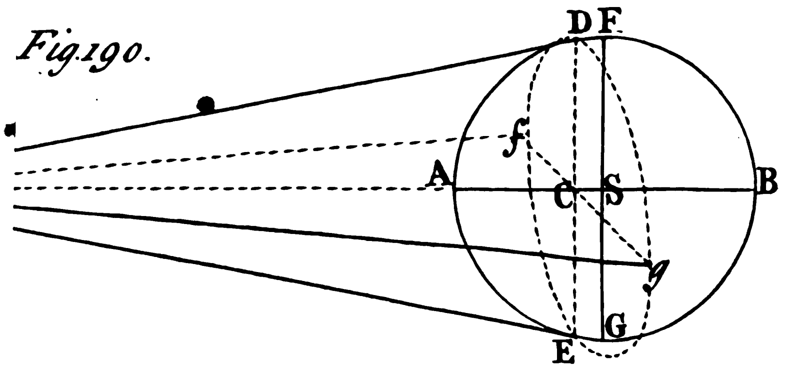


Fig. 191.

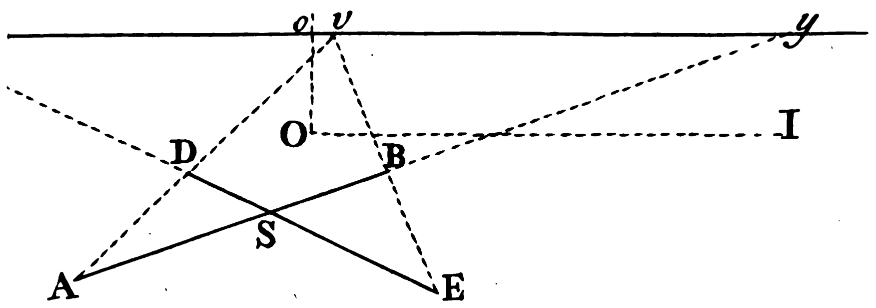
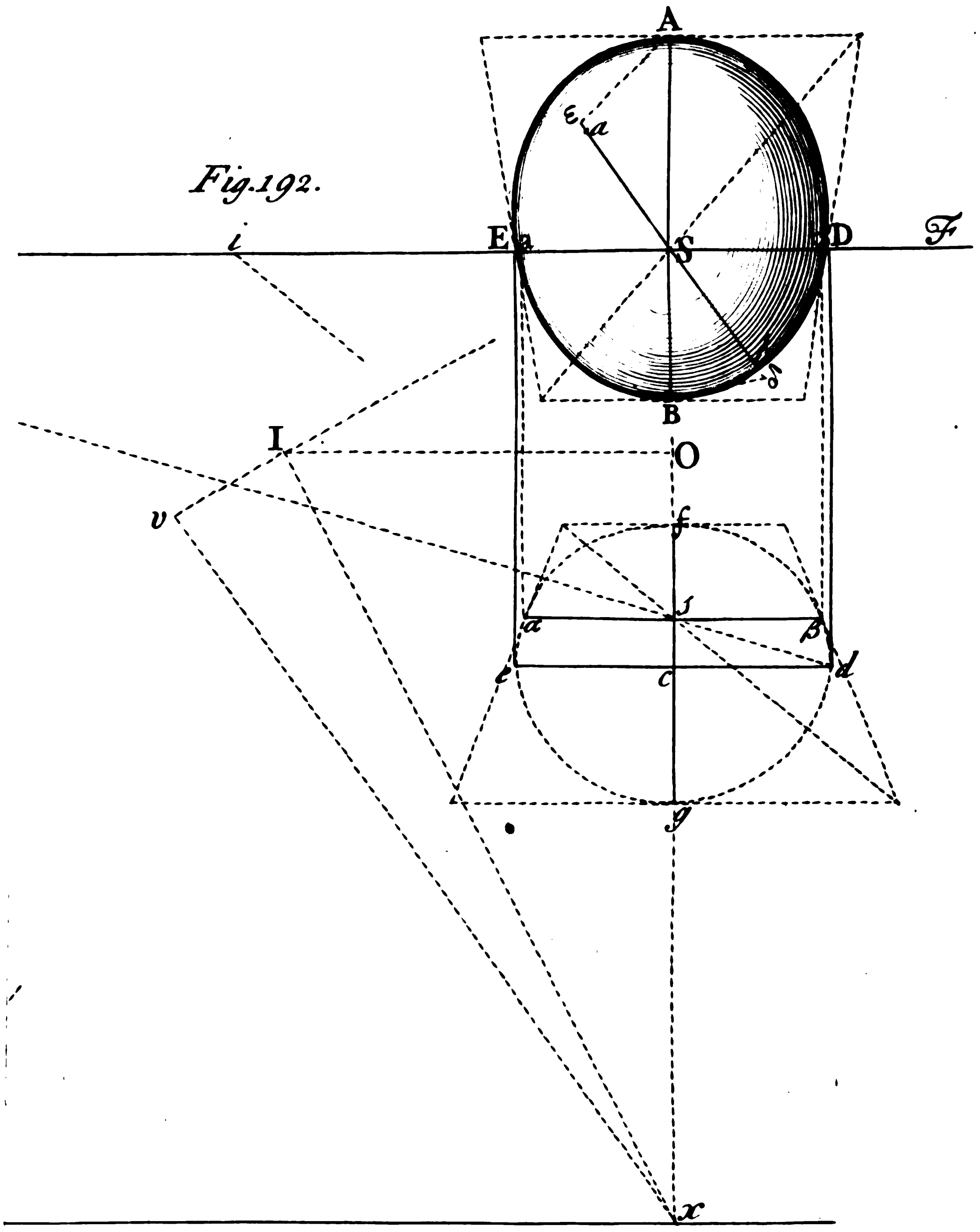


Fig. 192.



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is at the Foot of the Eye's Director, it is therefore the Seat of the Chord of the Tangents to the great Circle of the Sphere from the Eye, the Diſtance between the Foot of the Eye's Director and the Center s of the ſubſtituted Circle, being equal to that between the Eye and the Center S of the Sphere; wherefore DE is the Image of that Chord[a]: which Chord being the Diameter of the ſmaller Circle of the Sphere which terminates its viſible Part, and that Circle lying in a Plane perpendicular to the Line S[b], the Image ADBE of a Circle deſcribed with the apparent Diameter DE in the Plane xyDE repreſents that Circle, and is therefore the viſible Outline of the Sphere. *Q. E. I.*

COR. 1.

If the Center of the Sphere be in the Center of the Picture, the Plane of the terminating Circle will be parallel to the Picture, and the Image of that Circle will therefore be a Circle; but the Diameter of that Circle muſt be found by the help of a ſubſtituted Plane as before.

COR. 2.

If the Center of the Sphere be not in the Center of the Picture, the Image of the terminating Circle will always be an Ellipſis, whoſe Tranſverſe Axe AB will paſs through the Center of the Picture, and to which Axe DE will be a double Ordinate.

For the Viſual Cone being Right, if its Axe be not perpendicular to the Picture, the Section of that Cone by the Picture muſt be an Ellipſis, the Tranſverſe Axe of which Section always paſſes through that Point of the cutting Plane where a Perpendicular from the Vertex of the Cone meets it; and the Eye being here the Vertex of the Cone, a Perpendicular from thence to the Picture cuts it in its Center.

SCHOL.

Here, the Circle $\alpha f\beta g$ may be conſidered as the Baſe of a Cylinder formed on the great Circle of the Sphere which lies in the Plane EF, having the Line Ss parallel to the Picture for its Axe, and the Tangents in d and e for its terminating Sides; which Sides meet the upper Face of the Cylinder repreſented by the Line DE, in D and E where the terminating Circle ADBE of the Sphere cuts EF.

For DE being the Section of the terminating Circle by the Plane EF which paſſes through the Eye, the Line DE is the Chord of the Tangents from the Eye to the great Circle of the Sphere which lies in that Plane[c], and which makes the upper Face of the Cylinder.

COR. 3.

If the Vaniſhing Line xy ſhould be ſo far diſtant, that the Point of Diſtance y of the Vaniſhing Point x could not be conveniently marked on that Line, thereby to determine the Length of the Axe AB; that Diſtance may be ſet off from x at v on any other Line xv, to which a Parallel $\delta\varepsilon$ being drawn through S, equal to DE, $v\delta$ and $v\varepsilon$ will give the ſame Points A and B, and conſequently the Axe AB as before[d].

COR. 4.

The Image ADBE of the terminating Circle of a Sphere, being given; thence to find the Diameter of that Sphere.

Having found DE the Chord of the Tangents to the terminating Circle from x, transfer DSE to dce in any ſubſtituted Plane $EFde$ parallel to the Plane EF which paſſes through the Eye and DE; find in EF a Vaniſhing Point v perpendicular to the Vaniſhing Point D, and draw vd cutting Sc in s, and s will repreſent the Center, and sd a *Radius* of the Circle $dfeg$, which is the Seat of the great Circle of the Sphere that lies in the Plane EF; by the help of which *Radius* the Diameter $\alpha\beta$ of that Seat may be found, and thence the Diameter ab of the Sphere.

For dD being a Tangent to the Circle $dfeg$ in d, and D being its Vaniſhing Point, dv which repreſents a Perpendicular to that Tangent, paſſes through s the apparent Center of that Circle[e], of which Circle ds therefore repreſents a *Radius*.

PROB. XVIII.

The Center and Diſtance of the Picture, and the Image of any Diameter of a Sphere, with the Seat of its Center on any Plane, being given; thence to find the Shadow of the Sphere on that Plane from a given Luminous Point, whoſe Seat on the ſame Plane is given.

Let

[a] Prop. 41. B. IV.

[b] Lem. 13.

[c] Cor. Lem. 13.

[d] Lem. 1. B. II.

[e] 18 El. 3.

Fig. 193. Let O be the Center, and IO the Diſtance of the Picture, FG the given Diameter, and s the Seat of the Center S of the Sphere on the Plane EFs; and let Σ be the Luminous Point, and T its Seat on the ſame Plane.

Having drawn ΣS and found its Vaniſhing Point z[a], and the Vaniſhing Line xy of Planes perpendicular to that Point[b], draw a Diameter FG of the Sphere parallel to xy, and conſequently to the Picture[c]; and having through z drawn a Vaniſhing Line zv parallel to xy or FG, on FG as a Diameter deſcribe the Image AFBG of a Circle in the Plane zvFG; and having found DE the Chord of the Tangents to this Circle from Σ, on DE as a Diameter deſcribe the Image DaEb of a Circle in the Plane xyDE, and the Projection $\delta\alpha\varepsilon\beta$ of this Circle on the Plane EFs from the Point Σ will be the Shadow of the Sphere on that Plane.

Dem. For the Original of AFBG being a great Circle of the Sphere paſſing through ΣS, and the Vaniſhing Point of its Diameter AB being at z the Center of the Vaniſhing Line zv[d], the Diameter FG of the Sphere repreſents another Diameter of that Circle perpendicular to the Original of AB, to which Diameter FG, the Chord DE of the Tangents from Σ is therefore parallel; and this Chord being therefore alſo parallel to the Vaniſhing Line xy, its Original lies in the Plane xyDE, which Plane being perpendicular to the Line ΣS, the Circle aDbE deſcribed on the Diameter DE in the Plane xyDE, repreſents the terminating Circle of the Sphere from Σ[e], the Projection of which Circle on the Plane EFs from the Point Σ is therefore the Shadow of the Sphere on the Plane propoſed. *Q. E. I.*

S C H O L.

Here, the Line yp is the Line of the *Foci* of the Projections of all Lines in the Plane xyDE on the Plane EFs, and p is therefore the *Focus* of the Projections of ab, ln, and mr, whoſe Vaniſhing Point is x[f], and DE, lm, and nr being parallel to xy, their Projections $\delta\varepsilon$, $\lambda\mu$, and $\nu\varrho$ tend to t the Parallel Seat of Σ on the Plane EFs with reſpect to the Plane xyDE[g], through which Point yp alſo paſſes; and Ts being the Oblique Seat of ΣS on the Plane EFs, c is the Interſection of ΣS with that Plane, and is conſequently the Projection of C the apparent Center of the Circle aDbE; and the apparent Diameters $\alpha\beta$, $\delta\varepsilon$, of the Projection, being drawn through c to their reſpective *Foci* p and t, their Extremities are determined by the Projecting Lines Σa, ΣD, Σb, and ΣE, whereby the Oblique *Trapezium* $\lambda\mu\nu\varrho$ is found, which being conſidered as the Image of a Parallelogram in a Plane whoſe Vaniſhing Line is yp, and ſubdivided accordingly[h], the Projection $\alpha\delta\beta\varepsilon$ is thereby obtained as uſual[i].

C O R.

If the Luminous Point Σ be infinitely diſtant; it being then conſidered as a Vaniſhing Point, the Vaniſhing Line of the Plane of the Circle AFBG muſt paſs through Σ, perpendicular to ΣO, that Σ may be the Center of that Vaniſhing Line; and the terminating Circle from Σ will be a great Circle of the Sphere in a Plane perpendicular to the Vaniſhing Point Σ, ſo that C will coincide with S; but the Practice in all other reſpects is the ſame as before, regard only being had to the different manner of finding the Projection of the terminating Circle, according to the different Caſes of the Situation of the Projecting Point, as formerly ſhewn[k].

S C H O L.

Here, the Luminous Point when infinitely diſtant, is only conſidered as the Vaniſhing Point of the Rays of Light which are ſuppoſed parallel, and therefore enlighten exactly one Moiety of the Sphere, ſo that the terminating Circle is a great Circle of the Sphere; whereas the Rays of Light proceeding from the Sun or Moon, and Shining on any ſmaller Sphere, enlighten more than its half, and form a Conical Shadow whoſe Vertex lies on the oppoſite Side of the Sphere from the Luminary; and the dark Part of the Sphere is terminated by a ſmaller Circle, the Diameter of which is the Chord of the Tangents to the Sphere from the Vertex of the Conical Shadow, or the Point of Convergence of the Rays of Light; which Point lies in the Line drawn from the Center of the Luminary through the Center of the Sphere: but although this be of neceſſary Conſideration for determining the Quantity and Duration of Eclipſes, and for other Aſtronomical Purpoſes, yet with regard to any near Objects which are propoſed to be deſcribed, the Diſtance between them and their Shadows by the Sun or Moon, is ſo ſmall, that the Difference is not perceivable whether the Rays of Light be taken as Parallel or Converging.

Nevertheleſs, if greater Exactneſs were required, the *Penumbra* might be determined, by

[a] Prop. 40. B. IV.

[b] Prop. 21. B. IV.

[c] Lem. 14.

[d] Cor. 1. Prop. 22. B. IV.

[e] Lem. 13.

[f] Meth. 4. Prob. 6. B. V.

[g] Cor. Meth. 1. Prob. 6. B. V.

[h] Cor. 3. Meth. 4. Prob. 6. B. V.

[i] Meth. 1. Prob. 24. B. II.

[k] Caſe 1, 2, 3, 4. Prob. 6. B. V.

Plate 115 Book 6. Sect. 4.

ig.193.

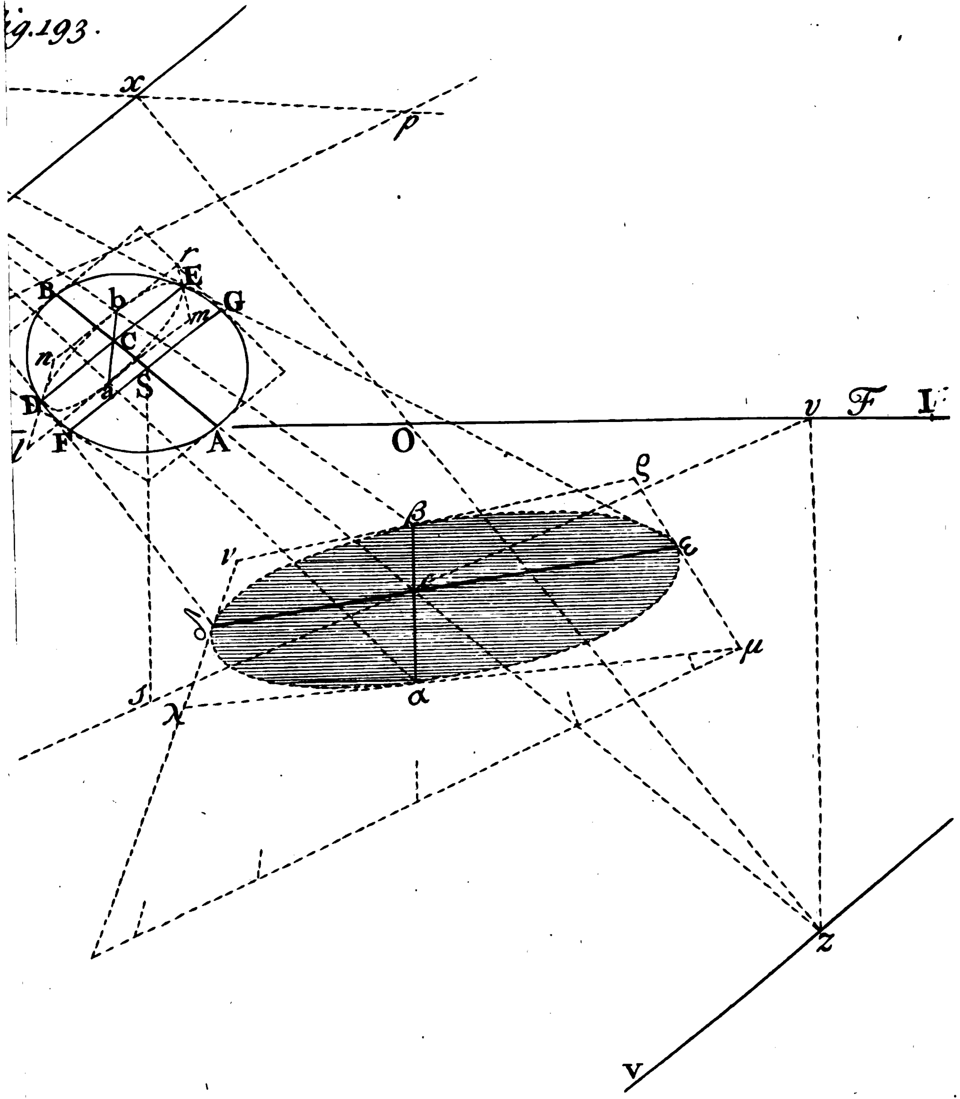
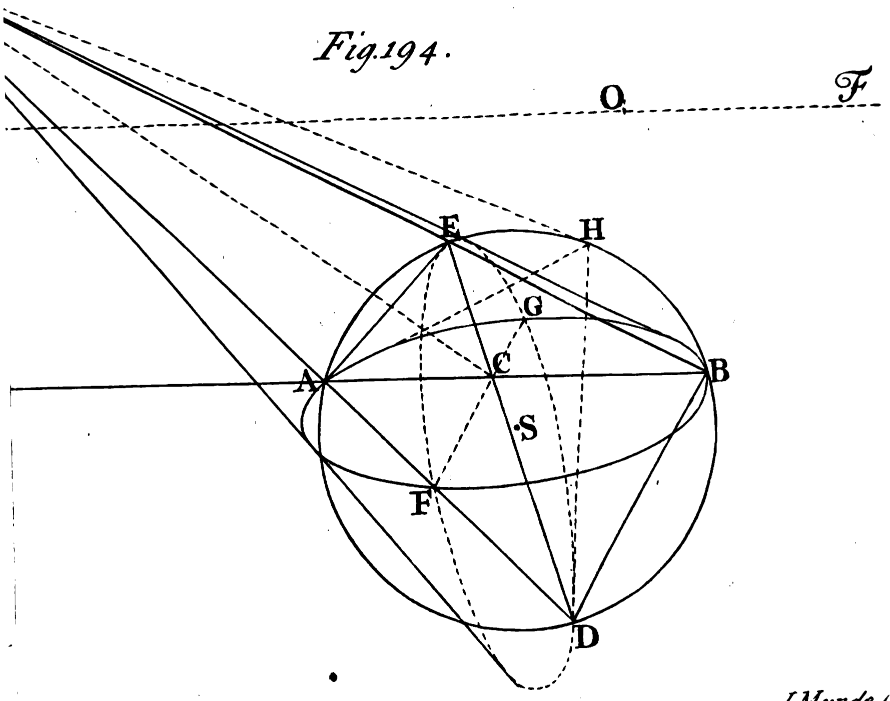


Fig.194.



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by drawing two Projections, the one of the terminating Circle of the Sphere from the Center of the Luminary, and the other of the terminating Circle of the Sphere from the Vertex of the Conical Shadow, the Space between which two Projections would be the Extent of the *Penumbra.*

L E M. 15.

If a Cone have any Circle of a Sphere for its Baſe, other than the terminating Circle from its Vertex; that Cone produced will again meet the Sphere in another Circle.

Let S be the Center, and AGBF any Circle of a Sphere, and VAGBF a Cone, having that Circle for its Baſe; and let T be the Perpendicular Seat of the Vertex V of the Cone on the Plane of that Circle, and VTB a Plane perpendicular to that Plane paſſing through the Axe VC and the Sides VA, VB, of the Cone, and cutting the Sphere in a great Circle AEBD: it muſt be ſhewn that the ſame Cone produced, will alſo cut the Sphere in another Circle EFDG. Fig. 194.

Dem. Produce the Sides VA, VB, of the Cone, which if they be not Tangents to the great Circle AEBD of the Sphere (as by the Suppoſition they are not) muſt again cut it in D and E, and thereby mark two Points in the Surface of the Sphere where the Cone cuts it, and DE will be a Line in the Baſe of this new Section: now, in the Triangles VAB, VED, the Angles VBA, VDE, are equal, as inſiſting on the ſame Chord AE of the Circle AEBD[a], and the Angle at V is common to both, wherefore theſe two Triangles are ſimilar; and conſequently if the given Cone be cut by a Plane paſſing through DE perpendicular to the Plane VTB, the Section EFDG will be ſubcontrary, and conſequently a Circle[b], of which DE will be a Diameter; but the ſame Plane will likewiſe cut the Sphere in a Circle, having the ſame Line DE for a Diameter[c], wherefore both theſe Sections coincide; and conſequently the Cone VAGBF which hath the Circle AGBF of the Sphere for its Baſe, alſo cuts the Sphere in another Circle EFDG. *Q. E. D.*

C O R. 1.

If the propoſed Cone have for its Baſe the Circle of the Sphere whoſe Diameter is AE, then DB will be the Diameter of the other Section of the Sphere by the Cone, which Section will likewiſe be a Circle.

For in the *Trapezium* AEBD, the Angles AEB, ADB, are together equal to two Rights[d], but the Angles AEB, AEV, are alſo equal to two Rights, wherefore the Angles ADB, AEV, are equal, and conſequently the Triangles VAE, VBD, are ſimilar; wherefore the Section of the Cone by a Plane paſſing through DB perpendicular to the Plane VAE is ſubcontrary, and therefore agrees both with the Cone and the Sphere.

C O R. 2.

If AH be the Diameter of the Circle which forms the Baſe of the Cone, and its Side VH be a Tangent to the Sphere in H, the other Section will be a Circle, of which DH will be the Diameter.

For VH being a Tangent to the Circle AEBD in H, the Angles VHA, VDH, are equal[e], and conſequently the Triangles VHA, VDH, ſimilar.

C O R. 3.

If the Vertex of the Cone be within the Sphere as at C, and have AE for the Diameter of its Baſe; the oppoſite Cone will be cut ſubcontrarily by the Sphere, and have DB for the Diameter of its Baſe.

For the oppoſite Cones whoſe Sections by the Perpendicular Plane are CAE, CDB, will be ſimilar; the Angles CAE, CDB, of thoſe two Triangles being equal, as inſiſting on the ſame Arch BE, and their Angles at C, oppoſite.

The ſame may be ſhewn of the oppoſite Cones, whoſe Sections by the Perpendicular Plane are the Triangles CAD, CEB.

S C H O L.

This Lemma is equally applicable to the Sections of a Sphere by a Cylinder, there being very little Difference in the Demonſtration, whether the Lines VA and VB meet in a Point V, or be parallel to each other.

P R O B. XIX.

The Center and Diſtance of the Picture, and any Portion of a Con-

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[a] 21 El. 3.

[b] Con. Sec. Art. 7. B. III.

[c] Def. Lem. 13.

[d] 22 El. 3.

[e] 32 El. 3.

cave Sphere terminated by a Circle, together with the Center and *Radius* of the Sphere, being given; thence to find the Boundary of the Light on the Concave Surface of the Sphere, which can enter it through that Circle from any given Luminous Point, whoſe Perpendicular Seat on the Plane of that Circle is given.

Fig. 195. Let O be the Center, and IO the Diſtance of the Picture, S the Center, and SD a *Radius* of the Sphere, whoſe given Portion is AFBGH terminated by the Circle AFBG whoſe Center is *s*, and the Vaniſhing Line of whoſe Plane is *EF*; and let Σ be the Luminous Point, and T its Perpendicular Seat on that Plane.

Through S draw an Indefinite Diameter SV of the Sphere, perpendicular to the Plane of the Circle AFBG, which will therefore alſo paſs through its Center *s*, and by the help of the given *Radius* of the Sphere find the Extremities D and E of that Diameter[a]; from T through *s* draw the Diameter AB of the Circle AFBG, and produce it to its Vaniſhing Point *y*, through which draw a Vaniſhing Line *yz* of Planes perpendicular to the Plane *EF*; and by the help of the Diameter DE of the Sphere, draw the Image of its great Circle AEBD in the Plane *yz*: Produce DE, and in it find the Point V where the Tangents to this great Circle in A and B meet that Line, which they muſt do ſomewhere if they be not parallel to it, the Originals of DE and AB being perpendicular[b]; then having drawn ΣV cutting TB in *t*, find FG the Chord of the Tangents to the Circle AFBG from *t*, cutting AB in C, and having drawn ΣA and ΣB cutting the Circle AEBD in *a* and *b*, draw *ab* till it cut ΣV in σ; from σ through F and G draw σF, σG, and from *w* the Vaniſhing Point of FG draw *wa*, *wb*, and thereby an Oblique *Trapezium* $\nu\lambda\mu\varrho$ will be formed, within which the Image *a*F*b*G of a Circle being deſcribed as uſual, it will be the Boundary of the Light required, of which the Part F*a*G will be that which falls within the given Portion of the Concave Sphere, and the remainder F*b*G will only be Imaginary.

Dem. Becauſe the Original of SV is perpendicular to the Plane of the Circle AFBG, and V is the Point of Concourſe of the Tangents to the Sphere in A and B, the Circle AFBG is the terminating Circle of the Sphere from the Point V, and all Lines drawn from V to any Points of the Circumference of that Circle, are Tangents to the Sphere in thoſe Points[c], wherefore VF and VG are Tangents to the Sphere in F and G; and becauſe FG is the Chord of the Tangents to the Circle AFBG from *t*, *t*F and *t*G are Tangents to that Circle, and conſequently to the Sphere in F and G; wherefore a Plane V*t*G paſſing through V*t* and the Tangents VG and *t*G muſt touch the Sphere in G, and for the ſame Reaſon a Plane V*t*F paſſing through V*t* and the Tangents VF and *t*F touches the Sphere in F, and therefore σG and σF drawn from σ (a Point in the common Interſection V*t* of thoſe two Planes) to G and F, are Tangents to the Sphere in thoſe two Points, and conſequently to the Circle of the Sphere formed by its Section with the Plane σFG.

Again, the Lines AB and *ab* being Diagonals of the *Trapezium* A*b*B*a* inſcribed in the Circle ADBE, and the Sides A*a*, B*b*, meeting in Σ, the Sides A*b* and *a*B will alſo meet in ſome Point *v*, if they be not parallel; and if a Line Σv be drawn, the Tangents to the Circle ADBE in A and B will alſo meet in ſome Point of Σv[d]; but V is by Conſtruction the Point of meeting of the Tangents in A and B, therefore the Point V is in the Line Σv, and if the Diagonal AB be produced till it cut that Line in *t*, it will be Harmonically divided in *t*, A, B, and its Interſection with the other Diagonal *ab*[e]; but becauſe FG is the Chord of the Tangents to the Circle AFBG from *t*, *t*B is Harmonically divided in *t*, A, B, and its Interſection C with FG[f], and the Points *t*, A, B, being the ſame in both Diviſions, the fourth Point C is alſo the ſame[g]; that is, the Diagonal *ab* and the Chord FG cut each other in the ſame Point C of the Line AB, wherefore *ab* and FG are in the ſame Plane σFG; and the Points *a*, F, *b*, G, being each in the Surface of the Sphere, they are four Points of the Circle formed by the Section of the Sphere with that Plane, of which Circle *ab* is an apparent Diameter, the Plane σFG being perpendicular to the Plane of the great Circle ADBE[h]: and becauſe FG is the Chord of the Tangents to the Circle *a*F*b*G from a Point σ in its Diameter *ab*, the Originals of the Tangents to this Circle in *a* and *b* are parallel to that Chord, wherefore *wa* and *wb* which have the ſame Vaniſhing Point *w* with FG, are the Images of thoſe Tangents; which Tangents being in the ſame Plane with FG and *ab*, they are in the Plane σFG, and therefore

meet

[a] Lem. 14.

[b] Cor. Lem. 4. B. II.

[c] Lem. 13.

[d] Cor. 2. Lem. 20. B. III.

[e] Lem. 22. B. III.

[f] Lem. 11. B III.

[g] Lem. 2. B. III.

[h] 18 El. 11. and Def. Lem. 13.

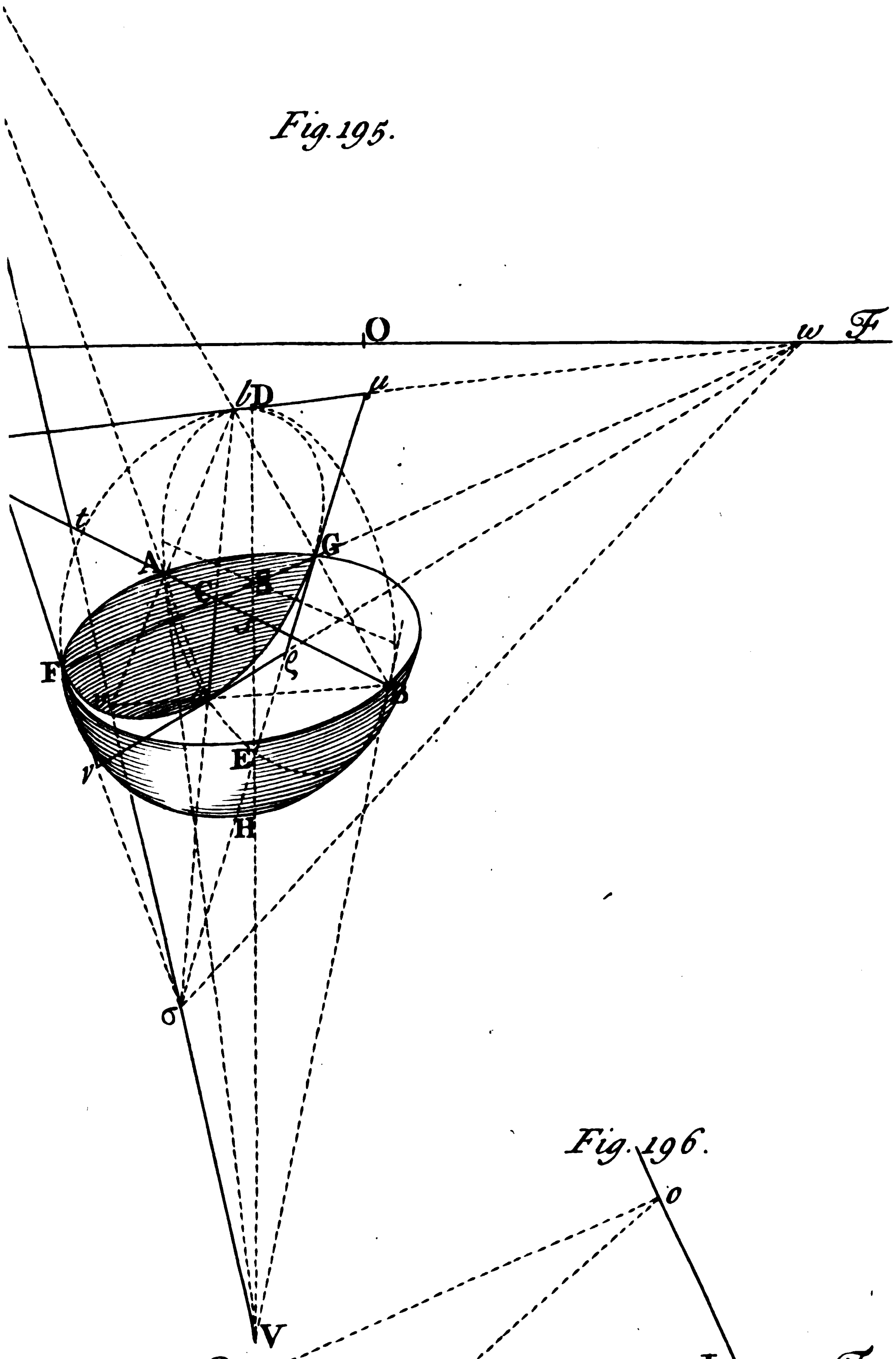

Fig. 195.

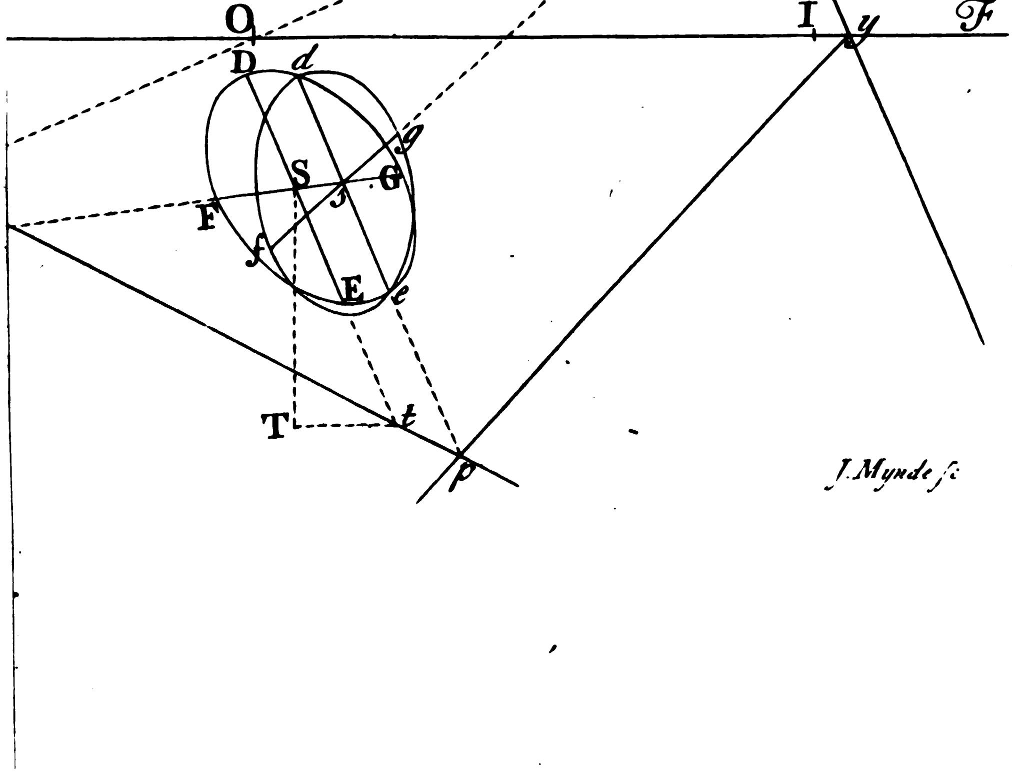

Fig. 196.

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meet the Tangents σF and σG in ν, λ, ϱ, and μ; and conſequently a Curve deſcribed in the uſual manner within the *Trapezium* $\nu\lambda\mu\varrho$ thus formed, will be the Section of the Sphere by the Plane σFG.

Laſtly, becauſe the Cone of Light from Σ, hath the Circle AFBG of the Sphere for its Baſe, that Cone alſo meets the Sphere in another Circle whoſe Diameter is ab[a]; and this being the ſame with the Circle $aFbG$ before found, that Circle is therefore the Boundary of the Light required, of which the Part FaG is all that falls within the given Portion AFBGH of the Concave Sphere propoſed. *Q. E. I.*

SCHOL.

The Subdiviſions of the Oblique *Trapezium* $\nu\lambda\mu\varrho$ are found by conceiving it to lye in a Plane whoſe Vaniſhing Line is σw; which indeed is not the Plane of the Circle Boundary of the Light, the true Vaniſhing Line of which Plane paſſes through w and the Vaniſhing Point of ab in the Line zy: nevertheleſs, as by this Conſtruction, a Conick Section $aFbG$ is formed, to which σF, σG, wa, wb, are Tangents in F, G, a, and b, and theſe Lines being the Images of the Tangents to the Original Circle formed by the Section of the Sphere with the Plane σFG in the ſame four Points, the Image of which Circle is a Conick Section, and in regard no two different Conick Sections can touch theſe four Tangents in the ſame Points[b], the Curve $aFbG$ muſt neceſſarily be the Image of the Circle required.

COR. 1.

If the Circle AFBG be a great Circle of the Sphere, that is, if its given Portion AFBGH be a Hemiſphere; then S coinciding with s, AB will be an apparent Diameter of the Sphere, the Originals of the Tangents in A and B will be parallel to SV, and the Point V will be either the Vaniſhing Point of Perpendiculars to the Plane of the Circle AFBG, or elſe it will be Infinitely diſtant, according to the Poſition of that Plane with reſpect to the Picture; and in either Caſe the Points T and t will coincide: but this will make no material Difference either in the Conſtruction or Demonſtration.

COR. 2.

If from v through C, a Line be drawn terminated by the great Circle ADBE, it will be the Chord of the Tangents to that Circle from Σ[c], and alſo the apparent Diameter of the terminating Circle of the Sphere from that Point[d], and conſequently of the Circle which terminates the Light on the Convex Surface of the Sphere from the given Luminous Point, the Projection of which Circle on any propoſed Plane alſo determines the Shadow of the Sphere on that Plane[e].

COR. 3.

The Points Σ, t, v, and σ, divide the Line ΣV Harmonically[f], if neither of thoſe Points be Infinitely diſtant; but if either of them be Infinitely diſtant, that is, if the Line which ſhould produce that Point ſhould be parallel to ΣV, then ΣV will be biſected by the other three Points.

COR. 4.

If ab be parallel to ΣV, the Point σ being then Infinitely diſtant, the Tangents σF and σG will alſo be parallel to ΣV.

PROB. XX.

The Center and Diſtance of the Picture, and the Image of any Diameter of a Sphere, with the Seat of its Center on any Plane, being given; thence to find the Section of that Sphere by any other Plane, whoſe Vaniſhing Line and Interſection with the firſt Plane are given.

Let O be the Center, and IO the Diſtance of the Picture, DE the given Diameter, and T the Seat of the Center S of the Sphere on the Plane EFT; and let yo be the Vaniſhing Line of the cutting Plane, and yp its Interſection with the Plane EFT. Fig. 196.

Having found x the Vaniſhing Point of Perpendiculars to the Plane oyp, draw xv parallel to yo cutting EF in v, and having found the Diameter DE of the Sphere parallel to the Picture and to the Vaniſhing Line xv, if that be not the Diameter already given, on that Diameter deſcribe the Image of a great Circle FDGE in the Plane xv;

[a] Lem. 15.

[b] Con. Sec. Art. 19. B. III.

[c] Lem. 14. B. III.

[d] Lem. 13.

[e] Prob. 18.

[f] Lem. 20. B. III.

then having drawn Tt parallel to EF, produce DE till it meet Tt in t, and draw vt cutting yp in p; from p draw ps parallel to DE cutting FG in s, and the Circle FDGE in d and e; laſtly, on de as a Diameter deſcribe the Image $fdge$ of a Circle in the Plane oyp, and that will be the Section of the Sphere by the Plane propoſed

Dem. For the Plane xv being perpendicular to the Plane oy, and FDGE being the Image of a Circle in the Plane xv, the Plane of that Circle is therefore perpendicular to the cutting Plane; and DE being a Line in the Plane xv parallel to the Picture, and T being the Seat of S a Point of that Line, on the Plane EFT, t is therefore the Interſection of DE with that Plane, and conſequently vt is the Interſection of that Plane with the Plane xvDE; and vt cutting yp in p, p is a Point in the Interſection of the Planes xvDE and yop; wherefore ps drawn parallel to xv and yo, is the common Interſection of thoſe Planes[a], and conſequently de is the Interſection of the cutting Plane oyp with the Circle FDGE; but FDGE being a great Circle of the Sphere, and the Section of the Sphere by the Plane oyp being perpendicular to the Plane of that Circle, the Image $fdge$ of the Circle in the Plane oyp formed on de as a Diameter, is therefore the Section required[b]. *Q. E. I.*

COR. 1.

The Axe FG of a Sphere with its Vaniſhing Point x, being given; thence to deſcribe the *Equinoctial* Circle, or any other Parallels of Latitude of that Sphere.

Having found the Vaniſhing Line yo of Planes perpendicular to the Vaniſhing Point x, that will be the Vaniſhing Line of the Planes of the *Equator*, and of all Parallels of Latitude; having therefore drawn the Image FDGE of a great Circle of the Sphere in the Plane xv, which will repreſent a *Meridian* Circle, the Images of the *Equinoctial* Circle and of all Parallels of Latitude will be found, by drawing a Parallel to DE through that Point of the Axe FG which is cut by the Plane of the propoſed Circle, which Line terminated both ways by the *Meridian* Circle FDGE, will be the Diameter on which the Image of the propoſed Circle is to be drawn in the Planes oy.

COR. 2.

If the *Equinoctial* Circle of a Sphere be given, the Axe of the Sphere and thereby any propoſed *Meridian* Circle may be thence found after the like manner.

Thus, if FDGE were the *Equinoctial* Circle of the Sphere in the Plane xv, 'tis evident the Vaniſhing Point of the Axe of the Sphere muſt be at o the Vaniſhing Point of Perpendiculars to the Plane xv, and the Image of a Circle deſcribed in the Plane oy on the Diameter DE will be a *Meridian* Circle and terminate the Axe; and as the Planes of all *Meridian* Circles are perpendicular to the Plane of the *Equator*, the Vaniſhing Lines of all their Planes muſt paſs through o[c], and all theſe Circles will have the Axe of the Sphere for a Diameter; and the Inclination of any propoſed *Meridian* to another being given, the Vaniſhing Line of its Plane may be thence found by the Methods formerly ſhewn[d].

COR. 3.

The *Equinoctial* Circle FDGE in the Plane xv, and in it the Diameter DE which paſſes through the *Equinoctial* Points, being given; thence to find the *Ecliptick* Circle.

Having drawn FG repreſenting a Diameter perpendicular to the given Diameter DE, find the Vaniſhing Line xo of the *Meridian* Circle which paſſes through FG[e], and in xo find a Vaniſhing Point ſubtending with x the Vaniſhing Point of FG, an Angle of 23 Deg. 29 Min. being the Angle of Inclination of the *Ecliptick* to the Plane of the *Equator*, and through this Point draw a Vaniſhing Line of Planes perpendicular to the Planes xo; then a Circle deſcribed in this Plane on the Diameter DE will be the *Ecliptick* Circle deſired.

For it is evident, the Plane of the *Meridian* Circle which paſſes through FG alſo paſſes through the *Solſtitial* Points, and is therefore perpendicular to the Planes of the *Equator* and *Ecliptick*, and conſequently the Arch of that Circle intercepted between theſe two Planes, meaſures their Angle of Inclination.

Here, as two Points may be found in xo ſubtending with x the Angle required, it is neceſſary to know on which Side the Inclination of the *Ecliptick* lies with reſpect to the Points F and G, in order to chuſe the Right.

After this manner may be obtained the Image of any great Circle of the Sphere paſſing through any Diameter of any other great Circle, and inclining to it in any Angle propoſed.

[a] Theor. 15. B. I.

[b] Def. Lem. 13.

[c] Cor. 3. Prop. 20. B. IV.

[d] Prop. 25. B. IV.

[e] Cor. 2.

[f] Cor. 3. Prop. 20. B. IV.

SCHOL.

In the preceeding Problems relating to a Sphere, the Eye is ſuppoſed to be at ſo'me certain Diſtance from it, in which reſpect, the Projections of the Sphere and its Circles, ſo formed, differ from the uſual Projections made for Mathematical Purpoſes.

Theſe laſt are of three Sorts, and are diſtinguiſhed by the Names of the *Orthographick*, *Stereographick*, and *Gnomonick Projections* of the Sphere.

1. The *Orthographick* Projection is deſcribed on a Plane paſſing through a great Circle of the Sphere, the Eye being ſuppoſed to be at an Infinite Diſtance perpendicular to that Plane; ſo that every Point in the Surface of the Sphere is projected on that Plane by Perpendicular Lines, and the Deſcription thus formed is therefore a Geometrical Deſcription, the Nature of which has been formerly explained[a].

In Projections of this Sort, all Circles of the Sphere whoſe Planes are perpendicular to the Plane of the Projection, are projected into ſtraight Lines; all Circles whoſe Planes are parallel to the Plane of the Projection, are projected into Circles, equal to their Originals; and all other Circles of the Sphere are projected into Ellipſes.

Let AGBH be a Section of the Sphere by a Plane paſſing through its Center S, perpendicular to the Plane of the Projection, which it cuts in GH; and let AB, AC, AD, and AE, be the Sections of the Plane AGBH with ſeveral Circles of the Sphere whoſe Planes are perpendicular to that Plane. Fig. 197. N°. 1.

'Tis evident, that if the Plane of the Circle whoſe Diameter is AB, be perpendicular to the Plane of the Projection, the whole of that Circle muſt be projected into a ſtraight Line paſſing through *a*, perpendicular to GH, and equal to AB.

If the Plane of the Circle whoſe Diameter is AE, be parallel to the Plane of the Projection, that Circle muſt alſo be projected into a Circle, whoſe Diameter is *ac* equal to AE, the Projections of A and E being at *a* and *c* where the Perpendiculars A*a* and E*c* cut GH; and if the Plane of the Circle AD incline to the Plane of the Projection, that Circle muſt be projected into an Ellipſis, whoſe ſhorter Axe will be *ad*, and its Tranſverſe Axe will be equal to AD; and in like manner, if AC be the Diameter of a great Circle of the Sphere, the Ellipſis formed by its Projection will have *ac* for its ſhorter Axe, and AC for its longer; in regard that the Diameters of thoſe Circles which are reſpectively perpendicular to the Diameters AC and AD, will be parallel to the Plane of the Projection, and conſequently equal to their reſpective Projections.

'Tis evident alſo that no other Section beſides an Ellipſis, can be produced by the Projection of any Circle whoſe Plane inclines to the Plane of the Projection; for the Projecting Lines of every ſuch Circle muſt produce a Scalene Cylinder having the propoſed Circle for its Baſe, and this Baſe inclining to the Plane of the Projection, the Section of that Plane with the Cylinder cannot be a Circle, and is therefore an Ellipſis.

2. In the *Stereographick* Projection, the Eye is ſuppoſed to touch the Sphere in the Extremity of ſome Diameter; and the Plane of the Projection, or Picture is ſuppoſed to paſs through the Center of the Sphere perpendicular to that Diameter, and conſequently parallel to a Plane paſſing through the Eye and touching the Sphere in that Point; or the ſame Projection may be made on any other Plane parallel to the former, either touching or cutting the Sphere, or at any Diſtance from it; the Projections in all theſe Caſes being ſimilar, and differing only in Size, as formerly obſerved[b].

In this kind of Projection, all Circles of the Sphere are projected into Circles, except only ſuch whoſe Planes paſs through the Eye, and which are therefore projected into ſtraight Lines.

Let I be the Place of the Eye in one Extremity of the Diameter IO of a Sphere, and let IBOA be a Section of that Sphere by a Plane paſſing through IO, and conſequently perpendicular to the Plane of the Projection; and let us ſuppoſe this laſt Plane to paſs through O the other Extremity of the Diameter IO, and ba to be its Interſection with the Plane IBOA parallel to I*k* the Interſection of the ſame Plane with the Plane which paſſes through the Eye, or the Directing Plane; and let AB be the Section of the Plane IBOA with any Circle of the Sphere whoſe Plane is perpendicular to it. Fig. 197. N°. 2.

Draw IA and IB, cutting ba in a and b. Then, it is evident that the Circle whoſe Diameter is AB, is the Baſe of a Cone having I for its Vertex, and that the Section of this Cone with the Plane of the Projection, gives the Projection of that Circle.

Now the Plane IBOA being ſuppoſed to cut the Cone perpendicularly to its Baſe, and alſo to be perpendicular to the Plane of the Projection, all that is neceſſary to

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[a] Sect. 3. B. I.

[b] Art. 24. Sect. 3. B. I.

ſhew that the Projection will be a Circle, is to prove that the Triangles IBA, Iab, are cut Subcontrarily[a], which is thus done.

Draw the Line BO, then the Triangles IBO, IOb, being Rectangular at B and O[b], and having their Angle at I common to both, they are therefore ſimilar, and the Angles IOB, IbO, are equal[c]; but the Angles IAB and IOB are equal, as inſiſting on the ſame Chord IB[d], wherefore the Angle IAB is equal to the Angle Iba, and conſequently the Triangles IBA, Iab, are cut Subcontrarily.

This Demonſtration holds good, whether the Plane of the Circle whoſe Diameter is AB, be perpendicular or not to the Plane IBOA; for the Axe IO remaining the ſame, there may always be a Plane correſponding to IBOA drawn through it, perpendicular to the Plane of any Circle in the Sphere, as well as to the Plane of the Projection.

Likewiſe, what is demonſtrated of the Projection when its Plane paſſes through O, is equally true when that Plane paſſes through o the Center of the Sphere, or through any other Point of IO indefinitely produced, ſo long as thoſe Planes remain perpendicular to that Line; ſeeing the Angle at I remaining, the Triangle Iba is ſimilar to every other Triangle formed by the Sides Ib and Ia with any Baſe parallel to ba.

3. The *Gnomonick* Projection of the Sphere ſuppoſes the Eye to be in the Center, and the Projection to be made on a Plane touching the Sphere, and conſequently perpendicular to a *Radius* drawn from the Eye to the Point of Contact; or it may be made on any other Plane perpendicular to that *Radius*, as was ſaid of the Stereographick Projection.

In this Projection, all great Circles of the Sphere are projected into ſtraight Lines, their Planes all paſſing through the Center, where the Eye is ſuppoſed to be placed; all Circles which are parallel to the Plane of the Projection, are projected into Circles, as is ſufficiently evident; and all other Circles are projected, either into Ellipſes, *Parabola's*, or *Hyperbola's*, according to their different Poſitions in the Sphere with reſpect to the Plane of the Projection.

Fig. 197. N°. 3. Let GCOk be a great Circle of the Sphere, formed by its Section with a Plane paſſing through the Center I and the Poiut of Contact O of the Sphere with the Plane of the Projection, and bd the common Interſection of thoſe Planes; and let Ck parallel to bd be the Section of the Plane GCOk with a Plane paſſing through the Eye at I, parallel to the Plane of the Projection, and therefore repreſenting the Directing Plane; and let AB, AC, AD, be Sections of the Plane GCOk with ſeveral Circles of the Sphere, whoſe Planes are perpendicular to it.

Then, if the Diameter AB lye wholly on the ſame Side of Ck, the Projection of that Circle will be an Ellipſis; becauſe both Sides of the Cone IAB by which the Circle is ſeen from I, are cut in a and b by the Plane of the Projection, on the ſame Side of its Vertex I, and the Triangles IAB, Iab, can never be ſimilar, unleſs AB and ab be parallel, the Triangle IAB being always Iſoſceles.

Here, it is evident that ab is the Tranſverſe Axe of the Ellipſis thus formed, and ab being biſected in c, c is its Center; if then Ic be drawn cutting AB in γ, and through γ a Chord $\delta\varepsilon$ of the Circle GCOk be drawn perpendicular to Ic, two Lines Iδ, Iε, will cut de drawn through c parallel to $\delta\varepsilon$, in d and e, and thereby determine de the Length of the Conjugate Axe, by which the Ellipſis may be compleated[e].

For if the Triangle Ide be imagined to be turned on the Line Ic till its Plane become perpendicular to the Plane GCOk, the Line $\delta\varepsilon$ will come into the Plane of the Circle whoſe Diameter is AB, and be a Chord of that Circle paſſing through γ perpendicular to AB, and parallel to the Plane of the Projection; and the Line de in this Poſition, will be the Interſection of this laſt Plane with the Plane of the Triangle I$\delta\varepsilon$, and conſequently d and e will be the Projections of δ and ε.

If the Diameter AC meet Ck in its Extremity C, the Projection of that Circle will be a *Parabola*; in regard that the Cone ICA is cut by the Plane of the Projection parallel to one of its Sides IC.

And here it is manifeſt, that a is the Vertex, and ab indefinitely produced beyond b, is the Axe of the *Parabola* thus formed; and if through r the Interſection of AC with IO, a Chord pq of the Circle GCOk be drawn parallel to Ck, Ip and Iq will cut ab in p and q, and thereby determine pq the Length of the double Ordinate to the Axe ab of the *Parabola* in the Point O, whence that Section may be deſcribed[f].

For if the Triangle Ipq be turned on the Line IO till its Plane become perpendicular to the Plane GCOk, the Line pq will come into the Plane of the Circle, whoſe Diameter is AC, and be a Chord of that Circle paſſing through r perpendicular to AC, and parallel to the Plane of the Projection; and the Line pq in this Poſition, will be

[a] Con. Sec. Art. 7. B. III.

[b] 31 El. 3.

[c] 8 El. 6.

[d] 21 El. 3.

[e] Prop. 16. B. III.

[f] Prop. 17. B. III.

197 N°1

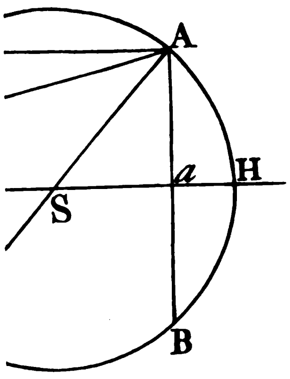


Fig. 197 N°2

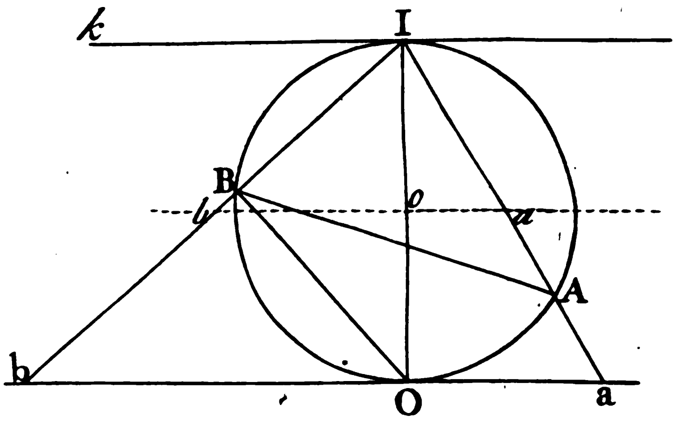


Fig. 197 N°3

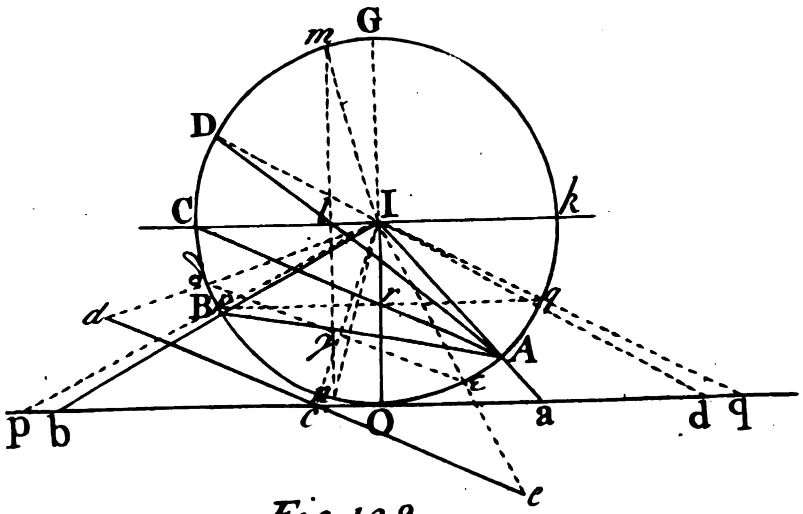
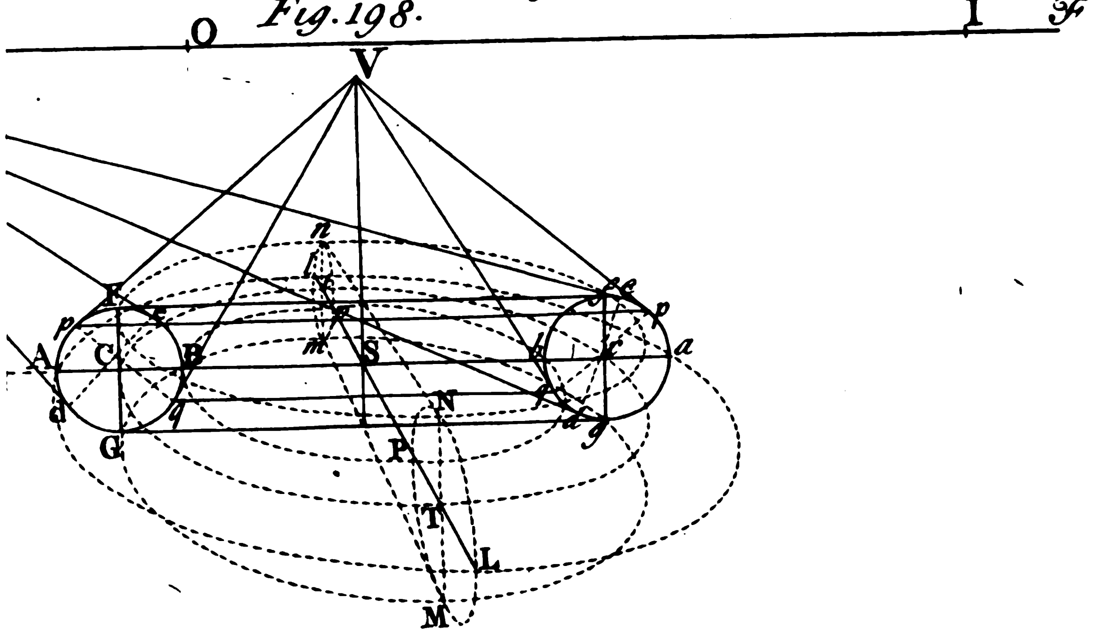


Fig. 198.



J. Mynde sc.

be the Interſection of this laſt Plane with the Plane I*pq*, and p and q will therefore be the Projections of *p* and *q*.

Laſtly, if the propoſed Diameter be AD cutting C*k* any where between C and *k*, in *l*; the Projection of that Circle will be two oppoſite *Hyperbola's*, the one formed by the direct Projection of ſo much of the Circle as lies between A and *l*, and the other by the Tranſprojection of the remainder of that Circle which lies between *l* and D: ſeeing the Plane of the Projection meets the Side DI of the Cone IAD produced beyond its Vertex I, at d.

It is here alſo evident, that ad is the Tranſverſe Axe of the *Hyperbola's* thus formed; and if through *l* a Line *mn* be drawn parallel to IO, the Angle *m*I*n* will be the inward Angle of the *Aſymptotes*, by the help of which theſe Sections may be deſcribed[a].

For the Diameter AD of the forming Circle cutting the Eye's Director I*k* in *l*, the Plane of that forming Circle will meet the great Circle of the Sphere through which the Directing Plane paſſes, in a Chord equal to *mn*, and conſequently *m*I*n* will be the inward Angle of the *Aſymptotes* of the *Hyperbola's* formed by the Projection of the Circle whoſe Diameter is AD[b].

Theſe Rules may be ſhewn to be applicable to all other Poſitions of the Circles in the Sphere, after the ſame manner as thoſe for the *Stereographick* Projection.

This Kind of Projection is uſed in Dialling (from whence it takes its Name) where the Point of the Style repreſents the Center of the Earth, and the Dial-plate a Plane touching its Surface in any propoſed Point, the Style or *Gnomon* itſelf repreſenting the Earth's Axe.

It might be eaſy from theſe Principles, and from what has been formerly ſhewn more at large in Book III, to deduce Methods of Projecting all Sorts of Dials, or other Projections of the Sphere for Aſtronomical Uſes: but this being Foreign to our purpoſe, and having been ſufficiently treated of by other hands, we ſhall purſue it no farther.

SECTION V.

Of the Annulus and its Image.

DEF.

Fig. 198.

IF a ſtraight Line SC in a given Plane *EFS* be moved round its Extremity S as a Center, until its other Extremity C deſcribe the Circle CT*ct* in that Plane, and at the ſame time carry round with it a ſmaller Circle AFBG whoſe Center is C, and whoſe Plane paſſing through SC, continues always perpendicular to the Plane *EFS*; the ſolid Figure generated by the Circumvolution of the Circle AFBG with the Line SC, is called an *Annulus*.

The Circle AFBG is called the *Generating Circle*, and the Point S the Center of the *Annulus*; A*a* is its greateſt Diameter, B*b* its leaſt, and C*c* its mean Diameter, and a Line SV drawn through S perpendicular to the Plane of the Circle CT*ct*, is the Axe of the *Annulus*.

If the *Annulus* be cut by any Plane MN*mn* paſſing through its Axe SV, 'tis evident its Section by that Plane will be two Circles LMPN, *lmpn*, each equal to the Generating Circle AFBG; and all Circles thus formed are called *Generating Circles* of the *Annulus*.

If the Diameter FG of the Generating Circle AFBG be parallel to the Axe SV, the Points F and G will by their Revolution round SV deſcribe two Circles FN*fn*, GM*gm*, each equal and parallel to the Circle CT*ct*; the Planes of which Circles will touch the Annular Surface at Top and Bottom all round, and compleatly cloſe its inner Cavity; and the Solid thus terminated is the ſame with the *Tore* of a Column, of which the Circles FN*fn*, GM*gm*, are called its upper and lower Faces.

That Part of the *Annulus* which is formed by the Revolution of the Semicircle FAG may be called its Exterior Surface, and is the whole of the Annular Surface which can appear in the *Tore* of a Column; and that Part which is formed by the Semicircle FBG may be called its Interior Surface, which laſt is hid in the Solidity of the *Tore*.

As all Points in the Circumference of the Generating Circle AFBG, do by their Circumvolution round the Axe, form Circles in the Annular Surface parallel to the Exterior

[a] Prop. 18. B. III.

[b] Prob. 9. B. III.

terior Circle A L*al* formed by the Point A, theſe are all called *Parallel Circles* of the *Annulus*; thoſe formed by the ſeveral Points of the Semicircle FAG are called *Exterior Parallels*, of which the great Circle A L*al* is the largeſt, and thoſe generated by the Points of the Semicircle GBF are *Interior Parallels*, of which the Circle BP*bp* is the leaſt; and the Exterior Surface is divided from the Interior, by the Circles FN*fn*, and GM*gm*, the laſt of which is called the Baſe of the *Annulus.*

SCHOL.

From the Nature and Generation of the *Annulus*, the following Articles may be deduced.

1. That through every Point of any Generating Circle of an *Annulus* a Parallel Circle may be drawn, and through every Point of a Parallel Circle a Generating Circle may paſs.

2. That if any Line drawn from the Eye touch the Surface of an *Annulus*, it muſt touch that Surface in ſome one Point, common to a Generating Circle and a Parallel Circle.

3. That if the Eye be in the Axe of an *Annulus*, and elevated above it, as at V; all Tangents, V*p*, V*p*, from the Eye to the Exterior Surface, touch it in *p*, *p*, Points of the ſame Exterior Parallel Circle, whoſe Diameter is *pp*; and the Tangents V*q*, V*q*, from the Eye to the Interior Surface, touch it in *q*, *q*, Points of the ſame Interior Parallel Circle, having *qq* for its Diameter; which Tangents therefore form two Conick Surfaces, of which the Eye V is the common Vertex, and the two Parallel Circles whoſe Diameters are *pp* and *qq*, are the reſpective Baſes; the Planes of which Circles are parallel to each other, but the outward Circle is nearer the Eye than the inward.

4. That if the Eye be elevated above the *Annulus*, and out of its Axe, as at Σ, and the *Annulus* be cut by a Plane F*f*G*g* paſſing through the Axe and the Eye; no two different Tangents from the Eye to the Annular Surface, on the ſame Side of the cutting Plane, can touch the ſame Parallel Circle of the *Annulus*; but every ſuch Tangent touches a different Parallel Circle, and conſequently a different Generating Circle; in regard that no Generating Circle of the *Annulus* can be touched by more than two Tangents from the Eye to the Annular Surface, one of which muſt touch it in its Exterior, and the other in its Interior Part, which two Points of Contact cannot therefore lye in the ſame Parallel Circle.

5. If two Tangents Σd, Σ*e*, from the Eye be drawn, touching the Generating Circles AFBG, *afbg*, formed by the cutting Plane, in their Exterior Parts at d and *e*; the Tangent Σd to the nearer Circle AFBG will touch a lower Parallel Circle, and the Tangent Σ*e* to the farther Generating Circle *afbg* will touch a higher Parallel Circle of the *Annulus*, than can be touched by any other Tangent from the Eye to any other Point of the Exterior Annular Surface.

6. If two Tangents Σe, Σ*d*, from the Eye be drawn, touching the Generating Circles formed by the cutting Plane in their Interior Parts e and *d*; the Tangent to the nearer Generating Circle will touch a higher Parallel Circle, and the Tangent to the farther Generating Circle will touch a lower Parallel Circle of the *Annulus*, than can be touched by any other Tangent from the Eye to any other Point of the Interior Annular Surface: Or if no Tangent could be drawn from the Eye to the inner Part of the farther Generating Circle, but the Tangent Σe to the Interior Part of the nearer Generating Circle AFBG ſhould cut the farther Generating Circle *afbg* in any Point, that Point will be in a lower Parallel Circle, than can be touched or cut by any other Tangent from the Eye to any other Point of the Interior Surface of the *Annulus.*

7. The Points d and *e* where the Tangents from the Eye meet the Exterior Parts of the nearer and farther Generating Circles, are the loweſt and higheſt Limits of the viſible Part of the Exterior Surface; and the Points e and *d* where the Tangents meet the Interior Parts of the nearer and farther Generating Circles, are the higheſt and loweſt Limits of the viſible Part of the Interior Surface; or if the Tangent Σe to the Interior Part of the nearer Generating Circle, cut the farther Generating Circle in any Point, that Point is then the loweſt Limit of the viſible Part of the Interior Surface.

8. Every Tangent from the Eye to the Exterior Surface of the *Annulus*, as it is more diſtant from the loweſt Limit d of that Surface, touches a higher Parallel Circle, than one that is nearer that Limit; and ſo on continually, till the Tangent reaches the higheſt Limit *e*: and in the ſame manner, every Tangent from the Eye to the Interior Surface, as it is more diſtant from the higheſt Interior Limit e, touches or cuts a lower Parallel Circle, than one that is nearer that Limit; and ſo on continually, till the Tangent arrives at the loweſt Limit *d* of that Surface.

9. If

Plate 118 Book 6
Sect. 5.

Fig. 199.

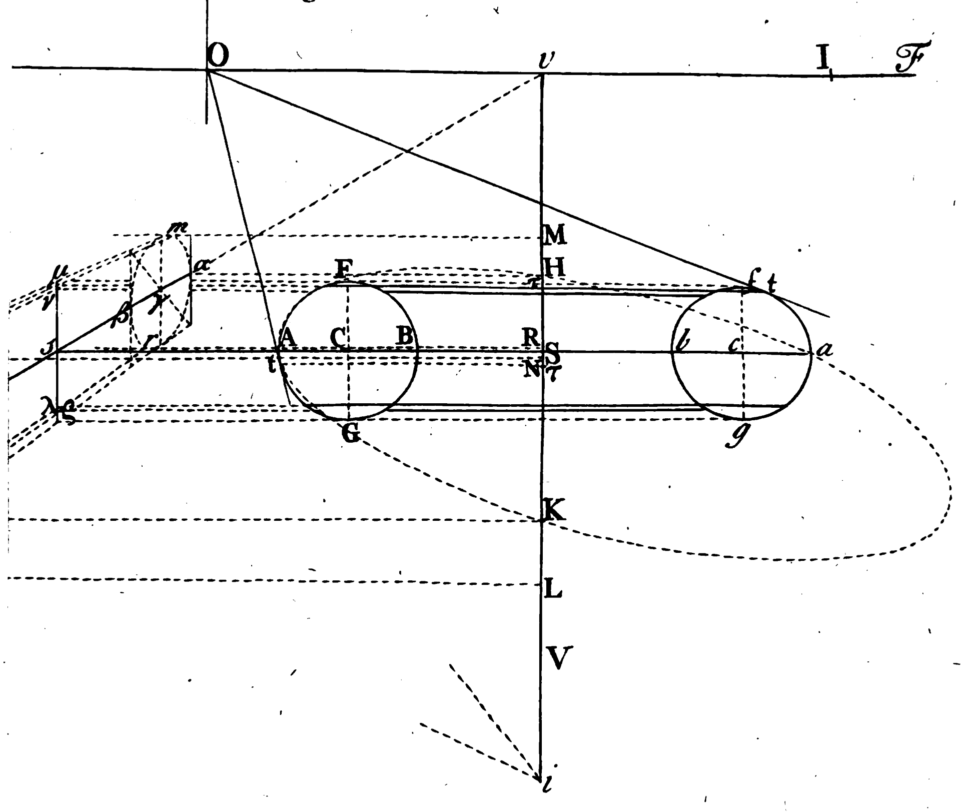
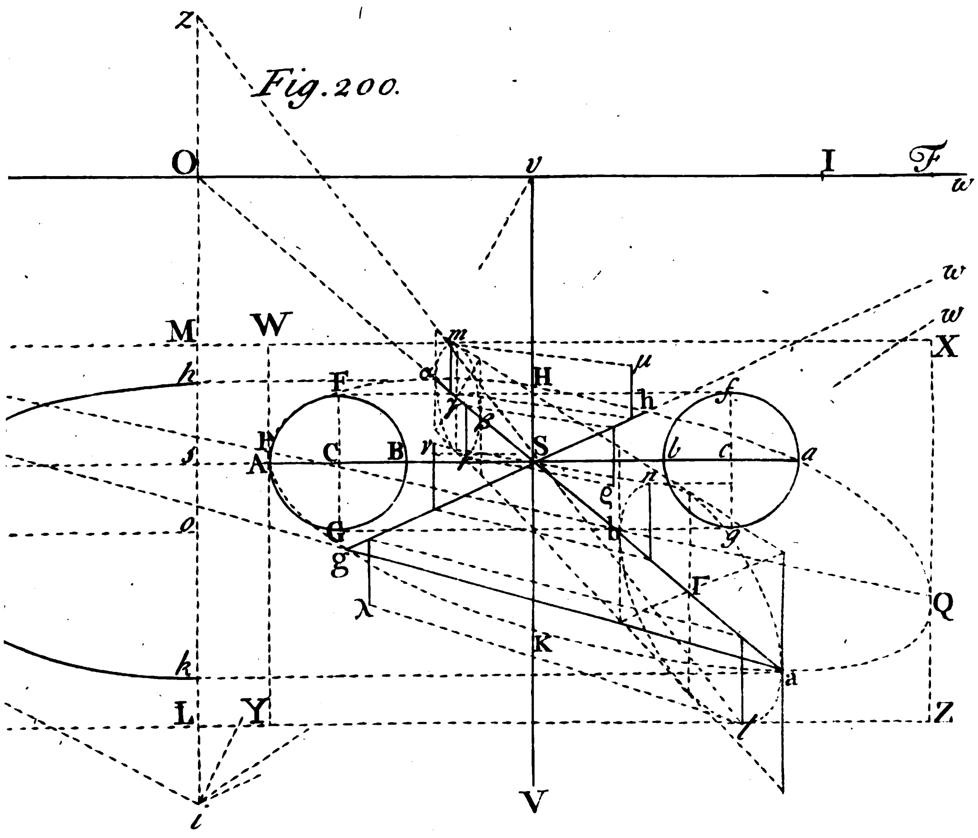


Fig. 200.



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9. If any two Tangents from the Eye, be equally diſtant on each Side from the ſame Limit of the Surface which they touch, they will both touch the ſame Parallel Circle; and no Parallel Circle can be touched by more than two Tangents from the Eye, which Tangents muſt be equally diſtant from the ſame Limit, and conſequently muſt lye in a Plane perpendicular to the cutting Plane *FfGg* which paſſes through the Eye and the Axe of the *Annulus*, in which Plane all the Limits lye, and muſt alſo incline in equal Angles to that Plane.

10. If the Eye be in the Plane of any Parallel Circle of the *Annulus*, no Part of its Interior Surface can be viſible.

11. In whatever Point any Tangent from the Eye to the Annular Surface, meets or cuts any Parallel or Generating Circle, that Point will be viſible, and be one Boundary or Limit of the viſible Parts of thoſe Circles.

12. If a Plane touch the Exterior Annular Surface in any Point, the Tangents in that Point to the Generating and Parallel Circles which paſs through it, will both lye in the touching Plane, and will be perpendicular to each other.

13. If the Eye be any where in the touching Plane, the Point of Contact will be one extreme viſible Point, common to the Generating and Parallel Circles which paſs through it.

PROB. XXI.

The Center and Diſtance of the Picture, and the Vaniſhing Line of the Plane of the Baſe of an *Annulus*, being given, together with the Section of the *Annulus* by a Plane paſſing through its Axe, cutting its Baſe in a Line parallel to the Picture; thence to deſcribe the Exterior and Interior Boundaries of its viſible Surface.

The viſible Part of an *Annulus* not being terminated by Geometrical Curves (except only when the Eye is in its Axe[a]) but by Curves of a different Kind, not reducible to any Planes, its Figure muſt be found by determining the Extremities of the viſible Parts of the Generating or Parallel Circles of which its Surface is compoſed, a ſufficient Number of which being found, the whole of it may be thence deſcribed.

METHOD I.

By the Generating Circles of the *Annulus*.

CASE I.

When the Eye is elevated above the *Annulus*, and ſituated out of its Axe.

Let O be the Center, and IO the Diſtance of the Picture, AFBG*afbg* the given Section of the *Annulus*, S its Center, *EF* the Vaniſhing Line of the Plane of its Baſe, and SV its Axe, here ſuppoſed parallel to the Picture. Fig. 199.

1. To find the higheſt and loweſt Limits of the viſible Parts of the Exterior and Interior Surfaces.

With the Center S and *Radius* SA deſcribe the Image AK*a*H of the great Circle of the *Annulus*, and produce the Axe SV till it cut that Circle in K and H, and the Vaniſhing Line *EF* in *v*; then KH will repreſent the Section of the *Annulus* by a Plane paſſing through its Axe and the Eye, the intire Image of which Plane, and of all Lines and Figures in it, being only the Line *v*V itſelf[b], nothing more can be deſcribed in it, and Recourſe muſt therefore be had to a ſubſtituted Plane.

Having therefore produced the Diameter A*a* of the great Circle at Pleaſure, from *v* draw any Line *vs* cutting it in *s*; and having alſo drawn Ka, H*α*, parallel to *EF* cutting *vs* in a and *α*, on a*α* raiſe a ſubſtituted Plane parallel to the Plane *v*K, and in it find the Images a*l*b*n*, *αrβm*, of two Circles equal to the Generating Circle of the *Annulus*, having the Extremes of their Diameters ab, *αβ*, at a and *α*; and theſe will be the Oblique Seats of the Generating Circles in the Plane *v*K on the ſubſtituted Plane: To theſe ſubſtituted Circles draw the Tangents *l*L, *m*M, *n*N, and *r*R, parallel to *EF*, and meeting *v*K in L, M, N, and R; if then theſe Circles be conſidered as the Baſes of two Scalene Cylinders in the ſubſtituted Plane, having their Axes parallel to the Picture, and whoſe upper Faces are the Generating Circles in the Plane *v*K, 'tis evident, that *l*L, and *n*N, which touch the ſubſtituted Circle a*l*b*n* in *l* and *n*, will alſo be Tangents in L and N to the hither Generating Circle in the Plane *v*K, and will terminate the viſible Part of that Cylinder, and conſequently of the

Yyyy Generating

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[a] Art. 3.

[b] Cor. 1. Theor. 17. B. I.

Generating Circle which makes its upper Face[a]; and in like manner, the Tangents mM and rR will give the Points M and R which terminate the viſible Part of the farther Generating Circle in the Plane vK: wherefore all ſuch part of the hither Generating Circle as lies between L and N, and hath lan for its Seat on the ſubſtituted Plane, will be viſible, and the Remainder whoſe Seat is lbn will be hid; and that part of the farther Generating Circle which hath $m\beta r$ for its Seat, will likewiſe be its only viſible Part, unleſs the Tangent nN ſhould riſe above the Tangent rR, in which Caſe the Point R of the farther Generating Circle would not be viſible, but N would mark the loweſt Point of that Circle that could then be ſeen[b]: and thus L and M will be the Images of the loweſt and higheſt viſible Limits of the Exterior Surface, on the hither and farther Part; and N and R (or N alone) will be the higheſt and loweſt viſible Limits of the hither and farther Interior Surface[c]. *Q. E. I.*

2. To find the Diameters of the Parallel Circles which paſs through the higheſt and loweſt Limits of the Exterior viſible Surface.

From v through m and l the Seats of the higheſt and loweſt viſible Limits of the Exterior Surface, draw vm, vl, cutting $\mu\lambda$ drawn through s parallel to vK, in μ and λ, from whence draw Parallels to Aa terminated by the Exterior Peripheries of the Generating Circles AFBG, $afbg$; and theſe will be the Diameters of the Parallel Circles of the *Annulus* which paſs through M and L, the firſt of which will be totally viſible, and the latter will be totally hid, except only its Point L: For it is evident that μ and λ are the Seats of the Centers of thoſe Parallel Circles on the ſubſtituted Plane, which Centers are in vK; and as the farther Point M of the upper Circle can be ſeen, the whole of that Circle muſt be viſible, and as the neareſt Point L of the lower Parallel Circle is the loweſt Point of the *Annulus* which can appear to the Eye, it is evident no other Point beſides L in that Circle can be viſible[d]. *Q. E. I.*

3. To find the Diameters of the Parallel Circles which paſs through the higheſt and loweſt Limits of the Interior viſible Surface.

From v through n and r the higheſt and loweſt viſible Limits of the Interior Surface, draw vn, vr, cutting $\mu\lambda$ in ν and ϱ, from whence draw Parallels to Aa terminated by the inner Peripheries of the Generating Circles AFBG, $afbg$; and theſe will be the Diameters of the Parallel Circles which paſs through N and R, of which that which paſſes through N, will be totally viſible, and that which paſſes through R will be only viſible in that Point: for the neareſt Point N of the upper Interior Parallel Circle being viſible, the whole of it muſt be ſo, and the Point R being the loweſt moſt diſtant Point of the inner Surface from the Eye that can be ſeen, no other Part of the Parallel Circle which paſſes through R can be viſible[e].

But in Caſe the Tangent rR fall below the Tangent nN, the Point R itſelf not being then viſible, a Line muſt be drawn from v through the Interſection of nN with the nearer Part of the ſubſtituted Circle $\alpha m\beta r$, by which the Diameter of the Interior Parallel Circle which paſſes through N will be obtained as before, of which the Point N will be the only viſible Point. *Q. E. I.*

4. To find the Limits within which all the Parallel Circles lye, which are either totally viſible or totally hid.

Produce vm till it cut the outward Periphery of the ſubſtituted Circle albn in p; then all Parallel Circles of the *Annulus* which paſs through any Points of the hither Generating Circle in the Plane vK, whoſe Seats are in the upper Arch pn, will be totally viſible; the Circles which paſs through M and N the Extremes of that Compaſs been both wholly ſeen: and if vr be produced till it cut the ſame ſubſtituted Circle in its inner Periphery at q, all Parallel Circles of the *Annulus* which paſs through any Points of the hither Generating Circle in the Plane vK, whoſe Seats are within the lower Arch lq, will be totally hid; ſeeing the Circles which paſs through L and R the Extremes of that Compaſs, are only viſible in their Points L and R[f]. *Q. E. I.*

5. To find the Limits within which all Parallel Circles lye, which are partly viſible and partly hid, and to determine their viſible Parts.

All Parallel Circles of the *Annulus* which paſs through any Points in the hither Generating Circle, whoſe Seats are in the Arches lap, and nbq, will be partly viſible and partly hid, the viſible Parts of the Exterior Parallels being next the Eye, and thoſe of the Interior Parallels fartheſt from it.

It

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[a] Prob. 12.

[b] Art. 6.

[c] Art. 7.

[d] Art. 8.

[e] Art. 8.

[f] Part 2. and 3.

It remains therefore to ſhew how to determine the Extreme viſible Points of ſuch Parallel Circles.

Fig. 200.

The ſame things being ſuppoſed as before; find any other Seƈtion of the *Annulus* by a Plane paſſing through its Axe, as for Example, by a Plane OiSV perpendicular to the Piƈture; which is done by drawing from O through S, a Diameter aα of the great Circle AKaH of the *Annulus*, and deſcribing thereon the Images albn, $\alpha r\beta m$, of the two Generating Circles formed by that Seƈtion, having the Extremes of their Diameters ab and $\alpha\beta$, at a and α: To theſe Generating Circles draw Tangents parallel to EF, meeting them in l, n, m, and r; then lan will be the viſible Part of the hither Generating Circle albn, and $m\beta r$ will be the viſible Part of the other; except the Tangent in n cut the farther Circle above r, which if it does, then that Interſeƈtion will determine the loweſt viſible Point of that Circle.

From a to y, a Vaniſhing Point in EF perpendicular to the Vaniſhing Point v, draw ay meeting the great Circle AKaH in g, and through g and S draw a Diameter gh of the great Circle, cutting the Vaniſhing Line EF in w; then, becauſe ag repreſents a Chord of the great Circle AKaH perpendicular to its Diameter KH, 'tis evident that the Diameters gh and aα of that Circle which paſs through g and a, will make equal Angles with the Diameter KH, and conſequently that a Plane paſſing through gh and the Axe SV of the *Annulus*, will incline to the Plane vSV which paſſes through the Eye and that Axe, in the ſame Angle as the Plane OiSV doth; and that therefore the Generating Circles in the Planes OiSV and wSV, are alike ſituated with reſpeƈt to the Eye on each Side, and conſequently that the ſame correſponding Parts of theſe Generating Circles will be viſible to the Eye; wherefore the Parallel Circles which paſs through the Extreme viſible Points l, n, m, r, of the Generating Circles in the Plane OiSV, will alſo paſs through the correſponding Points of thoſe in the Plane wSV[a]; and all theſe Points being to be found in Lines perpendicular to the Plane vSV, Lines drawn from l, n, m, and r, to the Vaniſhing Point y, will mark thoſe Points by their Interſeƈtions with the Plane wSV; which Interſeƈtions may be found without drawing the Generating Circles in that Plane, in the following manner:

Through l, n, m, and r, draw Parallels to Oi till they meet the Diameter aα, and from thence draw Lines to y cutting the Diameter gh, and through theſe Interſeƈtions draw Parallels to Oi, which will be cut by Lines from y to l, n, m, and r, in λ, ν, μ, and ϱ, the Points required; as is ſufficiently obvious, if the hither Generating Circle anbl and the correſponding Circle in the Seƈtion gh, and the farther Generating Circle and its Correſpondent, be conſidered as Seƈtions of two Scalene Cylinders, whoſe Axes paſs through Γ and γ, and have y for their Vaniſhing Point.

Then having found the ſeveral Centers of the Parallel Circles which paſs through l, n, m, and r, by the Interſeƈtion of SV with Lines drawn from thoſe Points to O, and thence their Diameters, and the Parallel Circles being deſcribed accordingly; the Interſeƈtions of thoſe Circles with the Planes OiSV and wSV, will terminate their viſible Parts; that is, ſo much of the hither Parts of the Exterior Parallel Circles which paſs through l and m, as are terminated by $l\lambda$ and $m\mu$, will be viſible, and ſo much of the farther Parts of the Interior Parallels as are terminated by $n\nu$ and $r\varrho$, will be their viſible Parts[b].

And thus, four Extreme viſible Points l, λ, m, μ, of the Exterior viſible Surface of the *Annulus*, and the like Number n, ν, r, ϱ, of the Interior Surface are had, beſides the Points L, M, N, R, before found[c]: and after the ſame manner, ſo many more Points may be obtained in thoſe Surfaces, by the help of Seƈtions of the *Annulus* by any two other Planes paſſing through its Axe, and inclining equally to the Plane vSV, as may be neceſſary for deſcribing the viſible Boundary of the *Annulus* to any required Degree of Exaƈtneſs; obſerving only that when the cutting Plane correſponding to OiSV, on which the Generating Circles are to be deſcribed, is not perpendicular to the Piƈture, the Tangents whereby the Points correſponding to l, n, m, and r, are found, muſt not be drawn parallel to EF, but muſt be made to tend to the Vaniſhing Point of Perpendiculars to that cutting Plane[d]; but the other Plane which correſponds to wSV, and the Points therein which correſpond to λ, ν, μ, and ϱ, are found in the ſame manner as before, by Lines tending to y the Vaniſhing Point in EF which is perpendicular to v. *Q. E. I.*

COR. 1.

Fig. 199.

If from O the Vaniſhing Point of Perpendiculars to the Plane of any Generating Circle AFBG or $afbg$, a Tangent Ot to the Exterior Periphery of its Image be drawn, meeting it in t; the Line Ot will be a Tangent to the Exterior viſible Boundary of

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[a] Art. 9.

[b] Art. 8. and 11.

[c] Part 1. Fig. 199.

[d] Caſe 2. Prob. 12.

of the *Annulus*, and the Point of Contact t will be an Extreme viſible Point of the Generating and Parallel Circles which paſs through it.

For the Original of the Tangent Ot being perpendicular to the *Radius* $t\tau$ of the Parallel Circle of the *Annulus* which terminates at the Point of Contact, and being in the ſame Plane with it, it is therefore a Tangent to that Parallel Circle; and if a Plane be imagined to paſs through this Tangent and the Eye, its intire Image being only the Tangent Ot itſelf[a], that Plane muſt touch the propoſed Generating Circle in the ſame Point t, in which Plane the Real Tangent to that Generating Circle therefore lies; wherefore this Plane paſſing through the Tangents to the Generating and Parallel Circles of the *Annulus* in their Point of Interſection t, it muſt touch the Exterior Annular Surface in that Point, and the Eye being by Suppoſition in this Plane, that Point will be one of the Limits of the viſible Parts of thoſe Circles[b], and conſequently a Point in the viſible Outline of the *Annulus*.

SCHOL.

If from the Vaniſhing Point of Perpendiculars to the Plane of any Generating Circle, a Tangent to the Interior Periphery of its Image be drawn, a Plane paſſing through the Eye and that Tangent will alſo paſs through the Tangent to the Parallel Circle in the ſame Point; but as this Plane muſt neceſſarily cut the *Annulus*, although it doth only touch the Parallel and Generating Circles in their common Point of Interſection, this Point will not be an extreme viſible Point of thoſe Circles, unleſs the Eye be ſo ſituated in this Plane, as that a Line from thence to the Point of Contact, may be drawn without obſtruction from a nearer Part of the Annular Surface.

COR. 2.

Fig. 200. If two Tangents WY and XZ be drawn to the great Circle $AKaH$ perpendicular to EF, touching that Circle in P and Q, and the Tangents in l and m be produced till they meet them in Y, Z, W, and X; a Parallelogram $WXZY$ will be thereby formed, which will incloſe the viſible Outline of the *Annulus*, and touch it in the Points P, Q, l, and m.

For P and Q being the Points of Contact of the great Circle $AKaH$ with Lines from the Foot of the Eye's Director relating to the Plane EF, if a Cylinder were formed on that great Circle with the ſame Axe SV, WY and XZ the Tangents in P and Q, would mark the viſible Bounds of that Cylinder[c]; and as no Part of the *Annulus* can touch that Cylinder, ſave in the great Circle $AKaH$, theſe Tangents therefore can only touch the Annular Surface in P and Q; in the next Place, WX and YZ the Tangents in l and m, being Tangents to the Annular Surface in two thoſe Points[d], and being parallel to EF, 'tis evident that l is the loweſt, and m the higheſt Point of the Annular Surface that can appear in the Picture; conſequently the Parallelogram $WXZY$ formed by the Tangents in P, Q, l, and m, incloſes the viſible Outline of the *Annulus*, and touches it in thoſe four Points.

COR. 3.

If the Axe of the *Annulus* coincide with Oi the Vertical Line of the Plane EF, then the Plane vSV which paſſes through the Axe and the Eye, will alſo coincide with Oi, and the Point v then coming into O, the Point y will be infinitely diſtant; wherefore all Lines which ſhould tend to that Point will be parallel to EF; but this will make no other Difference in the Practice.

COR. 4.

If the Plane of the Baſe of the *Annulus* be not perpendicular to the Picture (as it is here ſuppoſed to be) 'tis evident that the Section of the *Annulus* by a Plane paſſing through its Axe, and cutting its Baſe in a Line parallel to the Picture, will not be parallel to the Picture, but will have its Vaniſhing Line parallel to EF, and paſſing through the Vaniſhing Point of Perpendiculars to that Plane; and the Images of the Generating Circles thereby formed, will not be Circles but Ellipſes, and their Diameters GF and gf, as alſo the Axe SV, muſt all tend to that Vaniſhing Point: But all the Difference in the Operation ariſing on this Score, having already been ſo fully conſidered in former Problems, it will be unneceſſary here to enlarge upon it.

COR. 5.

If the *Tore* of a Column were to be deſcribed, ſo much of the Work as relates to the

[a] Cor. 1. Theor. 17. B. I.

[b] Art. 11, 12, and 13.

[c] Prob. 12.

[d] Cor. 1.

Fig. 201.

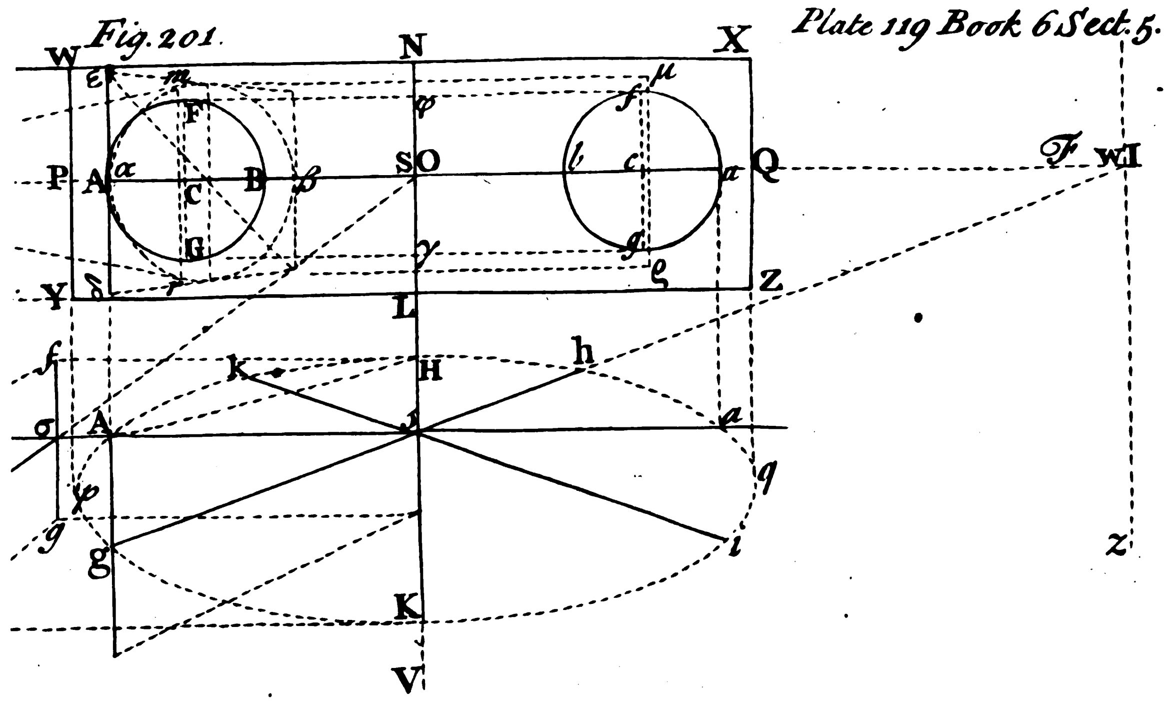
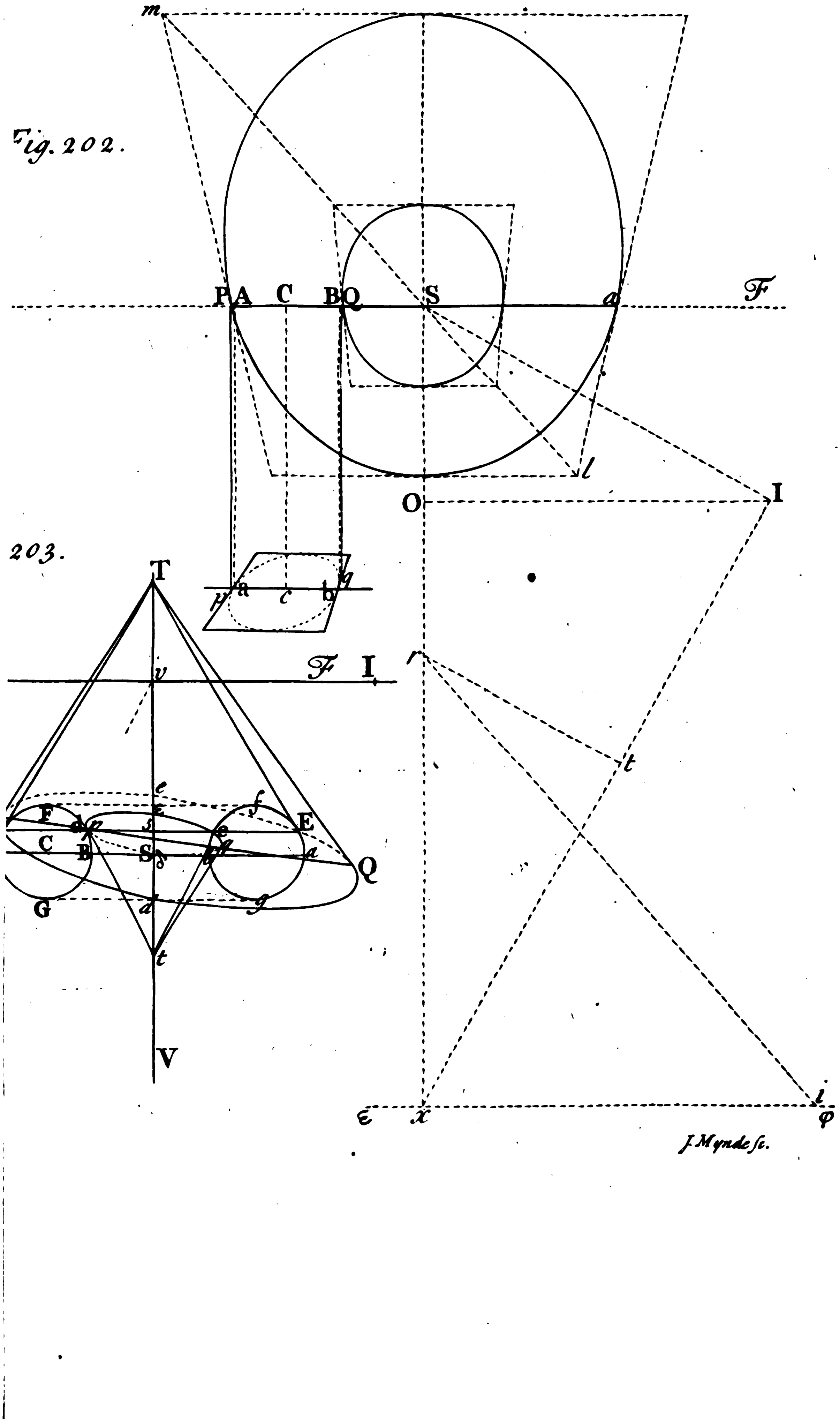


Fig. 202.

[Fig.] 203.



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the Interior Surface of the *Annulus* would be ſaved; and the Seat of the Column on its *Tore* will be found, by deſcribing the Image of a Circle on the Diameter Ff which divides the Exterior from the Interior Surface; and the viſible Outline of the Shaft of the Column being thence drawn[a], will ſhew how much of the *Tore* is hid by it; ſo that only the Remainder of the viſible Outline need be deſcribed.

SCHOL.

Although the viſible Outline of an *Annulus* be a Curve returning into itſelf, yet, when the Eye is not in the Axe of the *Annulus*, that Curve can be neither an Ellipſis nor a Circle.

For when the Axe SV of the *Annulus* doth not coincide with the Vertical Line Oi of the Plane EF, the Tangents in P, Q, l, and m, which form the Parallelogram WXZY which incloſes the viſible Outline[b], are none of them biſected by the Points of Contact, which they ought all to be in order to incloſe an Ellipſis or a Circle[c]; ſeeing the Line PQ which joins the Points of Contact P and Q, tends to the Vaniſhing Point y in EF, and cannot therefore be parallel to WX, nor biſect WY and XZ; and the Line lm which joins the Points of Contact l and m, being a Line in the Plane OiSV, it muſt tend to ſome Vaniſhing Point z in Oi, and cannot therefore be perpendicular to EF, nor conſequently parallel to WY, and ſo cannot biſect WX and YZ.

And when the Axe SV coincides with the Vertical Line Oi of the Plane EF, the Section $la\alpha m$ comes into the Poſition LkbM[d], and kpb will repreſent a Moiety of the great Circle AKaH of the *Annulus*; but although, in this Caſe, the Tangents in M and L will be biſected in M and L by the Line LM, and po the Semi-chord of the Tangents to the great Circle from the Foot of the Eye's Director, will be parallel to the Tangents in L and M, as being parallel to EF; yet LM will not be biſected by po, nor conſequently can the Tangent wy be biſected in p; in regard that po biſects the Diameter kb in o the Center of the Ellipſis formed by the Image of the great Circle, and Lk muſt always be greater than bM: ſo that in either Caſe, the Sides of the Parallelogram which incloſes the viſible Outline of the *Annulus*, not being all biſected in the Points of Contact, the Curve produced cannot be either a Circle or Ellipſis.

In the next Place, the Interſection of PQ with the Plane OiSV in which lm lyes, being where PQ cuts aα; if lm do not cut aα in the ſame Point (as it is apparent it doth not) then PQ and lm, and conſequently the Points P, Q, l, and m, cannot lye in one and the ſame Plane; and therefore the Original of the viſible Outline of the *Annulus* cannot be a Figure reducible to a Plane, as was obſerved at the beginning of this Problem.

But to examine into the Nature and Properties of Curves of this Sort, or to demonſtrate ſtrictly ſome of the Steps advanced in this Scholium relating thereto, *viz.* that Lk muſt always be greater than bM, and that PQ and lm do not interſect in the ſame Point of aα, as it is not within the Compaſs of our Subject, ſo it would require a longer and more difficult Analyſis than would be ſuitable to a Work of this Kind.

CASE 2.

When the Eye is in the Plane of the great Circle of the *Annulus*.

Here, the Eye being ſuppoſed to be in the Plane of the great Circle, the intire Image of that Circle will coincide with its Vaniſhing Line, and is therefore unfit to be uſed as before directed; and conſequently a ſubſtituted Plane muſt be imployed whereby the Deſcription of the viſible Outline of the *Annulus* may be obtained.

Let O be the Center, and OI the Diſtance of the Picture, AFBG$afbg$ the given Section of the *Annulus*, the Plane of whoſe great Circle Aa paſſes through the Eye, and therefore coincides with its Vaniſhing Line EF; and let SV be the Axe of the *Annulus*, and S its Center, which we ſhall ſuppoſe to coincide with O the Center of the Picture. Fig. 201.

Having drawn any ſubſtituted Line parallel to EF cutting SV in s, transfer the Extremities A and a of the given Diameter, to A and a in the ſubſtituted Line, by Perpendiculars to EF; and on Aa, as a Diameter, deſcribe the Image of a Circle AKaH in the ſubſtituted Plane EFs, cutting SV in K and H; then the Circle AKaH will be the Seat of the great Circle of the *Annulus* on that Plane, and KH will be the Seat of its Diameter which is perpendicular to the Picture.

Produce the ſubſtituted Diameter Aa at Pleaſure, and from O draw Oσ cutting it in σ; through σ draw fg parallel to SV, and make each Moiety σf, σg, equal to CF the *Radius* of the Generating Circle AFBG, and having from K drawn Kk parallel to EF cutting Oσ in k, find de the proportional Meaſure of fg on a Line ka parallel

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[a] Prob. 12.

[b] Cor. 2.

[c] Meth. 2. Prop. 16. B. III.

[d] Cor. 3.

parallel to it, drawn through k; produce de till it cut EF in a, and make ad, ae, equal to kd, ke, and on de deſcribe the Image of a Square incloſing a Circle anbl in the Plane OVak; draw two Tangents nN, lL, to this Circle, parallel to EF, touching it in n and l, and produce them till they meet SV in N and L; then N and L will be two Points in the viſible Outline of the *Annulus*, to which nN and lL will alſo be Tangents.

For by this Conſtruction, it is evident that de is the Proportional Meaſure of the Side of the Square which incloſes the hither Generating Circle of the *Annulus*, formed by its Section perpendicular to the Picture, of which Line, K is the intire Seat on the ſubſtituted Plane EFKk, and which Circle is repreſented by NL, and that anbl is the Seat of that Circle on another ſubſtituted Plane OVak; wherefore nN and lL are the Sides which terminate the viſible Part of the Cylinder of which anbl is the Baſe, and NL the upper Face, and conſequently N and L are the Extreme viſible Points of that Generating Circle.

In the next place, imagine the *Annulus* to be cut through its Axe by any other Plane wzSV; the Image of the hither Generating Circle formed by that Section may be found in this manner.

From w the Interſection of wz with EF, through s, draw the Diameter gh of the ſubſtituted Circle AKaH, and through its hither Extremity g draw gα perpendicular to EF cutting it in α; transfer the Extremities F and G of the Diameter FG of the Generating Circle AFBG, to φ and γ in the Line SV, by Parallels to EF, and draw wφ, wγ, cutting gα in ε and δ; and $\varepsilon\delta$ will repreſent the Side of the Square which incloſes the hither Generating Circle of the *Annulus* formed by its Section with the Plane wzSV, by the help of which the Image of that Square, and conſequently the Circle $\alpha m\beta r$ may be deſcribed.

For it is evident that gh is the Seat on the ſubſtituted Circle AKaH, of the Diameter of the great Circle of the *Annulus* through which the cutting Plane wzSV paſſes, and that g is the Seat of its hither Extremity α, and conſequently of the neareſt Extremity of the Diameter $\alpha\beta$ of the hither Generating Circle formed by that Section; and $\varepsilon\delta$ being the Proportional Meaſure of that Diameter on the Line gα, it therefore repreſents the Side of the Square which incloſes that Circle, and $\alpha m\beta r$ is therefore its Image.

Then from v the Vaniſhing Point of Perpendiculars to the Plane wzSV, draw two Tangents to the Circle $\alpha m\beta r$ meeting it in m and r, and having drawn the Chord mr, draw $\mu\varrho$ parallel to it, at an equal Diſtance on the other Side of O; which being terminated in μ and ϱ by $m\mu$, $r\varrho$, parallel to EF, the Points m, r, μ, ϱ, will be four Points more of the viſible Outline required.

For the Tangents vm, vr, being perpendicular to the Plane of the Circle $\alpha m\beta r$, m and r are its extreme viſible Points, and conſequently Points of the viſible Outline of the *Annulus*[a]; now the Axe SV being here ſuppoſed to paſs through O the Center of the Picture, if the *Annulus* be cut by another Plane paſſing through its Axe, and inclining the contrary way to the Perpendicular Section NL, in the ſame Angle as the Plane wzSV doth, and which Plane muſt therefore alſo cut the ſubſtituted Circle AKaH in its Diameter ik, which inclines to Ss in the ſame Angle as gh; it is evident that the hither Generating Circle formed by this Section, will have the like Situation with reſpect to O on the one Side, as the Circle $\alpha m\beta r$ hath on the other, and its extreme viſible Points will lye in the ſame Parallel Circles which paſs through m and r[b]; wherefore $\mu\varrho$ drawn parallel to mr, and at an equal Diſtance from O on the oppoſite Side, repreſents the Chord of the Tangents to that Generating Circle from the Vaniſhing Point of Perpendiculars to its Plane, and the Points μ and ϱ determined by $m\mu$ and $r\varrho$ drawn parallel to EF (the Point marked y in Fig. 200, being here infinitely diſtant[c]) are its Extremities, and conſequently two Points more in the viſible Outline of the *Annulus*: and after the like manner as many more Points of the viſible Outline may be obtained, four at a time, as may be deſired.

Laſtly, draw two Tangents to the ſubſtituted Circle AKaH perpendicular to EF, touching it in p and q, and produce them till they cut EF in P and Q; then P and Q will be the extreme viſible Points of the great Circle Aa of the *Annulus*, and thoſe Tangents will alſo be Tangents to its viſible Outline, and with the Tangents in N and L will form the Parallelogram WXZY which incloſes and touches that Outline in P, N, Q, and L[d], the Sides of which are in this Caſe biſected by thoſe Points. *Q. E. I.*

[a] Cor. 1. Caſe 1.

[b] Art. 9.

[c] Cor. 3. Caſe 1.

[d] Cor. 2. Caſe 1.

COR.

COR. 1.

If the Center of the *Annulus* do not coincide with the Center of the Picture, in which Caſe its Axe SV muſt croſs EF in ſome Point v different from O, the Chord $\mu\varrho$ of the Tangents will not be at an equal Diſtance from SV with mr; but its Place and Length may be obtained by transferring the Chord mr to the ſubſtituted Diameter gh, and by the help of a Point y perpendicular to v, finding its Place and Length on the Diameter ik which correſponds to gh, as was done at Part 5. Caſe 1. and thence transferring the Length of the Chord thus found, to its proper Place on EF.

COR. 2.

If the Eye were in the Plane of one of the Parallel Circles of the *Annulus*, in which Caſe, as well as when the Eye is in the Plane of the great Circle, no part of the Interior Surface can be ſeen[a], the ſame Method muſt ſtill be uſed; by reaſon that in ſuch a Poſition of the Eye, the Plane of the great Circle will have ſo little Depth, that the Extremities of its Diameters through which the cutting Planes are ſuppoſed to paſs, cannot be determined with ſufficient Exactneſs; which makes it neceſſary to imploy a ſubſtituted Plane of greater Depth, to which the Seat of that great Circle may be transferred: However, in this Caſe, the Image of the great Circle of the *Annulus* muſt alſo be drawn, to which the Extremities g and i of the Diameters gh, ik, *&c.* of the ſubſtituted Circle muſt be transferred[b], the Originals of thoſe Points being Points in the great Circle, whoſe Image doth not in this Caſe coincide with EF, the Eye not being ſuppoſed to be in that Plane; but in all other reſpects the Practice is the ſame as before.

[a] Art. 10.

[b] Cor. 2. Prop. 49. B. IV.

CASE 3.

When the Eye is in the Axe of the *Annulus*.

If the Eye be any where in the Axe of an *Annulus*, its Exterior and Interior viſible Parts will be terminated by Circles, the Diameters of which are found by Tangents from the Eye to the outward and inward Parts of any one of its Generating Circles[c]; but in order to determine theſe Points of Contact, and conſequently the Diameters of the Circles whoſe Images form the Bounds of the viſible Part of the *Annulus*, a ſubſtituted Plane muſt be imployed, as was done for finding the Circle Boundary of a Sphere[d].

[c] Art. 3.

[d] Prob. 17.

Fig. 202.

Let O be the Center, and OI the Diſtance of the Picture, Aa the Image of the greateſt Diameter of an *Annulus* parallel to the Picture, S the Center of the *Annulus*, and alſo the Indefinite Image of its Axe which paſſes through the Eye; and let AB be the Diameter, and C the Center of one of the Generating Circles formed by a Section of the *Annulus* by a Plane paſſing through the Axe and the Diameter Aa, and conſequently coinciding with EF the Vaniſhing Line of the cutting Plane.

Having at a convenient Diſtance from Aa drawn a ſubſtituted Line ab parallel to it, transfer the Points A, B, and C, to a, b, and c, in the ſubſtituted Line, by Perpendiculars to EF; on ab as a Diameter deſcribe the Image of a Circle in the ſubſtituted Plane EFab, and having drawn two Tangents to this Circle perpendicular to EF, touching the Circle in p and q, produce them till they cut EF in P and Q; then through x the Vaniſhing Point of Perpendiculars to the Plane EF, draw a Vaniſhing Line $\epsilon\varphi$ parallel to EF, and with the Center S and *Radii* SP, SQ, deſcribe the Images of two Circles in the Planes $\epsilon\varphi$, and theſe will repreſent the Exterior and Interior viſible Boundaries of the *Annulus*.

For the Circle apbq is the Oblique Seat of the Generating Circle whoſe Diameter is AB, on the ſubſtituted Plane, and the Tangents pP, qQ, to this ſubſtituted Circle, are alſo Tangents to the Generating Circle AB, and P and Q are therefore the Points of Contact of that Circle with Lines from the Eye; wherefore SP is the apparent *Radius* of the Circle formed by the Tangents from the Eye to the Exterior Surface of the *Annulus*, and SQ the *Radius* of the Circle formed by the Tangents to the Interior Surface; and as theſe Circles are in Planes perpendicular to the Plane EF, they have therefore $\epsilon\varphi$ for their Vaniſhing Line. *Q. E. I.*

SCHOL.

Here, as the Diſtance Ix of the Vaniſhing Point x is ſo great, that there is not room to ſet it off from x on the Line $\epsilon\varphi$, and yet the Diagonal lm of the Images of the Squares which incloſe the terminating Circles, muſt be made to tend to that Point; this Inconveniency is eaſily removed, by taking on xI any ſmaller Diſtance xt

which can be ſet off from *x* to *i* in εφ within the compaſs of the Paper, and from *t* drawing *tr* parallel to IS cutting *x*S in *r*; for then a Line *ir* being drawn, it will be parallel to *lm*[a].

COR.

If the Point S were in the Center of the Picture, the Images of the terminating Circles would alſo be Circles, they being then in Planes parallel to the Picture; but in either Caſe the *Radii* SP, SQ, of thoſe Circles muſt be found after the ſame manner, by the help of a ſubſtituted Plane.

METHOD 2.

By the Parallel Circles of the *Annulus*.

CASE 1.

When the Eye is elevated above the *Annulus*, and ſituated out of its Axe.

Fig. 203. Let AFBG*afbg* be the Section of an *Annulus* by a Plane paſſing through its Axe, and cutting its Baſe in a Line parallel to the Picture, as before.

Draw any Line DE parallel to the Diameter A*a* of the *Annulus*, cutting the Generating Circles AFBG, *afbg*, in D, d, e, and E, and through either of the Exterior Points D, draw a Tangent to the Circle AFBG, meeting the Axe SV in T; then with the Diameter DE deſcribe the Image D*d*E*e* of the Exterior Parallel Circle of the *Annulus* which paſſes through D and E, and from T draw two Tangents to that Image meeting it in P and Q; then P and Q will be the extreme viſible Points of that Parallel Circle, and the nearer Part P*d*Q will be its viſible Part, and the farther Part P*e*Q will be hid.

In like manner, through either of the Interior Points d, draw a Tangent to the Circle AFBG meeting the Axe in *t*, and having with the Diameter de drawn the Image of the Interior Parallel Circle which paſſes through d and e, from *t* draw two Tangents to that Image meeting it in *p* and *q*, and theſe will be the extreme viſible Points of that Parallel Circle, of which, the farther Part *pεq* will be viſible, and the hither Part *pδq* will be hid: and after the ſame manner, by drawing other Parallels to A*a*, terminated by the Exterior and Interior Peripheries of the Generating Circles AFBG, *afbg*, the viſible Parts of as many more Exterior and Interior Parallel Circles of the *Annulus* may be found, as may be ſufficient to deſcribe the Boundaries of its viſible Part.

Dem. For DE being parallel to the Diameters AB, *ab*, of the Generating Circles, it is evident that a Tangent to the Circle *afbg* in the Point E, will meet the Axe SV in the ſame Point T, where it is cut by the Tangent in D to the Circle AFBG; and that if from T, Lines be drawn to the ſeveral Points of the Parallel Circle D*d*E*e*, they will all be Tangents to the Annular Surface, and form a Right Cone, having the ſame Axe TS with the *Annulus*, and the Circle D*d*E*e* for its Baſe; the Limits of the viſible Part of which Baſe, are determined by Tangents to its Image drawn from the Vertex T[b]: and in regard that Lines drawn from the Eye to P and Q, are Tangents to this Cone in P and Q[c], they are therefore alſo Tangents to the Exterior Annular Surface in the ſame two Points, which Points are therefore viſible[d].

In like manner, the Tangents to the Generating Circles in d and e, meet in the ſame Point *t* of the Axe; and if from *t* Lines be drawn to the ſeveral Points of the Interior Parallel Circle dδeε, they will all be Tangents to the Interior Surface of the *Annulus*, and form a Cone of which *t* is the Vertex, and the Circle dδeε is the Baſe; and as Lines from the Eye to *p* and *q*, where the Tangents from *t* meet the Baſe, are Tangents to this Cone in *p* and *q*, they are therefore alſo Tangents to the Interior Annular Surface in the ſame two Points, which Points are therefore viſible.

Now, as all Tangents from the Eye to any Points of the Exterior Annular Surface, on the hither Side of P and Q, touch it in Parallel Circles, lower and lower than the Circle D*d*E*e* which paſſes through P and Q, till they arrive at the loweſt Limit on the hither Part[e], the Images of none of theſe Parallel Circles can poſſibly obſtruct the Sight of the Part P*d*Q of the Circle D*d*E*e*, and therefore that Part muſt be viſible; but the Parallel Circles touched by Lines from the Eye beyond P and Q, being all higher than the Circle D*d*E*e*, the Part P*e*Q of that Circle muſt neceſſarily be hid by them: On the contrary, the Interior Parallel Circles touched by Lines from the Eye any where beyond *p* and *q*, being all lower than the Circle dδeε, none of theſe Circles can obſtruct the Sight of the farther Part *pεq* of that Circle, which Part is therefore viſible; but the Interior Parallel Circles touched by Lines from the Eye on the hither

Side

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[a] Cor. 3. Meth. 2. Prob. 18. B. II.

[b] Prob. 8.

[c] Lem. 10.

[d] Art. 11.

[e] Art. 8.

Side of p and q, being all higher than the Circle $d\delta e\varepsilon$, they muſt therefore hide the Part $p\delta q$ of that Circle; and conſequently PdQ and $p\varepsilon q$ are the viſible Parts of the Exterior and Interior Parallel Circles DdEe and $d\delta e\varepsilon$. *Q. E. I.*

COR. 1.

The Chord of the Tangents from T or t to the Image of any Parallel Circle of the *Annulus*, always tends to the Vaniſhing Point in *EF* which is perpendicular to the Vaniſhing Point v, where the Axe SV croſſes that Line; and the Axe is always Harmonically divided by the Point T or t, and its Interſections with the Image of the correſponding Parallel Circle, and the Chord of its Tangents from T or t[a]; which furniſhes an eaſy Method of determining the Points of Contact P and Q, or p and q with greater Accuracy.

COR. 2.

If the greateſt or leaſt Diameter Aa or Bb of the *Annulus* be propoſed; then the Tangents to the Generating Circles in A and a, and B and b, being parallel to the Axe SV, the Point T or t becomes infinitely diſtant, and thoſe Tangents, which with reſpect to all other Parallel Circles produce Cones, will here produce Cylinders; and conſequently the Tangents to the Images of the greateſt and leaſt Circles which determine their viſible Parts, are parallel to SV.

COR. 3.

It is evident that the Vertices T of the Cones formed by the Tangents to the Exterior Parallel Circles whoſe Diameters lye above Aa, fall above S in the Axe SV, and thoſe of the Exterior Cones whoſe Baſes are below Aa fall below S; and as the viſible Parts of all ſuch Baſes are thoſe neareſt the Eye, it follows, that of the Images of the Exterior Parallel Circles which lye above Aa, more than a Moiety is viſible, and of thoſe which lye below Aa, more than a Moiety is hid, whilſt the Image of the great Circle whoſe Diameter is Aa, is biſected by the Line which ſeparates its viſible from its inviſible Part; that Line being a Diameter of the Ellipſis which repreſents that Circle, the Tangents at its Extremities being parallel.

On the contrary, the Vertices t of the Cones formed by the Tangents to the Interior Parallel Circles which lye above Aa, fall below S, and thoſe of the Interior Cones whoſe Baſes lye below Aa, fall above S; and as the viſible Parts of theſe Baſes are thoſe moſt diſtant from the Eye, it ſtill returns that of thoſe above Aa more than a Moiety is viſible, and of thoſe below Aa leſs, whilſt the Image of the leaſt Circle is biſected by its viſible Part; provided the Eye be not ſituated ſo low as to hide that Circle intirely.

COR. 4.

If the Vertex T or t of the Cone fall where the Image of the correſponding Parallel Circle which is its Baſe, croſſes the Axe SV; then if that Circle be above Aa, it will be the largeſt Parallel Circle of the reſpective Surface that can be totally ſeen; and if it be a Circle under Aa, no more of it than only its Point T or t will be viſible; which Circles will be the ſame with thoſe which paſs through M, L, N, and R, in Fig. 199[b]: and if the Point T or t fall within the Image of its correſponding Parallel Circle, then if that Circle lye above Aa it will be wholly ſeen, but if it lye below Aa it will be totally hid.

CASE 2 *and* 3.

As to the ſecond Caſe, when the Eye is in the Plane of the great Circle, the Application of this Method may be eaſily made, by the Aſſiſtance of ſubſtituted Planes; and with reſpect to the third Caſe, when the Eye is in the Axe of the *Annulus*, the Method before ſhewn remains the ſame. *Q. E. I.*

GENERAL COROLLARY.

The laſt Method of this Propoſition may be uſefully applied for finding the apparent Outline of any Vaſe, Urn, or other Object, whoſe Sections by Planes parallel to its Baſe are Circles, let its Elevation be of what Figure it will.

For a Section of the Object by a Plane paſſing through its Axe, cutting the Baſe in a Line parallel to the Picture, being deſcribed; Lines may be drawn a-croſs this Elevation parallel to its Baſe, through the remarkable Riſings and Sinkings of its Outline, which will ſerve as ſo many Diameters for drawing the Images of Circles parallel to the Baſe; the extreme viſible Points of which Circles may be determined by means of Tangents to the Outline of the Elevation at the Extremities of thoſe Diameters; for

[a] Prob. 3. B. III.

[b] Part 1. Meth. 1. Caſe 1.

thereby the Vertices of the Cones which have theſe Circles for their Baſes will be found, and thence their extreme viſible Points: and by the help of a ſufficient Number of Points thus determined, the viſible Outline of the Object may be deſcribed.

Likewiſe, the ſeveral Parallel Circles may be divided by Diameters, having any propoſed Inclination to the Diameters firſt ſuppoſed, whereby Points in thoſe Circles will be found, by which the Section of the Object by any Plane paſſing through its Axe and thoſe Diameters, may be deſcribed; and the viſible Surface may be ſo Reticulated by theſe Sections and the Parallel Circles, that any Figures or Ornaments on the Face of the Object may be thereby eaſily drawn.

SCHOL.

Although the Trouble of drawing the Images of ſo many Circles, as are neceſſary nicely to determine the viſible Outlines of Objects of this Sort, may be generally ſo much, that few Artiſts will care to undergo the Taſk, but will chuſe rather to truſt to their Eye in deſigning them; yet ſome little helps from *Stereography* may be neceſſary, and will greatly contribute towards deſcribing the propoſed Objects with more Exactneſs than can be done without its Aid: Nevertheleſs, after all the Aſſiſtance that *Stereography* can furniſh, there will be ſtill wanting a good Skill in the Art of Drawing, to give the Images that ſwelling Roundneſs which is requiſite to make them appear Natural and Juſt.

STEREOGRAPHY,

OR A

COMPLEAT BODY

OF

PERSPECTIVE,

In all its BRANCHES.

BOOK VII.

HAVING gone through what was at firſt propoſed, touching the Deſcription of the Images of ſingle Objects, whether Lines, plain Figures, or ſolid Bodies, as well Rectilinear as Curvilinear, and of their Projections or Shadows, and Reflections; and given ſuch Variety of Examples in all Kinds, as may furniſh ſufficient Methods for the Deſcription of any other Objects which may offer; we ſhall in this Book treat of ſome other matters relating to the general Practice of Painting, whether intended for private Houſes, or for Churches, Theatres, or other publick Buildings, either on plain or uneven Grounds, and amongſt other things lay down ſuch Directions for the Choice of the Diſtance and Height of the Eye, and the Size and Situation which ought to be given to Pictures in different Circumſtances, as may enable the Artiſt to put in Practice the Inſtructions before given, with greater Judgment, ſo that the Objects he deſcribes may appear in the moſt agreeable and beautiful Manner.

But, as what we ſhall advance on theſe Subjects, will be deduced from Principles already demonſtrated in the foregoing Part of this Work; it will not ſo naturally fall into the Form of a Series of Propoſitions, as into that of ſhort Eſſays, under the ſeveral Titles mentioned at the Heads of the following Sections.

SECTION I.

Of fixed or immoveable Painting on flat Grounds.

BY fixed Painting, is meant, all ſuch as is done on the Walls, or Ceilings of Rooms, or Buildings, on purpoſe to remain fixed and unmoved in the Place for which it was at firſt drawn.

The Rules for this Kind of Painting on flat Grounds, differ in nothing from thoſe already taught; ſave that in Detached Pictures, not Painted expreſſly for any particular Place or Situation, the Painter is at liberty to take what Height or Diſtance of the Eye he thinks fit; but for thoſe to be done on the Walls or Ceilings of Rooms, he is more confined in his Choice of thoſe Meaſures, as he is obliged to take ſome proper Point within the Room for his Station, from whence his intended Work may be ſeen to the beſt Advantage, and to place the Height of the Eye nearly at the uſual Height of a Man's Eye ſtanding on the Floor.

Therefore, all that is neceſſary in this Caſe, is, after having fixed upon the Point of Station, and the Height of the Eye, to prepare the Wall or Ceiling, by drawing thereon

thereon the neceſſary Lines and Points, as on the Plane of a Picture, in order for the Deſcription of the intended Objects.

How this is to be done univerſally, in all Situations of the Wall, whether Perpendicular, Parallel, Inclining, Declining, or in any other irregular Poſition with reſpect to the Floor, is ſhewn in the following Propoſition and its Corollaries.

P R O B.

The Perpendicular Seat and Support of the Eye on an Original Plane, and the Interſection of that Plane with any other Plane, together with their Angle of Inclination, being given; thence to find the proper Lines and Points neceſſary for the Preparation of a Picture on this laſt Plane with reſpect to the Eye and the Original Plane.

Fig. 204. Let ABCD be the Original Plane, and K the Seat of the Eye on that Plane, its Height above that Seat being known; and let GH be the Interſection of the Plane ABCD with another Plane inclining to it in any Angle Z.

Through K draw KP perpendicular to GH cutting it in P, and having drawn KI perpendicular to KP and equal to the given Height of the Eye above its Seat, draw I*k*, making the Angle I*k*K equal to the Angle Z; then transfer the Figure I*k*KP to any convenient Place apart[1], and through P draw P*o* parallel to I*k*; from I draw I*o* parallel to *k*P, and IO perpendicular to P*o*; meeting P*o* in *o* and O, and produce IK till it alſo meet that Line in *x*; and thereby the Vertical Plane I*ok*P will be found, by the help of which the Picture[2] may be prepared according to the Rules formerly taught[a].

Dem. For if the Triangle IK*k*[3] be turned up on the Line K*k* till IK become perpendicular to the Plane ABCD, the Point I will repreſent the Eye in its true Situation on that Plane; and the Plane IK*k* in this Poſition, being perpendicular to GH, it will alſo be perpendicular to the Plane on which the Picture is to be drawn, as well as to the Original Plane[b], and will therefore repreſent the Vertical Plane; the Interſections of which with thoſe two Planes, *viz.* the Vertical Line and the Line of Station, will make together an Angle equal to the Angle of Inclination Z of the two propoſed Planes[c], and the Angle I*k*K being by Conſtruction equal to that Angle, I*k* will therefore be parallel to the Vertical Line, and conſequently will repreſent the Eye's Director, and *k* the Point of Station; the reſt of the Conſtruction in Fig. 205 and 206 needs no farther Explanation, the Letters therein, denoting the ſame things as uſual. *Q. E. I.*

C O R. 1.

'Tis evident, that if the Angle Z were Right, the Point *k* would coincide with K, the Vertical Line P*o*[4] would be perpendicular to the Line of Station KP; the Center of the Picture O[5] would coincide with *o* the Center of the Vaniſhing Line *EF*, and *x* the Vaniſhing Point of Perpendiculars to the Original Plane would be infinitely diſtant.

C O R. 2.

The Vaniſhing Point of any Line BD in the Original Plane, is found by drawing from *k*[6] a Line *k*H parallel to BD cutting GH in H, and transferring the Diſtance PH from *o* to *z* in the Vaniſhing Line *EF*[7]: Or the ſame Point *z* may be found by transferring the Diſtance PH from P to H in the Interſecting Line, and erecting from H the Perpendicular H*z*[d].

C O R. 3.

If the intended Picture on the inclining Plane be to reach down to the Interſecting Line GH of the Original Plane[8], then the Meaſures to be taken on the Interſecting Line of the Picture, muſt be equal to thoſe in the Original Plane; but if the intended Picture is to be terminated by any Line above GH and parallel to it, another Line *gh* muſt be drawn in the Picture[9], at the ſame Diſtance above its true Interſecting Line GH, as the ſuppoſed Parallel in the inclining Plane is above the Interſecting Line of that Plane; and the Meaſures to be taken on *gh*, muſt only be the Proportional Meaſures with reſpect to thoſe of the Original Plane[e]: but in either Caſe, the ſame Diſtance of the Eye I*o* or *k*P muſt be retained.

By the help of this Propoſition, a Picture may be prepared and drawn to remedy or hide any Defect or Irregularity in a Room, or Building, in point of Height, Breadth, Length, or otherwiſe; ſo that by placing ſuch Picture in a propoſed Situation, it ſhall

tally

[1] Fig. 205.

[2] Fig. 206.

[a] Sect. 1. B. II.

[3] Fig. 204.

[b] 19 El. 11.

[c] Theor. 9. B. I.

[4] Fig. 205.

[5] Fig. 206.

[6] Fig. 204.

[7] Fig. 206.

[d] Cor. 1. Meth. 1. Prob. 1. B. II.

[8] Fig. 204.

[9] Fig. 206.

[e] Def. 3. Prob. 6. B. II.

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ig. 204.

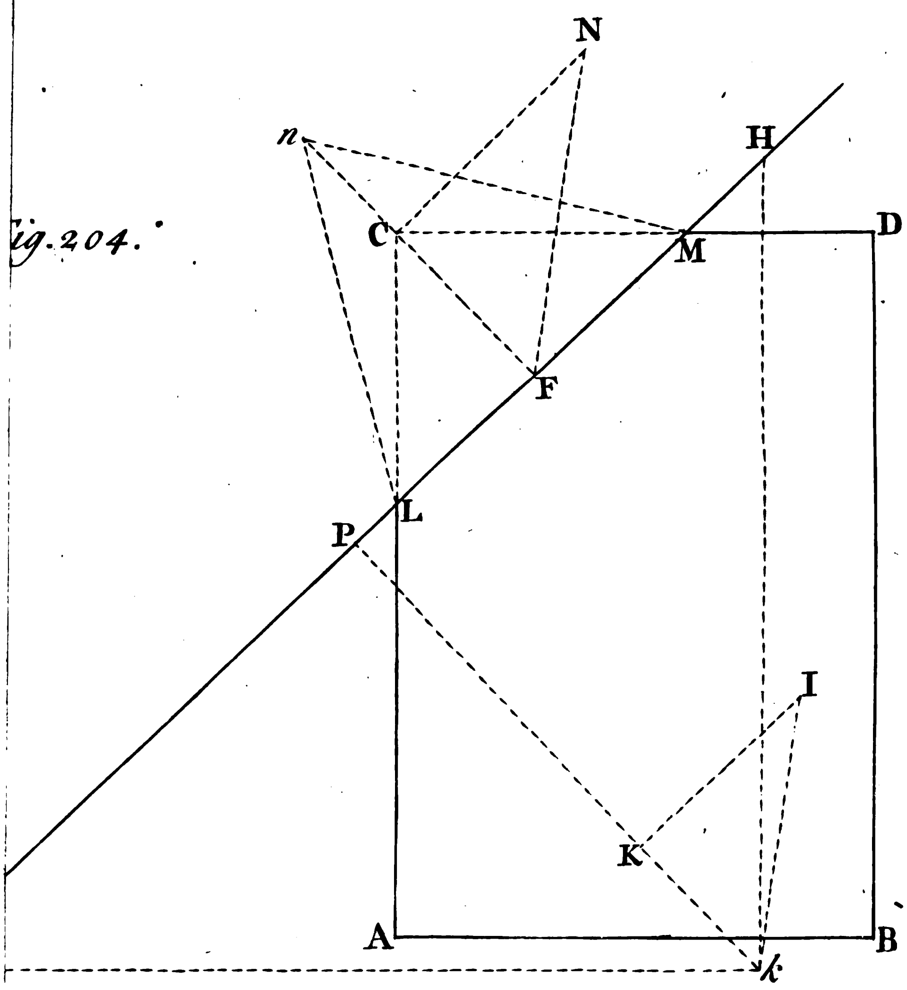
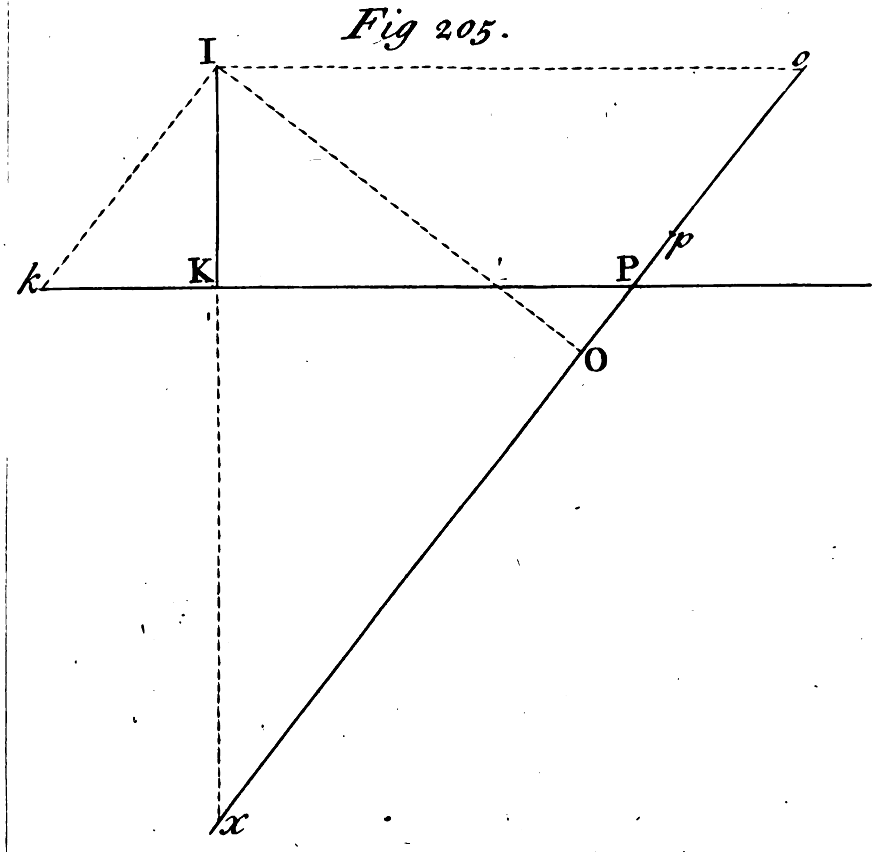


Fig 205.



J Mynde sc

tally and agree with the other part of the Building, and repreſent a Continuation of it in ſuch Manner as may be deſired.

Thus if the Original Plane ABCD were the Plan of a Room, irregular at one End Fig. 204. by reaſon of a ſloping Plane meeting the Floor in LM, and the two ſide Walls over AC and CD, in ſuch Manner as to form a Triangle cutting off the ſolid Angle at C; and it were required upon the Face of this inclining Plane, to draw ſuch a Picture as might ſupply the deficient ſolid Angle, and make the Room appear compleat to an Eye placed in a given Situation.

This may be effected in the following Manner.

From the deficient Corner C draw CF perpendicular to LM, and CN parallel to it, and equal to the Height above C where the ſloping Plane meets the Angle of the Room; then draw FN, and thereby NFC the Angle of Inclination of the ſloping Plane to the Floor will be found; laſtly, produce FC to *n* till F*n* be equal to FN, and the Triangle L*n*M will give the true Shape and Size of the ſloping Plane, as bounded by the Floor and the Sides of the Room.

For if the Triangle CFN be turned on the Line CF, till CN becomes perpendicular to the Plane of the Floor, CN will then repreſent the Interſection of the two ſide Walls over AC and DC, and the Point N will repreſent the Point of that Interſection where the ſloping Plane meets it, and FN will therefore be a Line in that Plane, and NFC the Angle of its Inclination to the Floor: And as FN in this Situation is perpendicular to LM, if FN be laid flat on the Floor ſo as to come into the Poſition F*n*, it is evident L*n* and M*n* will give the Triangle LM*n* equal and ſimilar to that Part of the Sloping Plane which is intercepted between the Points L, M, and N.

If the Height and Situation of the Eye be choſen on the Original Plane at diſcretion, and a Picture be prepared for this Sloping Plane according to the directions of Fig. 204. 206. the laſt Problem, as at Fig. 206; the true Part which is to be painted on, is determined by transferring the Diſtances of the Points L and M from P in the Original Plane, to L and M in the Interſecting Line GH of the Picture, and on LM making a Triangle LMN ſimilar and equal to LM*n* in the Original Plane, which will be the true Part of the Picture required: And the Vaniſhing Points *y* and *z* of the Sides DC and AC of the Floor, and *x* the Vaniſhing Point of Perpendiculars to the Original Plane being found[a], L*z*, M*y* and *x*C by their mutual Interſections will give LMNC the Repreſentation of the deficient ſolid Angle of the Room; which being placed in its true Poſition on the Sloping Plane, and ſeen by an Eye in the Situation before choſen, will have the deſired Effect.

The Sides LN and MN of the Triangle LMN may be alſo determined by drawing *xz* and *xy*, to which LN and MN are reſpectively parallel.

For *x* and *z* being the Vaniſhing Points of CN and LC, *xz* is the Vaniſhing Line of the Plane LNC; and LN being the Interſection of that Plane with the Picture, or the Interſecting Line of that Plane, it is therefore parallel to its Vaniſhing Line *xz*: and for the like Reaſon MN is parallel to *xy*.

And thus the Vaniſhing and Interſecting Lines of each of the deficient Planes being found, any propoſed Lines or Figures may be deſcribed in either of them, as may be required to compleat the Repreſentation intended.

'Tis evident that after this Manner, any Room may be made to appear enlarged in either of its Dimenſions.

Thus, if ABCD *abcd* were the Room, and it were required on the upright End- Fig. 207. Wall CD*cd* to draw a Picture to repreſent a Continuation of the Room to any greater Length beyond CD, the Foot of the Spectator being placed at K, and the Height of his Eye above that Point being IK.

Here, 'tis manifeſt that K is the Point of Station, and that KP drawn perpendicular to CD, gives the Diſtance of the Picture, the Height of the Eye being P*o* equal to KI; and that a Picture being prepared according to theſe Meaſures, and the required additional Part of the Building being deſcribed on it by the uſual Rules, it will, when placed in its true Situation, juſtly repreſent what is propoſed, when ſeen from the proper Station.

If it be required to give an additional Height to the Room, by painting on the Plane of the Cieling *abcd*; the only Difference is, that inſtead of taking the Seat K of the Eye on the Floor, its Seat *k* on either of the upright Walls CD*cd* of the Room muſt be taken, where the perpendicular I*k* from the Eye to that Wall meets it; which Wall muſt be conſidered as the Original Plane for which the Picture on the Ceiling *abcd* is to be prepared, taking I*k* for the Height of the Eye and I*w* equal to *kp*

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[a] Cor. 2.

for its Diſtance; for then a Picture being prepared according to theſe Meaſures, and the propoſed Objects being deſcribed thereon, according to their Situations with reſpect to the Original Plane CD*cd*, it will, when placed on the Ceiling *abcd*, give the deſired Appearance from I the Point of Sight propoſed.

For in all Pictures, ſome Original Plane, either real or ſubſtituted, muſt be uſed, to connect the Picture with the Directing Plane, by the help of which the proper Meaſures for the Preparation of the Picture may be obtained; and with regard to which Original Plane, the Situations of the Objects intended to be repreſented ought to be known, in order to their being deſcribed according to their Relations to that Plane.

The Original Plane choſen for this Purpoſe, is uſually the Ground conſidered as a Horizontal Plane, that being the moſt natural Seat of viſible Objects, as well as of the Spectator; and is generally the moſt convenient, except only when the Poſition of the Picture is parallel to the Ground, as in Paintings on flat Ceilings or on Pavements; in which Caſes, the Plane of the Ground becomes unfit to be uſed as an Original Plane, which renders it neceſſary to chuſe ſome other Plane which cuts the Picture, to ſupply its Place, to which Plane the Situation of the intended Objects may be more conveniently referred.

SECTION II.

Of Scenography.

SCENOGRAPHY is the Art of Painting on ſeveral Planes or Scenes at different Diſtances, and in various Poſitions with reſpect to the Eye, in ſuch Manner, that all thoſe different Scenes, when ſeen from one certain determinate Point, may correſpond with each other, and repreſent one intire View of the Deſign without Breaks or Confuſion, as if it were one continued Picture.

Fig. 208. Let QYSZ repreſent the Room intended for a Theatre, TYLλ the Plan of that Part of it which is allotted for the Spectators, and LλZR the Plan of the Remainder appropriated for the Stage, Scenes, and Performance.

Imagine a Plane ABCD to be raiſed over this laſt Part, parallel to the Horizon, at ſuch a Height from the Ground as to be at a convenient Diſtance below the Spectator's Eye when ſeated in his Place. This Plane ſhall be called the Horizontal Plane.

Let another Plane MNGH be erected perpendicular to the Horizon and to the Sides of the Room, meeting the Horizontal Plane ABCD in GH; in which let there be a Rectangular opening G*mn*H, the nearer to a Square the better, and of ſuch Dimenſions as ſhall be thought proper: This Opening may be called the *Aperture* of the Theatre, or the Curtain, it being uſually covered by a Curtain until the Performance begins, when it is drawn up and diſcovers to the Spectators the moveable Scenes, all which are placed beyond it within the Space HX, which, in a limited Senſe, may be called the Theatre, as being that wherein the intire Scene appears, and all Changes of the View are performed, and which therefore, ſo far as relates to the Scenography, is the principal Object of the Eye.

The Part ABGH of the Stage which lies before the Curtain, is called the *Proſcene*; this is generally built Horizontal, and is the chief Place where the Actors ſpeak their Parts, as being neareſt the Audience, and having the Advantage of being covered by a Ceiling, which prevents the Voice from being diſſipated; the other Part of the Houſe beyond the Curtain, being not ſo convenient for that Purpoſe, it being uſually open to the Roof of the Building, for the better management of the moveable Scenes and Machinery which are all placed there; and the Floor of this Part is not Horizontal, but made to incline upwards with a gentle Aſcent.

It is, however, in the Conſtruction of this Part of the Houſe, and the Diſpoſition of the Scenes within it, that the principal Art of *Scenography* conſiſts, and which is therefore what is here intended to be conſidered; and in order thereto the following Propoſition may be uſeful.

PROP.

Fig. 209. If a hollow Priſm or Parallelepiped HX be expoſed to an Eye I placed any where in a Line IO parallel to the Axe of the Priſm;

the

the Image of that Priſm will coincide with the Image of a Pyramid, having the ſame Baſe MNGH with the Priſm, and having its Vertex V any where in the Line IO.

In the firſt Place, if the Baſe MNGH be taken as the Plane of the Picture, 'tis evident that Io which is parallel to the Sides MS, HD, NX, and GC, of the Priſm, repreſents the Radial of thoſe Lines, and o their Vaniſhing Point; wherefore Mo, Ho, No, and Go, are their indefinite Images; which being terminated in s, d, x, and c, by the Lines IS, ID, IX, and IC, give $sxdc$ the Image of the oppoſite End SXDC of the Priſm, and thereby compleat the intire Image of the Priſm on the Plane MNGH, from the Point I.

Now, becauſe MS and IO are Parallel, the Side MV of the Pyramid MNGHV which joins thoſe Parallels, is in the ſame Plane with them[a], in which Plane Mo alſo lies, wherefore Mo is alſo the Image of MV, and o the Image of V; and in like manner No, Ho, and Go, repreſent the Sides NV, HV, and GV, of the Pyramid, and conſequently the Images of the Pyramid and Priſm coincide. *Q. E. D.*

COR. 1.

If in the Pyramid MNGHV, the Points μ, ν, χ, γ, be found, where its Sides are cut by IS, IX, ID, and IC, and the Figure $\mu\nu\gamma\chi$ be drawn, it will repreſent the farther End SXCD of the Priſm; and the Truncated Pyramid MG$\chi\nu$ will repreſent the intire Priſm HX, in ſuch manner that their Images on the Plane of the Baſe MNGH from the Point I will every way coincide.

COR. 2.

If the End SXCD of the Priſm were removed to any aſſignable Diſtance beyond O, it is evident a Figure correſponding to $\mu\nu\gamma\chi$ may be found in the Pyramid, between $\gamma\chi$ and its Vertex V, which ſhall repreſent that End, at how great Diſtance ſoever it be placed; ſeeing if its Diſtance were ſuppoſed infinite, its repreſentation in the Pyramid can never reach beyond V; the intire Pyramid repreſenting the ſame Image to the Eye at I, as the Priſm would do were its End SXCD at an infinite Diſtance.

COR. 3.

All Objects which lye between GH and CD in the Face HDCG of the Priſm, muſt appear ſomewhere between GH and $\gamma\chi$ in the Face GHV of the Pyramid; and on the contrary, all Objects placed between GH and $\gamma\chi$, may appear as lying in correſponding Parts of HDCG.

In the Antient *Greek* and *Roman* Theatres, there ſeems to have been very little of what is now termed Scenery; for that which was then called the Scene, was a Real Building fronting the Spectators, and ſeparating the Proſcene, or that Part of the Theatre where the Actors performed, from the Poſtſcene to which they retired when their Part was done.

This Building or Scene generally repreſented the Front of a Palace, or of ſome ſumptuous Edifice, enriched with Marble Columns, Pilaſters, Statues, and other Ornaments of Architecture; and had commonly three Doors or Paſſages in it, for the Performers to enter and go off the Stage: behind, or within which Paſſages, there were three Triangular Priſms fixed perpendicularly on their Axes, ſo as to be made to turn round and ſhew either of their Faces to the Spectators; upon each of which Faces were painted ſuch Deſigns as ſuited the Subject of the Action, whether Tragick, Comick, Satyrical, or Paſtoral.

In this Way, it is evident there could be no great Variety made in the View, the principal Scene always continuing the ſame ſo long as the Theatre ſubſiſted, it being Part of the Building itſelf; and all the Diverſity of the Proſpect being only what could be ſeen through the three Paſſages of the Scene, according as the different Faces of the painted Priſms were turned towards the Spectators; and conſidering the ſmall Skill in Perſpective which the Ancients had, it may be preſumed that the greateſt Beauty of their Theatres lay rather in the real Architecture than in the painted moveable Scenes.

In after-times, when Perſpective began to be better underſtood, *Scenery* (as it is now called) was improved, and became the principal Ornament of the Theatre; by which, not only a greater Variety of Proſpects could be repreſented, but alſo the great Expence of erecting a real Building to repreſent the Scene, was ſaved.

However,

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[a] 7 El. 11.

However, at firſt, the Diſpoſition of theſe painted Scenes on the Theatre, was very different from that which a longer Experience has introduced; the Stage indeed, beyond the Curtain, was made to riſe gently, as is ſtill done, for the reaſon we ſhall mention by and by, but the ſide Scenes were ſet nearly Parallel to the ſide Walls of the Houſe, only a little inclining inwards to lengthen the Proſpect; theſe were either one intire Scene on each Side, reaching from the Front of the Stage to the back Scene, repreſenting in ſome Sort the Sides MH$\chi\mu$ and NG$\gamma\nu$ of the Pyramid MHGNV[1]; or they were divided into two or more Scenes on each Side, leaving Paſſages between, for the Actors to enter at; but all of them on the ſame Side were in one and the ſame Plane; except that each of them had a return of ſome Breadth joined to the Edge next the Audience, which return was Parallel to the Front of the Stage, ſo as to make each Scene have a conſiderable apparent Thickneſs, and repreſent the ſolid Corners of Houſes or Buildings.

But even in this manner there could be no Variety of the ſide Proſpects made, with any conveniency, during the Time of Action; in regard that the ſide Scenes thus diſpoſed, could not eaſily be removed out of Sight to make place for others, ſo that the chief Alteration of the View muſt ariſe from the Change of the back Scene.

But as the Art of Perſpective hath within this laſt Century been brought to much greater Perfection than formerly, many new Improvements have alſo been made in the Diſpoſition of the Scenes for Theatres, by which all deſireable Variety of Views may with eaſe be exhibited, ſo as to make a total Alteration of the Proſpect as often as is required; to the facilitating of which, many Machines have been invented, as well for the eaſy moving and ſhifting of the Scenes, as for repreſenting flyings, ſinkings, and riſings of Objects, and in ſome Sort to make the Theatre appear as a moving Picture: but as the Machinery of the Theatre is intirely diſtinct from the Scenery, and founded on Principles foreign to the Subject of theſe Papers, we ſhall here only conſider what particularly regards the moſt convenient Diſpoſition of the Scenes, and the manner of Painting on them, according to the Rules of *Stereography*, which is what is properly called *Scenography*.

In all Dramatick Entertainments, the Stage is conſtantly taken to repreſent the Floor, Pavement, or Ground, on which the Objects deſcribed in the Scene, are ſuppoſed to ſtand, or to which at leaſt they have Relation; and this Ground or Floor is always ſuppoſed Parallel to the Horizon: That Part of the Stage which lyes before the Curtain is generally Horizontal, but that Part of it which lyes beyond the Curtain is made to incline upwards as already obſerved.

The Reaſon of this is, that if the Plane of that Part of the Stage which lies within the Theatre, were parallel to the Horizon, it could then only appear as any other Floor, or Pavement; and every Object placed upon it being made of its true Size and Shape, the whole would only be a Geometrical Model of what is intended to be repreſented, without reference to the Rules of Perſpective; ſeeing there could be no apparent foreſhortning in this Caſe, but what was the natural Effect of direct Viſion; and thus nothing upon the Stage would appear of any larger Extent, than what that Floor, or Piece of Ground might contain, and the whole Appearance of the Theatre could be no other than that of a Room, wherein the Real Objects were placed in their true Dimenſions and Situations: But the Art of the Conſtruction of a Theatre, conſiſting in making it appear of greater Extent than it is, that the Stage or Ground may ſeem enlarged, and the Diſtances between one Object and another increaſed; and that by this means, the Artiſt may be able on a ſmall Space of Ground, to repreſent a more ample and extended Proſpect, not barely as in a Picture painted on a flat Wall, but as ſomething more Real, having truly ſome Part of the Depth or *Enfoncement* which it repreſents; it becomes neceſſary to have recourſe to the Expedient already mentioned, of making the Stage riſe gradually upwards, ſo as to repreſent the Face GHV of the Pyramid[2]; by which means the contracted Space GH$\gamma\chi$ of the Stage, becomes capable of repreſenting the whole Space GHCD of the Horizontal Plane; and the ſame Conſtruction being obſerved with reſpect to the Sides and Top, the whole Room HX may be repreſented within the compaſs Hν.

Fig. 208. Let therefore GH*gb* repreſent the Plane of the Stage; the Angle *v*P*p* which that Plane makes with the Horizontal Plane GHDC, is termed the Angle of *Elevation* of the Stage; and on this, and the Height IK of the Eye above the Horizontal Plane, depends the Place of the Point V, which is called the *Center* of *Contraction*; it being the Vertex of the Pyramid formed by the Contraction of the Theatre, when intended to

[1] Fig. 209.

[2] Fig. 209.

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Fig. 208.

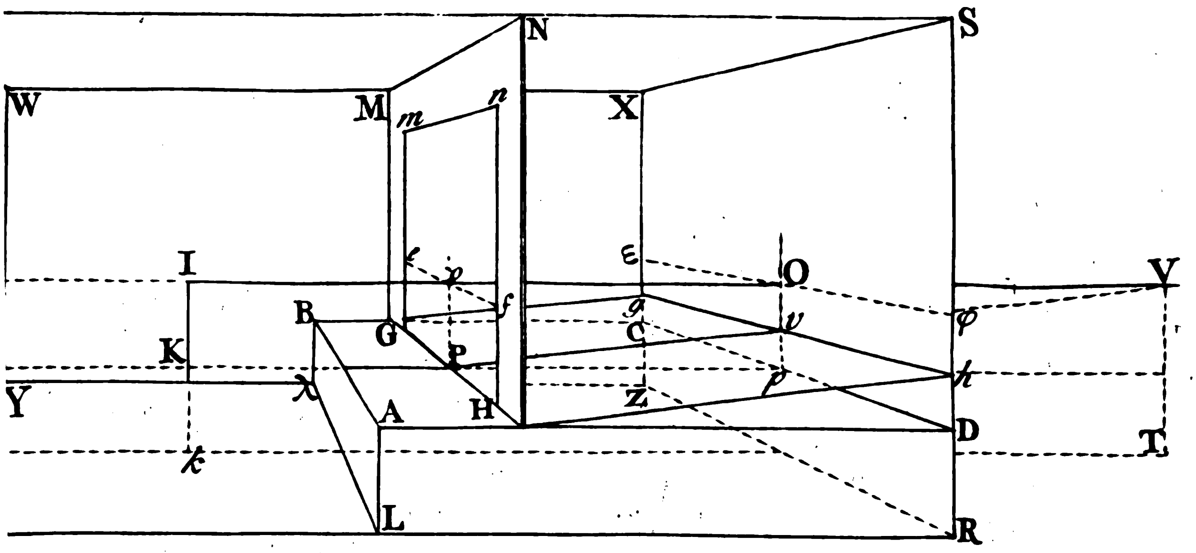


Fig. 209.

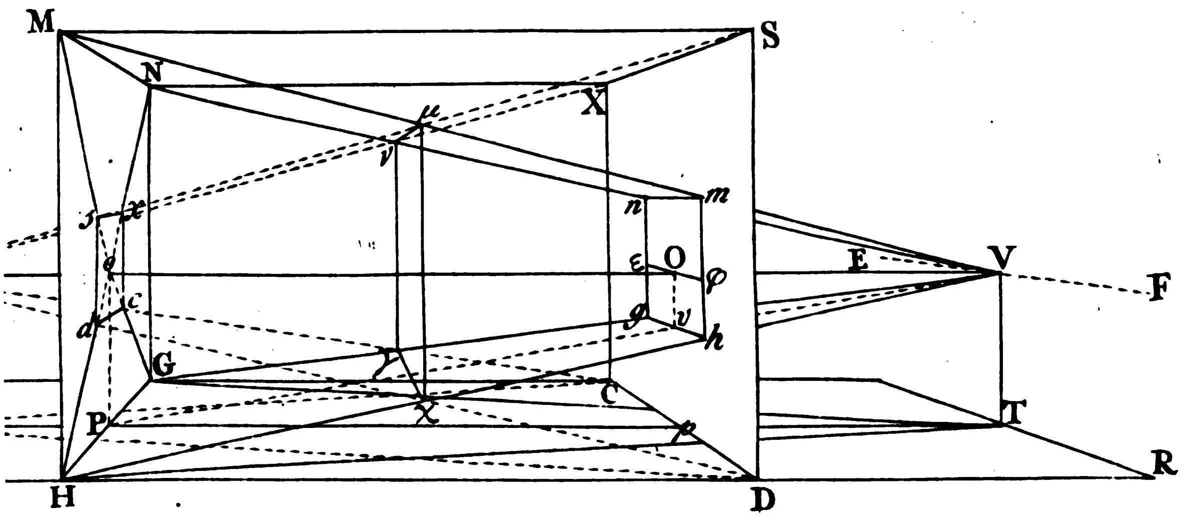
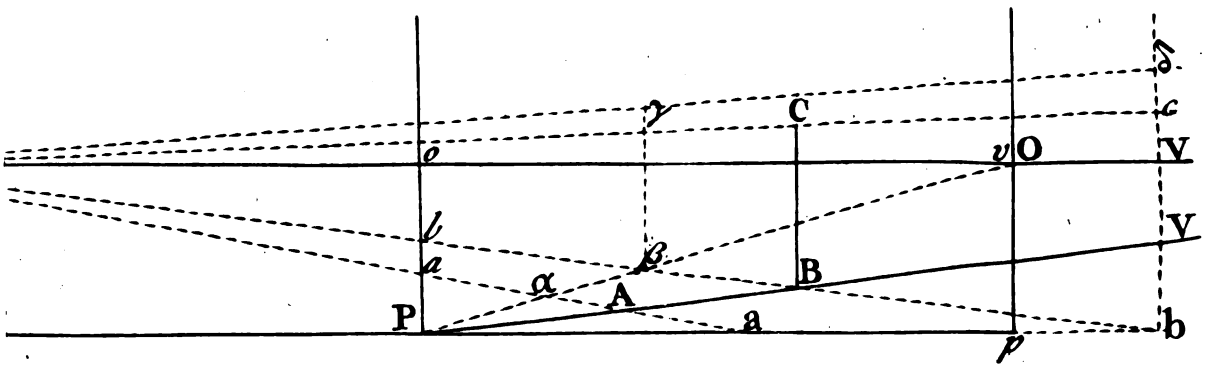


Fig. 210.



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to repreſent a Parallelepiped; the Point V being determined by the Interſection of the Line IO with the Plane of the Stage.

Now, as the Stage is intended not only to repreſent the Ground to the Eye, but alſo to ſerve as the Ground for the Actors to walk upon, and to perform their Parts; it muſt not be made ſo inclining as to be inconvenient for that Purpoſe, nor ſo as to take off from its appearing to be Horizontal; there muſt therefore be a due Medium kept in the Choice of the Angle of Elevation of the Stage, ſo that on the one Hand, the Proſpect may not be too much confined, and on the other, that the Declivity of the Stage may not be too apparent.

Fig. 210. Let $IOKp$ repreſent the Vertical Plane, Pp its Interſection with the Horizontal Plane, and PV its Interſection with the Stage, IO the Eye's Axe, IK its Height above the Horizontal Plane, and VPp the Angle of Elevation of the Stage.

It is evident that ſo long as PV can be conceived to be Horizontal, or to coincide with Pp, an Object placed at A or B will appear as if it ſtood at a or b, and conſequently that the Part AB of the Stage, will ſeem equal to the Part ab of the Horizontal Plane; and if the Elevation of the Stage were ſtill greater, ſo as to cut the Vertical Plane in Pv, the larger alſo would be the apparent Increaſe of the Parts of the Stage, the Part $\alpha\beta$ then repreſenting the ſame Space ab; and ſo on, till if the Stage were raiſed perpendicular to the Horizontal Plane, ſo as to cut the Vertical Plane in Po, it would coincide with the Curtain, and become the ſame as a Picture perpendicular to the Original Plane, and could no longer ſerve any the Uſes of a Stage: On the other Hand, if PV did coincide with Pb, there could then be no apparent Increaſe of the Parts of the Stage, but every Point of it would appear in its own natural Place, according to the Rules of plain Opticks. It is neceſſary therefore between theſe Extreams to chuſe a proper Mean, whereby both the propoſed Ends may be beſt ſerved.

For from the too great Elevation of the Stage, an Actor ſtanding at β, and by the Conſtruction of the Stage, appearing as if he ſtood at b, his Height $\beta\gamma$ will appear to be $b\delta$ and ſeem Gigantic, and as he moves towards the Front of the Stage, his Height will appear to decreaſe in too quick a Proportion; ſeeing that when he comes to P, he will there appear only of his own natural Size; whereas with the Elevation PV, an Actor of the ſame Height ſtanding at B, will appear at no greater Diſtance than the former, and his apparent Height will be only bc; which nevertheleſs is ſtill greater than the Life, an Inconveniency which cannot be totally avoided, while the Stage is anywiſe elevated above the Horizontal Plane: wherefore it may be obſerved, by the way, that it is beſt for the Actors to enter pretty near to the Front of the Stage, and not to go too remote from it, eſpecially when the Proſpect is long; the ſame is to be underſtood of all other living Objects, whoſe Size cannot be at Pleaſure accommodated to that due Degradation, which the Place they occupy on the Stage requires.

By what has been ſaid of the Elevation of the Stage, it appears that its Quantity cannot be confined to any certain determinate Rule, but muſt be guided by Judgment and Experience, according to the Size of the Theatre, the propoſed Diſtance of the Eye, and the Nature of the Deſign intended to be repreſented.

Andrea Pozzo, who in his Work, Intitled, *The Perſpective of Painters and Architects*, has treated this Subject, and did himſelf ſucceed very well in the actual Practice of it, gives theſe Rules for the general Conſtruction of a Theatre: That the Room in which it is to be erected, ſhould be divided, as to its Length, into two equal Parts, the one for the Spectators, and the other for the Theatre; That the Elevation of the Stage ſhould be after the Rate of one Foot to every nine or ten Feet of its Depth, that is, of about an Angle of ſix Degrees little more or leſs; and as to the Place of the Eye or Point of Sight, he makes its Diſtance from the Curtain to be equal to the Depth of the Theatre, that is, he places it at one End of the Houſe, and the Center of Contraction at the other; whence the Height of the Eye above the Horizontal Plane will be about an eighteenth or twentieth Part of the whole Length of the Houſe.

Fig. 210. Thus if $IOKp$ repreſent a Vertical Section of the Room lengthwiſe; Kp being biſected in P, gives the Part KP for the Spectators, and Pp for the Theatre; and pO being taken equal to one ninth or one tenth of Pp, PO is the Elevation of the Stage; and IK equal to Op, will be the Height of the Eye, equal to one eighteenth or twentieth Part of the whole Length Kp.

But as it is not neceſſary that the Place of the Eye ſhould be taken at the Extremity of the Houſe, but rather near to the Center of that Part which is allotted for the Spectators; that the Inconveniency neceſſarily ariſing from their different Situations out of the true Point of Sight, may be the more equally diſtributed, and that the better Sort of the Company may ſee the Proſpect to the moſt Advantage: So neither is it neceſſary that the Center of Contraction ſhould fall exactly on the oppoſite Wall O*p*, but rather at ſome Diſtance beyond it; to prevent the too quick decreaſe of the Back Scenes, whereby a conſiderable Part of the Depth of the Theatre might be rendered uſeleſs: it being evident[1], that the nearer the Center of Contraction V falls to the Curtain, the quicker is the Decreaſe of the *Scenographick* Pyramid, and conſequently of the Back Scenes, ſuch as $\mu\nu\gamma\chi$, and that when theſe become too ſmall, the Remainder of the Theatre behind them is of no farther Uſe for the Scenery.

The Elevation of the Stage and the Place of the Eye being then choſen at diſcretion, the Size of the Aperture of the Theatre muſt be next determined; and this muſt, in a good Meaſure, be governed by the Diſtance between the Eye and the Curtain: for as that Aperture is to be conſidered as the Frame of a Picture, within which the intire Scene appears, the Size of that Picture ought to bear ſuch a Proportion to the Diſtance of the Spectator, that his Eye may not be too much diſtracted, or forced to turn too much aſide to ſee its ſeveral Parts, but rather that he may be able to comprehend the whole Proſpect at one eaſy View. This poſſibly may be effected ſufficiently, if the Aperture be ſo made, that its Diagonal may not exceed double the Diſtance of the Eye; but in Matters of this Sort, altho' there be certain Extremes on both Sides to be avoided, yet no ſuch determinate Proportion is aſſignable, but that ſome Latitude may be allowable in the more or leſs; Experience therefore in ſuch Caſes is the beſt Guide.

As to the Height of the Eye above the Horizontal Plane, this alſo admits of ſome Variety, but in ſome ſort depends on the Height of the Aperture of the Theatre; it may be therefore ſufficient to remark on this Head, that the Height of the Eye ought not to be ſo great, as that its Axe may meet the Curtain higher than the Center of the Aperture, where its Diagonals croſs; nor ſhould it be ſo ſmall, as to be much below the Face of an Actor ſtanding on the Front of the Stage. But in regard that when the Elevation of the Stage is once fixed, the Diſtance of the Center of Contraction, and conſequently the Form of the *Scenographick* Pyramid, depend on the Height to be choſen for the Eye, (ſeeing it is by the Interſection of the Eye's Axe with the Plane of the Stage, that the Center of Contraction is determined) this Conſideration alſo may be of ſome Weight in the fixing a proper Height for the Eye.

It appearing from what has been advanced, that when the Elevation of the Stage, the Aperture of the Theatre, and the Height and Place of the Eye are ſettled, the intire *Scenographick* Pyramid is thereby determined; but that this laſt is variable by increaſing or leſſening the Height of the Eye, although the Aperture and Elevation of the Stage remain the ſame; the enlarging the Height of the Eye making the Center of Contraction fall more diſtant, as the lowering the Eye brings the Center nearer; both which will have correſponding Effects with reſpect to the apparent Diſtances of the ſeveral Parts of the Stage: This points out an eaſy Way of introducing great Variety of Scenery, without the Trouble of making any Alteration in the fixed Part of the Theatre. But how far it may be proper, in the different Scenes to be repreſented at the ſame Entertainment, to vary the Height of the Eye in one, from what it was in another, muſt be left to Experience to decide; it ſeems not to be ſtrictly allowable, it being too great a Strain on a Spectator's Imagination, to fancy himſelf raiſed higher, or ſet lower at every ſhifting of the Scene, while he is ſenſible he ſtill keeps the ſame Place.

Thus far touching the general Conſtruction of a Theatre. It remains to conſider of the Diſpoſition of the Scenes, and the manner of Painting on them.

The Scenes commonly uſed in a Theatre, are the *Side Scenes*, the *Hanging* or *Top Scenes*, and the *Back Scene*; all which together, make what is called a *Sett of Scenes*, and exhibit one intire Proſpect of all that is propoſed to be repreſented at one View, each Scene having its peculiar Part deſcribed on it.

The Side Scenes are uſually five or ſix on each Side of the Theatre, and ſerve to repreſent the Sides of the Proſpect to the Right and Left, as the Stage left open between them, repreſents the Ground; the Hanging Scenes are generally the ſame in Number with the Pairs of Side Scenes, and ſerve to connect the Tops of each Pair croſs the Theatre, and to repreſent the Ceiling or Sky, according to the Nature of the intended Proſpect; and the Back Scene is placed beyond all the reſt, and fills up the Space left open by the Side Scenes and Hanging Scenes, and cloſes the View.

All the Scenes are conſtantly ſet perpendicular to the Horizon, and are moveable

in

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[1] Fig. 209.

Plate 123 Book 7 Sect. 2.

Fig. 211.

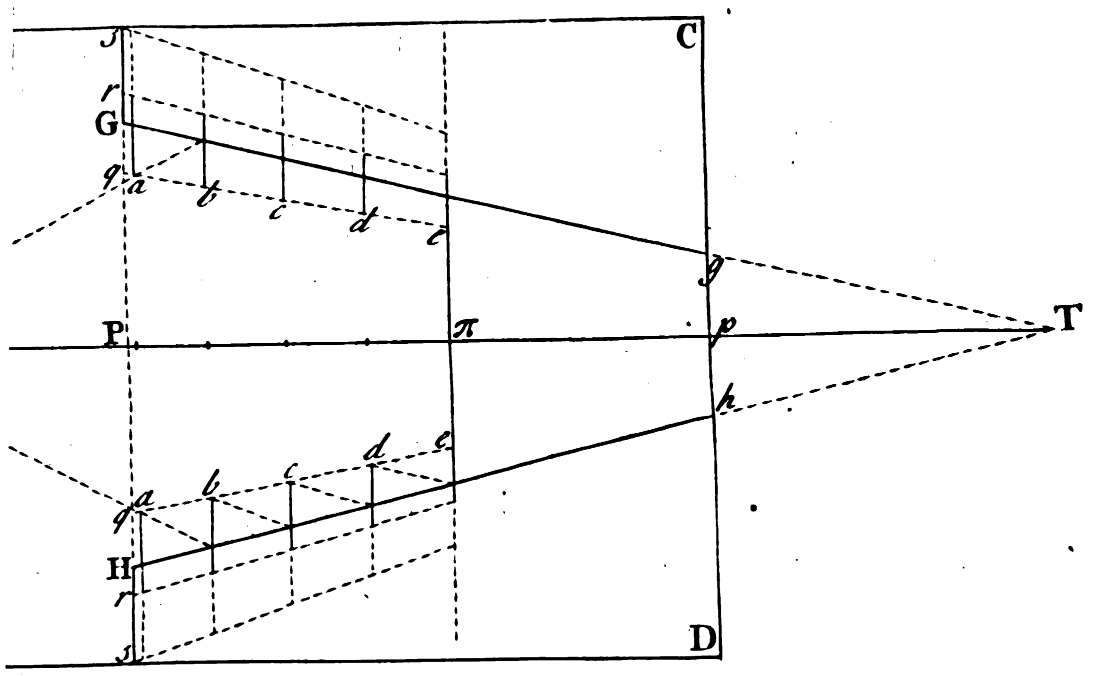


Fig. 212.

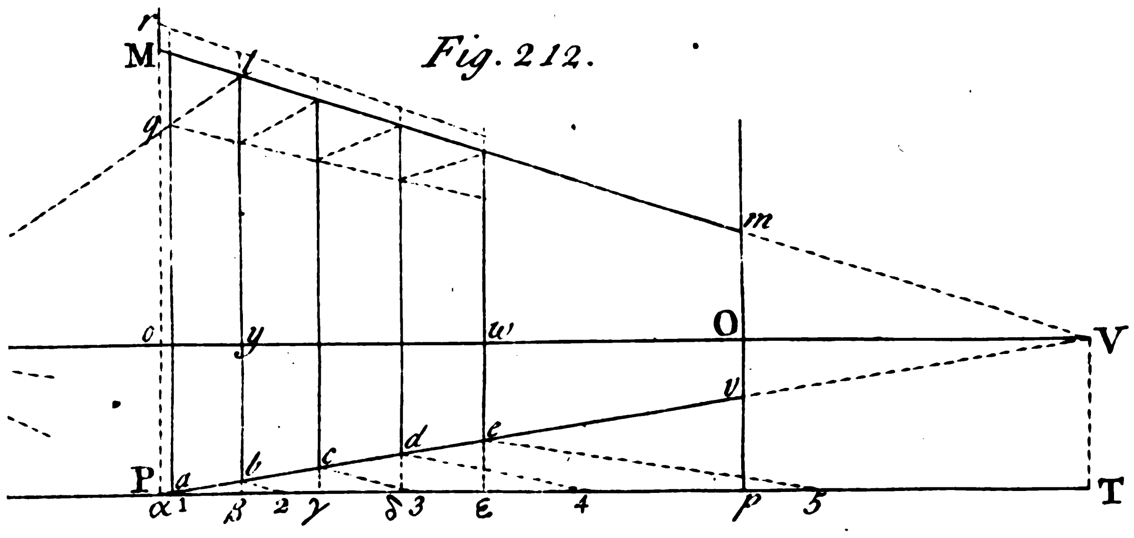


Fig. 213.

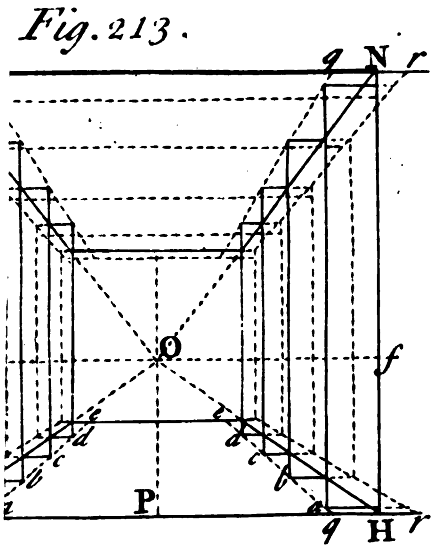
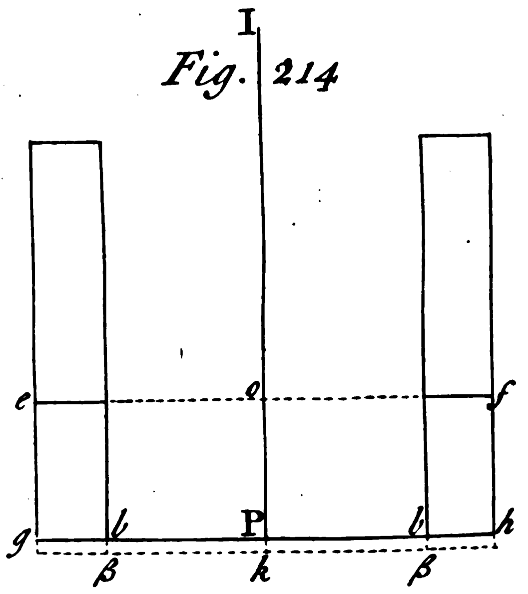


Fig. 214



J. Mynde sc.

in Channels, or Grooves made for them in the Floor, parallel to the Front of the Stage; that they may be drawn forward or backward, to be brought into, or removed out of Sight; except only the Hanging Scenes, which are made to be let down and drawn up from the Top of the Houſe.

The before named Author *Andrea Pozzo* mentions two different Methods of placing the Side Scenes; the one, of ſetting them parallel to the Front of the Stage; the other of making them incline inwards towards the Back Scene; the firſt is the *German* way, the other the *Italian*, which laſt this Author prefers, as he thinks it ſerves better to prevent the Appearance of the Prompter, and Servants imployed to move the Scenes: But as other Ways are eaſily found for avoiding that Inconveniency, and as painting on Scenes in an Oblique Situation, is more troubleſome than when they are Direct, this laſt Way of placing them is much the more eligible.

We ſhall therefore firſt treat of Scenes in a Situation parallel to the Curtain, and afterwards of ſuch as are ſet ſlanting; whereby the Advantage of the one Method over the other will appear.

The Situation of the ſeveral Scenes in the Theatre are principally governed by the Form of the *Scenographick* Pyramid; for as the Stage makes the lower Face of that Pyramid, ſo the Side Scenes ought to Range, that is, their Edges ought to be terminated by Planes Parallel to the ſide Faces of that Pyramid, as the Hanging Scenes ought to do with reſpect to the upper Face; and the Back Scene ought to form a Section of the Pyramid parallel to the Curtain, ſo as to cloſe the View at the farther End.

Let ABCD repreſent the Plan or Ichnography of the Houſe on the Horizontal Plane, GH the Breadth of the Aperture, or the front Line of the Stage, K the Seat of the Eye, and T the Seat of the Center of Contraction on the Horizontal Plane, and KT the Section of that Plane with the Vertical Plane, or the Line of Station. Fig. 211.

Draw GT and HT, and theſe will be the Seats of the Faces GNV, HMV, of the *Scenographick* Pyramid on the Horizontal Plane, as repreſented at Fig. 209: *a*, *b*, *c*, *d*, are the Channels or Grooves in which the Side Scenes are placed, which are generally terminated by Lines *q*T and *s*T, tending to T, to the End that the Edges of the Side Scenes which ſtand perpendicularly in thoſe Grooves, may lye in Planes parallel to the ſide Faces of the Pyramid.

Theſe Scenes are made to project beyond the Lines GT and HT more or leſs, according to the Nature of the Deſign, that each of them may receive the Deſcription, not only of ſuch Objects as ſtand directly fronting, but alſo of ſo much of the Depth of the Deſign as lyes between the apparent Place of each particular Scene, and that of the next Scene beyond it; and they are to have ſuch a Breadth *qr* allowed them, that a Spectator ſtanding any where in the uſual Places allotted for the Audience, may not be able to diſcover any opening or naked Part between them: and the Grooves in which they move, ought to be extended ſtill farther backward to the Line *s*T, to give leave for the Scenes to ſlip back out of Sight, when another Sett are to be brought forward to ſupply their Place and change the Proſpect.

Beyond the ſide Scenes is placed the Back Scene in the Groove *ee*; it is divided into two equal Leaves, which are brought together, and meet in *π*, forming one continued Plane; and the Groove is made of ſufficient Length to let each Leaf be drawn back out of Sight, to make way for a Change; and, laſtly, the Hanging Scenes are let down over each Pair of Side Scenes *aa*, *bb*, &c. to hide or join their Tops, and cloſe the Proſpect upwards.

The Seats of the ſeveral Scenes on the Horizontal Plane, with their Breadths, and the Lengths and Ranges of their Grooves being ſettled; the next thing is to determine their Height.

To this purpoſe let IVKT repreſent the Vertical Plane, KT and PV its Interſections with the Horizontal Plane and the Stage, V the Center of Contraction, and PM the Height of the Aperture of the Theatre; the Perpendiculars *a*, *b*, *c*, *d*, and *e*, are the Interſections of the Planes of the ſeveral Pairs of Side Scenes, and of the Back Scene with the Vertical Plane, meeting the Horizontal Plane in *α*, *β*, *γ*, *δ*, and *ε*, at the ſame Diſtances from each other and the Front of the Stage, as their reſpective Grooves ſtand in the Ichnography, and cutting the Stage in *a*, *b*, *c*, *d*, and *e*. Fig. 212,

Having then drawn MV, it will mark the Section of the Vertical Plane with the upper Face of the *Scenographick* Pyramid, and conſequently cut the *Profil* of all the Scenes at their proper Heights; and two other Lines *q*V, *r*V, being drawn at proper

Diſtances above and below MV, theſe will terminate the Depth of the Hanging Scenes.

By the help of theſe two Preparations, the intire Sett of Scenes may be put in their proper Places, and have their juſt Sizes given them; taking their Breadths from the Ichnography, and their Heights from the Elevation, their Diſtances from each other upon the Stage, being alſo taken from their Interſections with PV: all the Difficulty that can then remain in the Practice, will be to find the Lines of Ranges which terminate at V.

When that Point falls in the end Wall of the Houſe, as at O, there can be no Difficulty, that Point being then within reach; but when it falls beyond the end Wall, (as for the Reaſons already mentioned, it ought for the moſt Part to do) then the *Scenographick* Pyramid being cut by that Wall, in a Section ſimilar to its Baſe, that Section may be deſcribed on it accordingly, and which is had by its Height *vm* in the Elevation [1], and its Breadth *gb* in the Ichnography [2], the Interſection of the Stage with the end Wall which paſſes through *v*, giving the lower Side of that Section; and then the Angular Points of this Section being connected with thoſe of the Aperture of the Stage, will give the Lines of the Ranges required; and after the ſame manner any other Ranges, as *q*V, *r*V, may be obtained.

All this appears in Fig. 209, which alſo ſhews the conveniency of ſetting the Point V beyond the end Wall, that the Space μg of the Theatre may be the fitter to receive other Back Scenes beyond $\mu\gamma$, before they become too much contracted; whereas, if the end Wall did paſs through V, good part of that Space would be uſeleſs for Scenes, by reaſon of their Diminution.

When the Scenes are thus diſpoſed in the Theatre, they will, even when naked or unpainted, appear to the Eye in ſome meaſure to repreſent a Rectangular Parallelepiped, as in this Figure [3], which gives the *Stereographick* Appearance of the whole Sett of Scenes on the Plane of the Aperture MNGH of the Theatre.

And here it is to be obſerved, that as in the Ichnography [4], the Side Scene *a* projects ſo far, as that a Line K*a* meets the next ſucceeding Scene *b*, in the Point where it cuts the Range GT, and ſo of all the reſt, on to the Back Scene; ſo in this Figure [5], the upright Edge of the Scene *a*, cuts the Range GO in the ſame Point where the Scene *b* meets it: For the Lines K*a*, K*b*, &c. in the Ichnography, all proceeding from K the Foot of the Eye's Director with regard to the Curtain, the Images of all thoſe Lines on the Curtain, taken as a Picture, muſt be Perpendicular to its Interſecting Line GH, and conſequently coincide with the upright Edges of the Scenes; and it is evident that O is the Image of V [6], and therefore that GO and HO are the Images of the Ranges on the Stage, whoſe Seats in the Ichnography are GT and HT.

But as it is Painting that muſt enliven the Appearance, by repreſenting every Part of the Deſign with that due Strength of Colour, and Proportion of Size which it ought to have, according to the Diſtance it is intended to appear at; we ſhall now ſhew in what manner, and by what Rules that is to be performed.

Fig. 209. In the firſt Place, the Stage itſelf muſt be conſidered as the Picture of the Horizontal Plane, of which GH is the Interſecting Line, and a Line EF drawn through the Center of Contraction V parallel to GH, is the Vaniſhing Line, of which V is the Center, and IV the Diſtance; and although it is not uſual to paint any thing upon the Plane of the Stage, but it is generally left naked, yet nothing hinders but that Painting may be imployed upon it, ſo as to make it repreſent a rich marble Pavement, or a Parterre, diverſified in any manner the Fancy of the Artiſt may ſuggeſt, and which may be done by the common Rules of *Stereography*, as on any other Picture, whoſe Vaniſhing and Interſecting Lines EF and GH, together with the Diſtance of the Eye IV are given; and would doubtleſs greatly aſſiſt the Horizontal Appearance of the Stage, although, even without Painting, that Appearance is in ſome Sort preſerved by the Painting on the ſeveral Scenes.

As to theſe, every ſingle Scene is to be conſidered as the Plane of a Picture, which is to receive upon it ſuch Part of the Deſcription as falls within its Bounds; and as we have ſuppoſed them all to be parallel to the Front of the Stage, and perpendicular to the Horizontal Plane, the Interſection of the Plane of every Scene with the Axe of the Eye IO, will mark the Center of that Picture, through which the Horizontal Line muſt paſs parallel to the Ground; and the Length of the Axe between the Eye and that Center, will be its true Diſtance by which it muſt be painted: The only thing remaining, is to determine what Meaſures are to be given to the Objects on each par-

ticular

[1] Fig. 212.

[2] Fig. 211.

[3] Fig. 213.

[4] Fig. 211.

[5] Fig. 213.

[6] Fig. 212.

ticular Scene, ſo that the Sizes of the Objects on all the Scenes, when ſeen together, may correſpond and keep their due Proportion.

Now it is evident, that if every Scene ſtood directly upon the Horizontal Plane, the Section of each Scene with that Plane would be its true Interſecting Line; and that therefore, if the true Meaſures of the Objects were imployed on thoſe Lines in working, at whatever different Diſtances the ſeveral Pictures ſtood, the Objects repreſented on each of them would keep their due Proportions with reſpect to the Eye; the different Diſtances from the Eye to the ſeveral Scenes, giving the Objects deſcribed on them, their proper Diminution, although drawn according to the ſame Scale. But by reaſon of the Elevation of the Stage, each Scene meets the Stage before it reaches to the Horizontal Plane, whereby Part of the Bottom of that Scene is cut off, or hid by the Stage, in a Line parallel to its true Interſecting Line: If then inſtead of the true Interſecting Line of the Scene, the Parallel in which it cuts the Stage be uſed, the Meaſures to be taken on that Parallel muſt not be the true, but only the Proportional Meaſures of the Objects [a]; and the Objects thus deſcribed on every particular Scene, will all correſpond together, and appear with their proper Diminutions.

Thus to prepare the Side Scenes *b*, *b*, for painting: Having in any convenient Place drawn *gb*, repreſenting the Interſection of the Plane of the propoſed Scenes with the Stage, from any Point P in that Line, erect the indefinite Perpendicular P*o*; then having taken from the Ichnography [1], the Diſtance and Breadth of the Scenes on each Side of the Line of Station KT, ſet them off at *b*,*g*, and *b*, *b*, on each Side of P in the Line *gb*; from whence raiſe Perpendiculars equal to the Height *bl* of the Scene [2] from its Interſection *b* with the Stage, to its Interſection *l* with the Range MV. And thus the Pair of Side Scenes will be deſcribed in their true Dimenſions, and Diſtances from each other, as they are to ſtand in the Theatre when uſed.

Then having taken the Diſtance *by* [3] between the Foot of the Scene *b*, and its Interſection *y* with the Eye's Axe IV, ſet it off from P to *o*, and through *o* draw *ef* parallel to *gb*; from *o* ſet off *o*I on the Line P*o*, equal to I*y* [4] the Diſtance of the Eye from the Plane of the Scene, and take *ok* equal to the Height of the Eye IK or *βy*, and through *k* draw *ββ* Parallel to *gb*: Then *o* will be the Center, and *o*I the Diſtance of the propoſed Scenes, conſidered as a Picture, *ef* will be the Vaniſhing Line of the Horizontal Plane, and *ββ* the true Interſecting Line, and *gb* will be the Parallel on which the Proportional Meaſures are to be taken, by the help of which this Pair of Scenes is to be painted: which Meaſures are to the true Meaſures as P*o* to *ko* [b]. And thus the Scenes *b*, *b*, are fully prepared for the Deſcription of whatever Objects can fall within their Bounds, by the common Rules of Stereography: And the ſame Method ſerves in every reſpect for the Preparation of all the other Scenes.

But here it muſt be obſerved, that the Point *b* [5] where the Scene *bl* cuts the Stage, being the Image of the Point 2 of the Horizontal Plane or Ground, no Part of that Plane, nearer than the Point 2, can be repreſented on that Scene, but rather the Scene itſelf is judged to ſtand at 2; for which Reaſon, nothing ſhould be deſcribed on the Scene *b*, but what has its Seat on the Horizontal Plane at or beyond a Line parallel to the Front of the Stage, drawn through the Point 2: And whatever Objects lye between 2 and the Point 1, (the apparent Place of the Scene *a*) ought to be deſcribed on this laſt Scene, for which Purpoſe it ſhould have a due Breadth allowed it, as already mentioned. The ſame is to be underſtood of the ſucceeding Scenes *c*, *d*, *e*, whoſe apparent Places are 3, 4, 5. So that the Scene *a* ought to take in all the Depth of the Proſpect from 1 to 2, the Scene *b* that from 2 to 3, the Scene *c* that between 3 and 4, the Scene *d* that between 4 and 5; and laſtly, the Back Scene *e* ſhould take in the whole Remainder of the Proſpect beyond 5, be it ever ſo diſtant.

Hence, when any Object is to be repreſented nearer than the apparent Place of the Back Scene, and occupying the middle Parts of the View, to which the Side Scenes do not extend, ſuch as a Temple, a Triumphal Arch, a Tree, or any other Object detached from the reſt of the Deſign; it becomes neceſſary to raiſe a ſeparate, or ſtanding Scene on purpoſe for it, and to ſhape the Scene to the Outline or Extremities of the Figure, with proper Openings in all its vacant intermediate Spaces, that it may obſtruct no more of the View of the Scenes behind it, than what is abſolutely neceſſary to the Deſcription of the Object intended. Theſe kind of open or pierced Scenes have a ſurprizing good Effect, and add greatly to the Life of the Proſpect, making the whole appear more natural and real; as on every the leaſt Motion of the Spectator, he diſcovers through thoſe Openings new Parts of the more diſtant Objects, an Appearance nearly approaching to real Nature. And on this Principle it is, that even the

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[a] Def. 3. Prob. 6. B. II.

Fig. 214.

[1] Fig. 211.

[2] Fig. 212.

[3] Fig. 212.

[4] Fig. 212.

[b] Prob. 6. B. II.

[5] Fig. 212.

the Side Scenes are often made with uneven Edges, anſwering to the Outline of the Objects deſcribed on them; which however ought not to be carried too far, ſo as to prevent their being intirely hid when drawn back.

Fig. 209. The Side Scenes have hitherto been conſidered as all Ranging to V the Center of Contraction, whereby they appear to be bounded by Planes perpendicular to the Curtain and the Horizon, and in that Poſition are proper for the Deſcription of an *Arcade*, a Gallery, or the inſide of any Piece of Architecture, or a Street of Houſes, a Walk of Trees, or any other Subject, whoſe Sides are ſuppoſed to have that Situation; but if it were required to repreſent ſuch Objects in an Oblique Poſition with reſpect to the Curtain, the Scenes may be made to Range to any other Point in the Vaniſhing Line of the Stage, to be choſen according to the Obliquity propoſed: For as the Center of Contraction V is the Vaniſhing Point of all Lines in the Horizontal Plane, which are perpendicular to the Front Line of the Stage, ſo in the Vaniſhing Line EF of the Stage which paſſes through V, are found the Vaniſhing Points of all other Lines in the Horizontal Plane which incline in any Angle to the Front Line; which Points are determined by Radials drawn from the Eye to that Vaniſhing Line, parallel to the Lines propoſed; and every ſuch Radial will, by its Interſection with the Plane of each Scene, mark the correſponding Vaniſhing Point on that Scene; and the other proper Points, Lines, and Meaſures for each Scene, may be thence eaſily adjuſted by the Rules already given.

We ſhall only farther obſerve touching the placing the Side Scenes, that it ought always to be governed by the Nature of the Deſign; ſo that each Scene may, as near as can be, contain the whole of the ſame intire Object, and not to let part of it fall on one Scene, and part on another, which would unavoidably occaſion diſagreeable Breaks in the View; and therefore when the Artiſt has laid down the Plan of his Deſign, he ought to conſider of the Diſtribution of it amongſt the Scenes, in a manner leaſt liable to that inconveniency; and when he has drawn Lines in his Plan, where each Scene is to take Place, their Diſtances from the Front Line being transferred to the Line of Station KT[1], as at 1, 2, 3, 4, &c. the true Places *a*, *b*, *c*, *d*, &c. of the Scenes on the Stage, will be found by the Interſections of PV with Lines from I to thoſe Points.

[1] Fig. 212.

Having thus given Rules for the Diſpoſition and Painting of the Scenes in a Poſition parallel to the Curtain; it now remains to conſider how that is to be performed, when the Side Scenes are made to incline towards the Back Scene.

Fig. 215. Let ABCD be the Plan of the Houſe on the Horizontal Plane, in which the ſame Meaſures and Letters, and likewiſe the ſame Ranges for the Side Scenes, are retained as before; but inſtead of placing the Side Scenes parallel to the Front of the Stage, let them be ſet ſlanting and parallel to each other on each Side, ſo as their Seats on the Horizontal Plane may be as in the Figure.

Fig. 216. Let IVKT be the Vertical Plane alſo in the ſame Meaſures, and with the ſame Ranges for the Heights of the Scenes as in the former Figure.

Here, it is evident that each Side Scene muſt have two Lines of Elevation, the one for the Height of its outward Edge a, b, c, or d, and the other for the inner *a*, *b*, *c*, or *d*, and that none of them can be Square either at Top or Bottom, by reaſon of their meeting the upper and lower Faces of the *Scenographick* Pyramid in an Oblique Manner; ſo that in order to ſhape every Scene, its Section with the Stage and alſo with the upper Face of that Pyramid muſt be found.

Fig. 217. Now, if the Obliquity of the Side Scene whoſe Seat is a*a*, be ſuch, that a*a* being produced to the End-Wall CD, can meet it at any convenient Diſtance as L; then the Breadth *p*L being ſet off from *v* in the Interſection *gb* of the Stage with the End-Wall[2], will give the Point to which the Interſection of the Scene a*a* with the Stage muſt tend, and a Perpendicular from the ſame Point to *mn*[3], will cut it in the Point to which the Top of the Scene muſt run; and if a*a*[4] could alſo meet TR within reach, as at R, the Diſtance TR being ſet off from V in the Vaniſhing Line EF of the Stage[5], will give the Vaniſhing Point of the Section of the Stage with the Scene, and conſequently determine the apparent Angle it makes with the Front Line, to which Vaniſhing Point the Line which bounds the Top of the Scene will alſo tend: And thus the Bottom and Top of every ſingle Side Scene on the ſame Side of the Stage, may be found, each of them tending to a different Vaniſhing Point in the Line EF, or to correſponding Points in *gb* and *mn*; and the Tops and Bottoms of the correſponding Side Scenes on the other Side of the Stage, will tend to correſponding Points in EF on the contrary Side of V, or to Points in *gb* and *mn* on the contrary Side of O.

[2] Fig. 209.
[3] Fig. 209.
[4] Fig. 217.
[5] Fig. 209.

But as the Planes of the Oblique Side Scenes are, by thoſe who uſe them in the modern Way, made to incline in a very ſmall Angle to the Curtain, their Interſections LL[6]

[6] Fig. 217.

with

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Sect. 2.

Fig. 215.

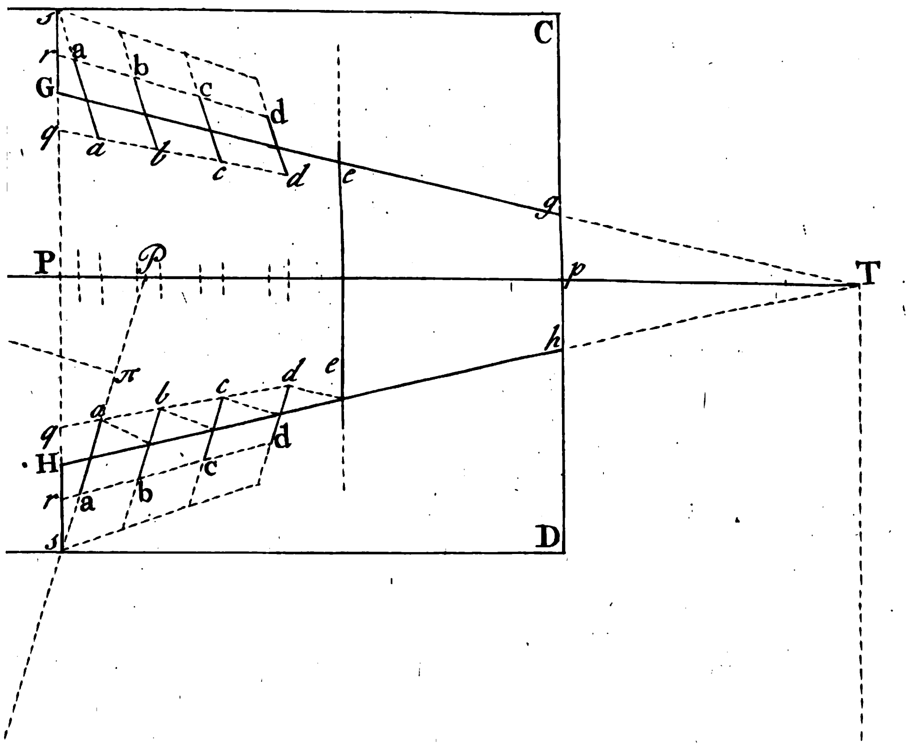
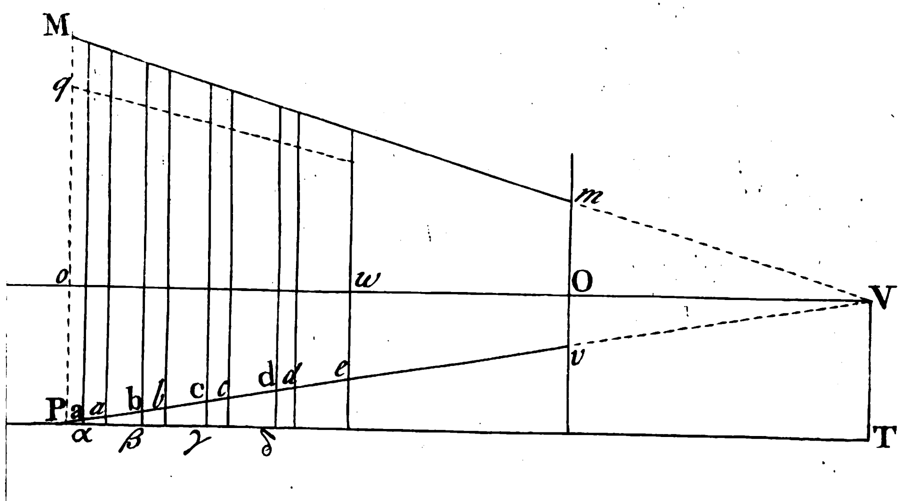


Fig. 216.



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with the End-Wall, fall at too great a Diſtance to be manageable; which makes it neceſſary to find another Way to give the Scenes their proper Shapes, and to determine their Interſections with the Stage, by the help of the Vertical Plane or Elevation.

Thus to deſcribe the Interſections of the Side Scenes with the Stage, their Ranges on the Stage muſt be firſt drawn, as already directed; then taking from P in the Line P*v* of the Elevation[1], the ſeveral Diſtances of the Edges a, *a*, b, *b*, *&c.* of the Scenes from P, ſet thoſe Diſtances off from P on the Vertical Line PV of the Stage[2], and through each of thoſe Points draw Parallels to the Front of the Stage, which will cut the reſpective Ranges in the Points through which the reſpective Edges of the ſeveral Scenes paſs, and thereby determine their Interſections with the Stage.

Then to ſhape each Scene, firſt draw it out ſquare, taking its Breadth from the Plan[3], and its Length from the Height of its longer Edge in the Elevation, from its Interſection with MV to its Interſection with PT[4]; and having marked at the Bottom of the Scene, the Heights at which its reſpective Edges cut P*v* above PT, and ſet off the Difference at the Top between the two Edges as cut by MV, the Obliquity of the Bottom and Top of each Scene will be found. And thus much for the forming the naked Scenes, and finding their Places on the Stage.

Now in order to prepare any ſuch Scene for painting; as for Example, the Scene whoſe Seat is a*a*[5]; that Seat muſt be conſidered as its Interſecting Line with the Horizontal Plane, K as the Seat of the Eye, and IK[6] as its Height. Then Kπ[7] drawn perpendicular to a*a*, gives π the Foot of the Vertical Line of that Scene, conſidered as a Picture; and a*a* being produced both Ways, till it meet KT in *P*, and a Line K*n* drawn perpendicular to KT, in *n*, *P* will be the Seat of the Vaniſhing Point of all Lines in the Original Plane which are perpendicular to the Front Line of the Stage, and *n* will be the Seat of the Vaniſhing Point of all ſuch as are parallel to that Line; and the Height of the Eye IK being ſet off perpendicularly from π on the Scene, will give its Center, through which the Vaniſhing Line of the Horizontal Plane muſt paſs parallel to a*a*; and the Vaniſhing Points whoſe Seats are *P* and *n*, being transferred by Perpendiculars to that Vaniſhing Line, that Scene will then be prepared for the Work; the Meaſures for which, if taken on a*a*, muſt be the true Meaſures, but if on any other Parallel to it, they muſt be the Proportional Meaſures[1].

But by reaſon of the Smallneſs of the Angle of Inclination of the Scene to the Curtain, the ſame Inconveniency recurs, with regard to the too great Diſtance of the Vaniſhing Point of Lines parallel to the Front of the Stage, whoſe Seat is *n*; which muſt in general be much wanted, and will therefore render the working troubleſome, unleſs the Deſign be ſo made as not to need it. It is, however, by Lines tending to that Vaniſhing Point, drawn at the Top and Bottom of each Scene from the Extremities of its ſhorter Edge, that it ought to be terminated, in caſe it be required to appear as Parallel to the Front of the Stage; which terminating Lines will neither agree with the Section of the Scene with the Stage, nor with the upper Face of the *Scenographick* Pyramid.

Beſides, in this Poſition of the Side Scenes, their Interſections with the Stage tending to different Vaniſhing Points in its Vaniſhing Line, they will neither appear parallel to each other, nor to the Front of the Stage; which muſt produce a diſagreeable Effect at the Bottom of each Scene: and the like will be the Effect at their Tops from the Hanging Scenes, which, as they muſt ſtill be parallel to the Curtain, will thereby appear to cut each Pair of Side Scenes in two different Oblique Lines, tending to make an Angle upwards; add to this, that each Pair of Side Scenes not being in the ſame Plane, but having contrary Inclinations, they cannot be worked upon together as one Picture (as the Parallel Scenes may be[8]) but muſt be work'd upon ſeparately.

Nevertheleſs, with regard to the firſt of theſe Inconveniencies, it might be avoided, if inſtead of making the Seats a*a*, b*b*, &c. of the Side Scenes parallel, as in the Figure[9], all thoſe on one Side of the Stage were made to tend to the ſame Point R in the Line TR on the contrary Side of T, and thoſe on the other Side of the Stage were drawn tending to another Point R at an equal Diſtance on the other Side of T; for then, the Interſection of the Stage with all the Side Scenes on one Hand, would have one and the ſame Vaniſhing Point, and thoſe on the other Hand would have another, and conſequently thoſe of each Side would appear parallel to each other, though not to the Front of the Stage.

Likewiſe with reſpect to the Grooves in which the Side Scenes ought to run; it would be very inconvenient to have them made ſlanting, in the ſame Manner as the Scenes themſelves; by reaſon of the Declivity of the Stage, which would hide more

[1] Fig. 216.

[2] Fig. 209.

[3] Fig. 215.

[4] Fig. 216.

[5] Fig. 215.

[6] Fig. 216.

[7] Fig. 215.

[1] Prob. Sect. 1. and Cor. 1.

Fig. 215.

[8] Fig. 214.

[9] Fig. 217.

or leſs of the Bottom of each Scene, as it was moved forward or backward, and alſo alter the apparent Diſtance of the Scene, and conſequently the Meaſures by which the Objects ought to be deſcribed on it, in thoſe ſeveral Poſitions; though this may be alſo avoided, by making the Grooves themſelves parallel to the Front, and fixing the Scene in its Frame with the Inclination deſired.

Upon the whole, there being ſo many inconveniencies in the uſe of ſlanting Scenes, as well in giving them their proper Shapes, and finding their Interſections with the Stage, as in Painting on them when that is done, and the whole when finiſhed not being free from ſome diſagreeable conſequences; all theſe can never be compenſated by the only ſuppoſed Advantage of better hiding thoſe who move the Scenes, which can be ſo eaſily provided for in another manner, by the help of proper Pullies and Machines under the Stage.

To conclude this Subject, we ſhall in the laſt Place propoſe a Method for painting Scenes, which may ſometimes be of uſe, and render the Work leſs troubleſome, eſpecially when the neceſſary Vaniſhing Points happen to fall at an inconvenient Diſtance; and this may be done by the help of a Model or Picture of the whole intended Deſign, drawn upon a ſingle Plane.

Thus, if a Picture of the whole Deſign be drawn on the Plane of the Curtain, by way of Model; the Part which each Scene ought to bear, may be found by drawing on this Model or Picture, the Appearance of the naked Scenes, as at Fig. 213, which will diſtinguiſh upon the Model what is proper for every Scene; and if a farther Aſſiſtance be deſired towards drawing on each Scene its own particular Part of the Deſign, the Model may be Reticulated, that is, divided Netwiſe by Lines perpendicular and parallel to the Horizontal Plane, at equal Diſtances from each other, ſo as to ſubdivide the whole into equal Squares or Parallelograms at Pleaſure.

Then to Reticulate each Scene ſo as to anſwer to the correſponding Reticulations of the Model; the Diſtances between the Diviſions of the Reticulations on every Scene, muſt be made in the ſame Proportion to thoſe in the Model, as the Diſtance of the Eye from each reſpective Scene is to its Diſtance from the Curtain[a]; or, which will amount to the ſame thing, each Scene being drawn out in its proper Dimenſions, divide that Scene by Lines parallel and perpendicular to the Horizontal Plane, in the ſame manner as its Image in the Model is divided by the Original Reticulation; and thereby the Scene will be ſo prepared, as that by transferring to each Subdiviſion of the Scene, the Objects which lye in the correſponding Subdiviſion on the Model, the Scene when ſo painted, and placed in its true Situation on the Stage, will repreſent ſuch part of the Deſign as belongs to it, in its due Proportions.

But this is to be underſtood only of ſuch Part of each Scene as is viſible from the Point of Sight; for as each Scene muſt have ſome additional Breadth given it, that a Spectator, though ſtanding out of the true Point of Sight, may ſtill ſee ſome farther Deſcription, and not diſcover any naked Part of the Scene, or any gap or opening between them; this additional Part muſt contain the Deſcription of ſome Objects which cannot appear in the general Picture of the Deſign, but muſt be ſupplied from the Original Plan; it being abſurd, that what falls on the viſible Part of one Scene, ſhould be repeated on the additional Part of that next behind it.

For this Reaſon, in Deſigns for Theatres, it is beſt ſo to contrive them, that there may be proper Breaks ſideways at certain intervals, accommodated to the Places of the Scenes, to let in the View of other Objects more remote from the Axe of the Eye, with the Deſcription of which, the additional Parts of the Side Scenes may be furniſhed; for although thoſe Side Objects cannot be ſeen from the true Point of Sight, yet they are of great Uſe and Beauty in an Oblique View, and conduce very much to the more natural Appearance of the whole Scene.

All this will be very eaſily perform'd while the Scenes are made parallel to the Curtain; but when the Side Scenes are Oblique, it will be more difficult, in regard that all the Lines of the Reticulation which ought to be parallel to the Horizontal Plane, muſt, in that Caſe, tend to the Vaniſhing Point of thoſe Parallels in each reſpective inclining Scene; and the Diſtance between thoſe Reticulations on either Edge of the Scene, muſt be taken in Proportion to thoſe in the Model, as the perpendicular Diſtance of that Edge from the Directing Plane, is to the Diſtance of the Eye from the Curtain: Or in other Words, if a Plane be imagined to paſs by the Edge of the Scene parallel to the Curtain, the Diſtance between all the Reticulations on that Edge, muſt be to thoſe in the Model, as the Diſtance of the Eye from this ſuppoſed Parallel Plane, is to its Diſtance from the Curtain or Model. Beſides, when the Side Scenes are ſlanting

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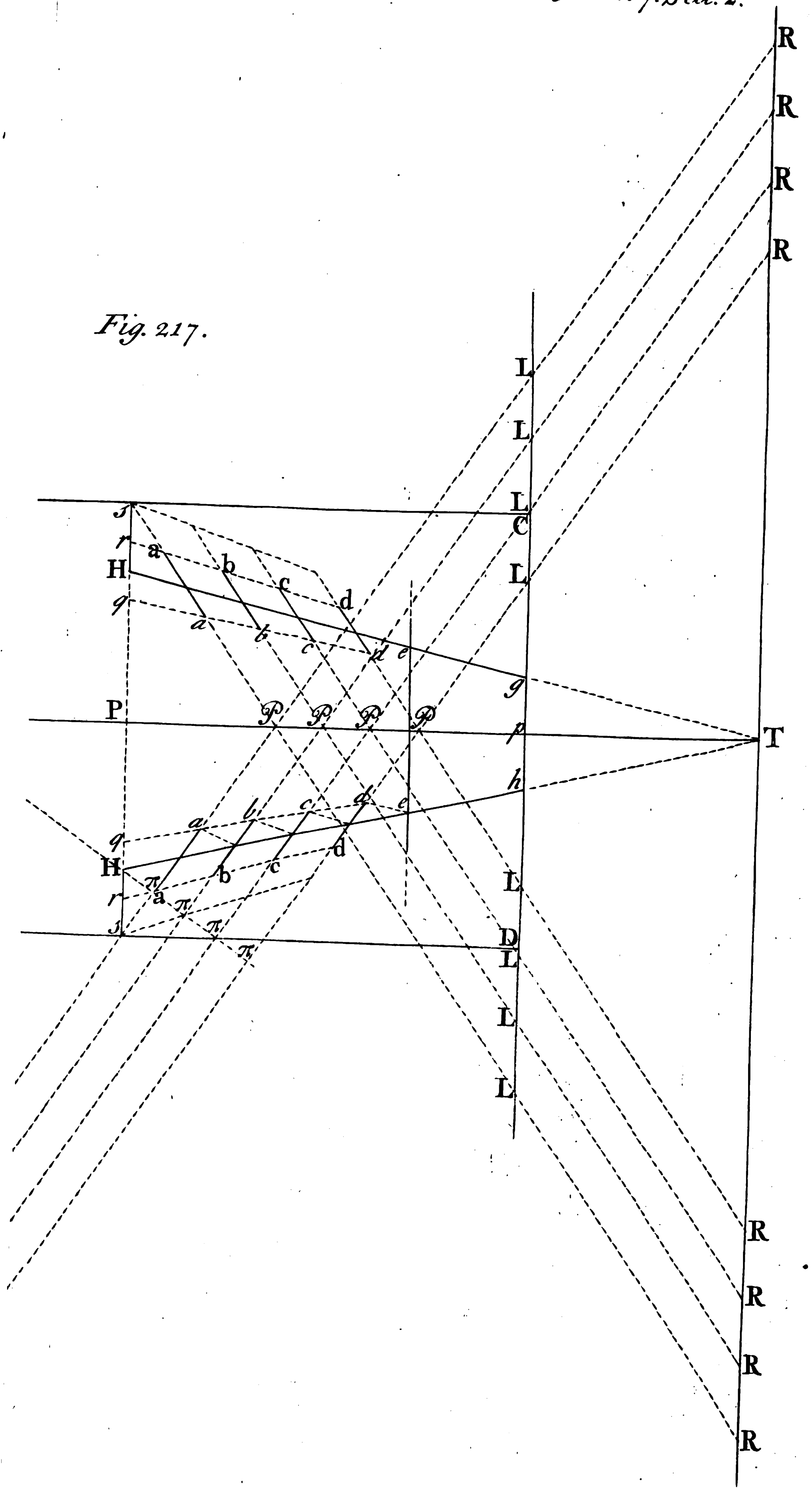
[a] Theor. 23. B. I.

Plate 125 Book 7. Sect. 2.

Fig. 217.



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ing, they will not ſo naturally fall in with the neceſſary Breaks in the Deſign for enlarging the View ſidewiſe, as abovementioned; which furniſhes ſtill a farther Reaſon againſt their being uſed, unleſs perhaps on a very extraordinary Occaſion, when the principal Lines of the Deſign are ſo ſituated, that their Vaniſhing Points on a Scene parallel to the Curtain, would fall at a more inconvenient Diſtance, than on one which inclines to it.

SECTION III.

Of Painting on Vaulted Ceilings, Dome's, Cupola's, or other Curvilinear or Uneven Surfaces.

WHEN the Surface to be painted on is not Plain, but Curvilinear, or otherwiſe uneven, the Rules of plain Stereography cannot be directly applied to it; in regard that ſuch a Surface cannot be fitted or prepared with the proper Lines and Points to direct the Deſcription; and the Images of all Objects on it are ſo diſtorted and bent, that it would be impracticable to trace them by the common Rules; Straight Lines in the Deſign being projected into Curves of all Sorts, or otherwiſe irregularly broken, according to the Shape of the Surface on which they are deſcribed.

So that although the Section of an uneven Picture by any Plane of Rays, might poſſibly be found, yet the Deſcription of a whole Deſign in that way, where almoſt every Point muſt be ſought out ſingly, would be ſo intolerably laborious and tedious, that no one could have Patience to undergo ſuch a Taſk.

It therefore becomes neceſſary in all Works of this Sort, to draw out on a Plane properly choſen, a Picture of the intended Deſign by way of Model; and then, having Reticulated the Model in the moſt convenient Manner, and projected that Reticulation on the propoſed Surface, to transfer ſuch Part of the Deſign as lies within each Cell of the Model, to the correſponding Cell of the Surface to be painted; whereby the intire Image of the Deſign on that Surface will be obtained, which being viewed from the proper Point, will have the deſired Effect.

This is uſually done by making the Reticulation of the Model regular, ſubdividing it into equal Squares or Parallelograms, and then by the help of an open Frame, divided in the ſame Manner by Threads or Lines, and placed properly over-againſt the propoſed Surface, and of a Light ſet at the Point of Sight, throwing the Shadows of the Lines on it, the correſponding Reticulation of that Surface is found and marked out.

But this Method being liable to many Inconveniencies, the leaſt Variation of the Place of the Light having a great Effect on the Place of the Shadows, and theſe of themſelves being neither ſteady nor well defined, beſides the Difficulty of tracing them out, by reaſon of the Irregularity of their Figure; It ſeems much eaſier and ſurer, to draw the Original Reticulation on the propoſed Surface, in ſuch Manner as may be beſt ſuited to its Shape, and can with the moſt Eaſe be done, and then to draw on the Model, the Image of that Reticulation, by the common Rules of *Stereography*; which will divide the Deſign on the Model, into ſuch Parts as are proper to be transferred into each correſponding Cell of the Original Reticulation.

Fig. 218.

Thus, if it were propoſed to paint any Deſign on an Arched or Vaulted Roof A*a*BC*a*D; a Reticulation may be made on the Vault itſelf, by drawing on it ſeveral ſtraight Lines *aa*, *bb*, *cc*, &c. lengthwiſe, parallel to the Walls AC, BD, from whence the Arch ſprings, and at equal Diſtances from each other, and by croſſing theſe with equidiſtant Sections $l\lambda l$, $m\mu m$, &c. of the Vault, perpendicular to the Horizon, all which Lines may be drawn on the Vault with great Eaſe; which being done, and a Model of the Deſign being deſcribed on a Plane ſuppoſed to paſs by the Bottom of the Arch ABCD, parallel to the Horizon, and of the ſame Dimenſions with the Plan of the Arch, the Image of the Reticulation of the Vault may then be deſcribed on the Model by the uſual Rules, as in the Figure[1].

But it muſt be obſerved, that the Center O of the Model, muſt be taken perpendicular to the ſuppoſed Place of the Eye; and the Diſtance to be worked with, muſt be the ſame as that between the Eye and the Plane ABCD[2], as well for deſcribing the Model itſelf, as for the Reticulation.

[1] Fig. 219.

[2] Fig. 218.

For the Place of every Point of the Painting on the Roof, being the Projection of the correſponding Point of the Model from the true Place of the Eye, it is neceſſary that every Line drawn on the Model, be truly projected on the Roof from the ſame Point of Sight; and as the Original Object and its Projection are Reciprocal, the Reticulation on the Model muſt be the true Perſpective of the Original Reticulation of the Roof from the ſame Point.

What is ſaid of the Diſtance of the Eye for the Model, is meant only when the Model is of the ſame Dimenſions with the Plan of the Roof ABCD; but if it be made leſs in any Proportion, (as it may often be convenient to do, ſtill retaining a Similitude in its Shape) the Diſtance to be taken for the Eye, muſt be leſſened in the ſame Proportion, the Model being then ſuppoſed to be brought ſo much nearer the Eye.

And here it may be obſerved, that the Eye being ſuppoſed to ſtand perpendicularly under the Section $m\mu m$ of the Roof, the Image of that Section in the Model becomes a ſtraight Line.

Thus alſo if a Deſign were to be painted on the Inſide of a Dome or Cupola, the Eye being ſuppoſed perpendicularly under its Center; a convenient Reticulation may be made in the Dome, by dividing the Circumference of its Baſe from whence it ſprings, into any even Number of equal Parts, and from every Diviſion and its Oppoſite raiſing a Section of the Dome by a Plane perpendicular to the Horizon, all which Sections will paſs through the Pole or Vertex of the Dome, which Sections may be afterwards ſubdivided by Horizontal Circles, drawn round the Inſide of the Dome at proper Intervals.

Then having made a Model of the intended Deſign on a Circular Plane, ſuppoſed to paſs by the Baſe of the Dome; the Image of the Reticulation of the Dome on this Model, will be found by dividing its Circumference into the ſame Number of Parts with that of the Dome; for all the perpendicular Sections of the Dome will become ſtraight Lines or Diameters in the Model, paſſing through its Center; and the Horizontal Circles in the Dome will alſo form Circles in the Model, having its Center for their common Center, the ſeveral Diameters of which Circles are eaſily found by the Help of a perpendicular Section of the Dome.

Fig. 220. Thus, let BAC be a perpendicular Section of the Dome, and *ab*, *cd*, *ef*, its Sections with the Planes of the Horizontal Circles; incloſe that Section with a Parallelogram BLMC properly ſubdivided, and having put that Parallelogram into Perſpective, as at BC*lm*, thereby the Lengths of the ſeveral Diameters $\alpha\beta$, $\gamma\delta$, $\epsilon\phi$ are found; Or if O be taken at the ſame Diſtance from BC, as the Eye is ſuppoſed to ſtand below the Plane of the Baſe of the Dome, Lines drawn from O to the Extremities of the Diameters *ab*, *cd*, *ef*, will, by their Interſections with BC, mark the Lengths of the Images of thoſe Diameters on the Model BDCE without farther Trouble; as in the Figure, where the ſeveral Diameters repreſent the perpendicular Sections, and the Concentrical Circles, the correſponding Horizontal Circles of the Dome.

All this is done with great Eaſe, when the Eye is ſuppoſed to be perpendicularly under the Center of the Dome; but if it were placed obliquely, the Reticulation of the Model would become a little more troubleſome; in regard that in ſuch a Poſition of the Eye, the perpendicular Sections of the Dome would not form ſtraight Lines in the Model, but Curves.

Fig. 221. Thus, let BAC be a perpendicular Section of the Dome, and *ab*, *cd*, *ef*, its Sections with the Horizontal Circles as before; and let I be the Place of the Eye.

Then Lines drawn from I, to the Vertex A of the Dome, and to the Centers and either Extremity *b*, *d*, *f*, of the Horizontal Diameters, will cut the Baſe of the Dome BC in Correſponding Points, which being transferred by Perpendiculars to the Diameter BC of the Model, will give the apparent Vertex O, and the Centers and *Radii* of the Images of the Horizontal Circles on the Model; and theſe being drawn, and each divided into the ſame Number of equal Parts, as the Baſe of the Dome is ſuppoſed to be, Curve Lines drawn through the correſponding Diviſions of theſe Circles, will give the Projections of the ſeveral perpendicular Sections of the Dome, as in the Figure.

For, as the Horizontal Circles are all ſuppoſed parallel to the Plane of the Model, it is evident their Projections will ſtill remain Circles, and their Subdiviſions will be equal, as are thoſe of their Originals.

And here all the perpendicular Sections of the Dome, form Curves in the Model, except the Section BAC, which is projected into the ſtraight Line BC, the Eye being ſuppoſed to lye in the Plane of that Section.

Plate 127 Book 7. Sect. 3.6.

.220.

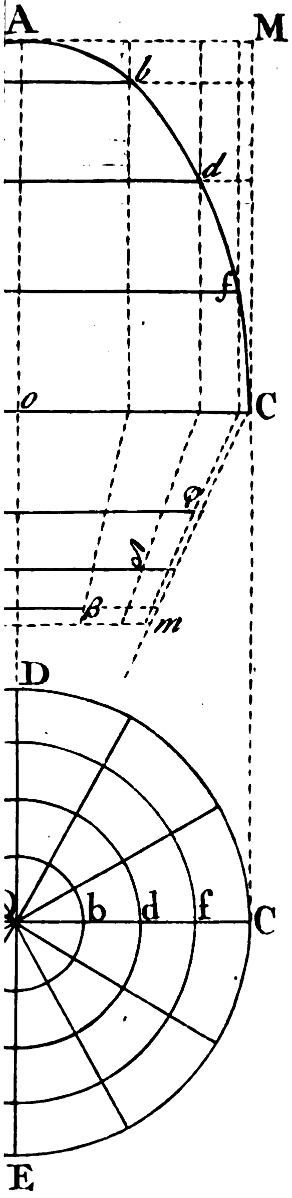


Fig. 221.

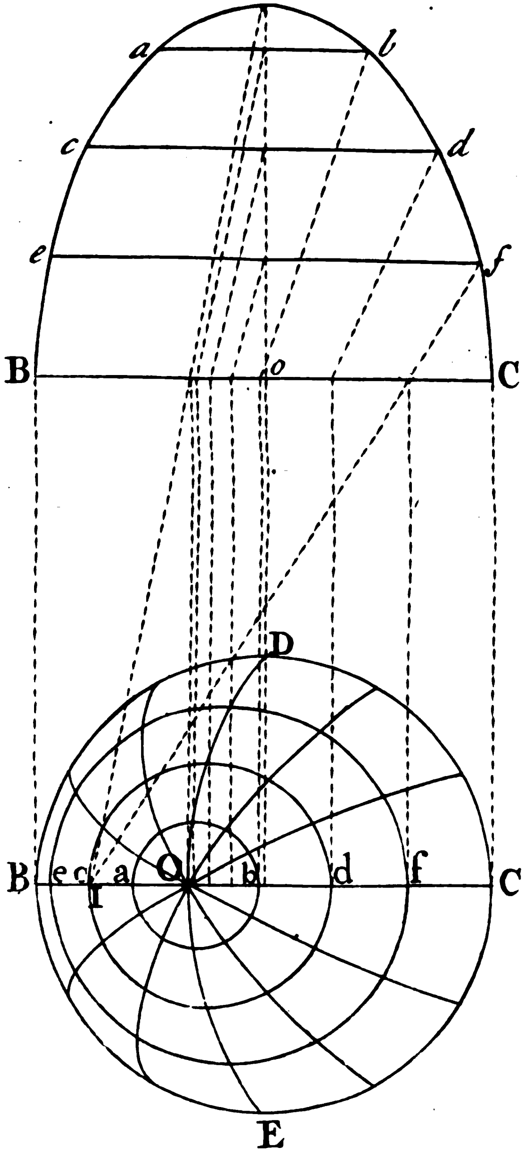


Fig. 222.

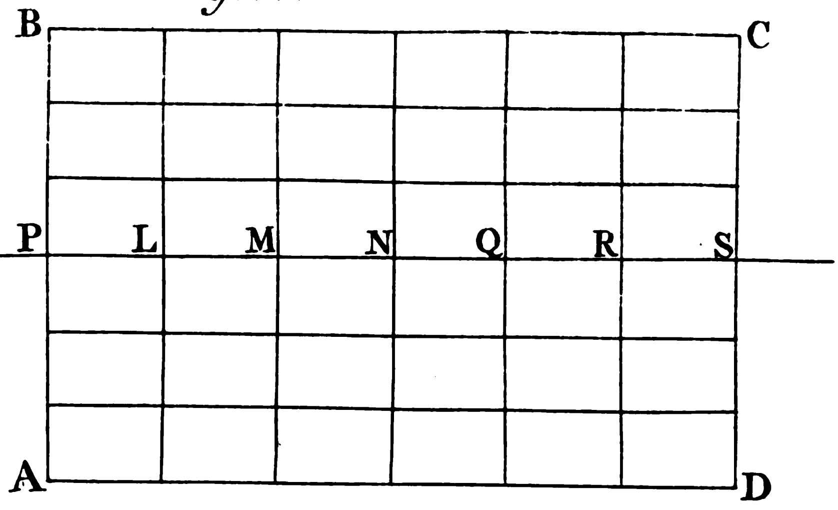
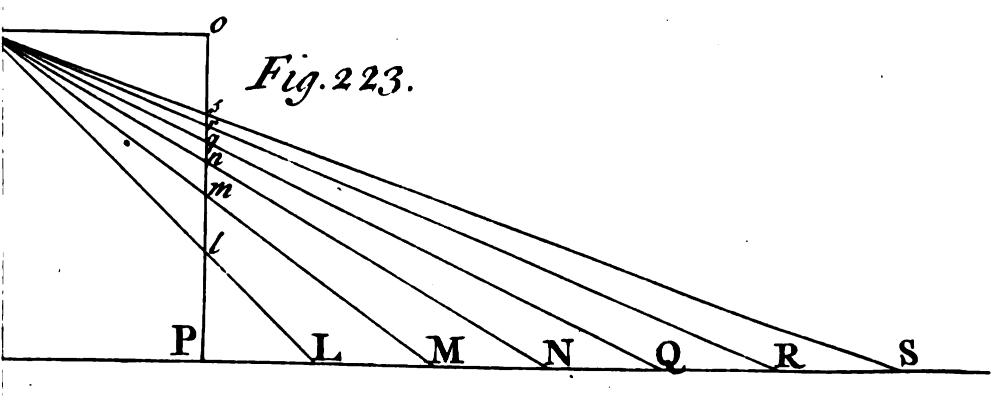


Fig. 223.



J. Mynde sc.

But in Painting on Curvilinear Grounds, the moſt direct Situation of the Eye ought always to be choſen, that the Deſign, when painted, may appear the more agreeably; and indeed, in all ſuch Works, the Deſign ought as much as poſſible to be ſuited to the Shape of the Surface, and to conſiſt principally of Ornamental Architecture fitted to it, putting the Hiſtorical Part into ſmall Copartments to be diſpoſed in proper Places; Or elſe of ſome Aereal View, where the Sky and Clouds with other Objects proper for that Situation may be deſcribed; in which caſe, the principal Objects not being confined to Regular Figures, there will be the leſs Danger of their appearing diſtorted by the Shape of the Surface painted upon.

But when a Cupola, Dome, or Vault is to be deſcribed on a flat Ground, there may be a greater Liberty taken in placing the Eye; which may have either a Direct or Oblique Poſition, as the Artiſt judges beſt for the View he intends to repreſent, and will not be liable to thoſe inconveniencies which attend Painting on an uneven Ground.

SECTION IV.

Of Aereal Perſpective, Chiaro Oſcuro, *and Keeping in Pictures.*

BY *Aereal Perſpective*, is meant the Art of giving a due Diminution or Degradation to the Strength of the Light, Shade, and Colours of Objects, according to their different Diſtances, the Quantity of Light which falls on them, and the Medium through which they are ſeen.

The *Chiaro Oſcuro* conſiſts more particularly in expreſſing the different Degrees of Light, Shade, and Colour of Bodies, ariſing from their own Shape, and the Poſition of their Parts with reſpect to the Eye and neighbouring Objects, whereby their Light or Colours are affected.

And *Keeping*, is the Obſervance of a due Proportion in the general Light and Colouring of the whole Picture, that no Light or Colour in one Part, may be too bright or ſtrong for another, but that a proper Harmony amongſt them all together may be preſerved.

All theſe are neceſſary requiſites to a good Picture, and may be properly enough included within the general Name of Aereal Perſpective, as they all relate to the different Degrees of Strength of the Lights and Colouring, according to the Circumſtances of the Shape and Poſition of Objects with regard to each other, the Eye, and the Light which illuminates them.

But as the Enquiry into the Effects of Light and Colours on each other, is not ſo properly the Subject of Mathematical Reaſoning as of Experiment and Obſervation, and for that Reaſon doth not lye directly within the Deſign of this Work; we ſhall only offer a few Hints and Conſiderations which may be uſeful to that purpoſe.

It has been obſerved in another Place[a] that the Eye does not judge of the Diſtance of Objects barely by their apparent Size, but alſo by their Strength of Colour and Diſtinction of Parts; it is not therefore ſufficient to give an Object its due apparent Bulk according to the Rules of *Stereography*, unleſs at the ſame time it be expreſſed with that proper Faintneſs and Degradation of Colour which the Diſtance requires.

Thus, if the Figure of a Man at a Diſtance, were painted of a due Size for the Place, but with too great a Diſtinction of Parts, or too ſtrong Colours; it will appear to ſtand forward, and ſeem proportionally leſs, ſo as to repreſent a Dwarf ſituated nearer the Eye, and out of the Plane on which the Painter intended he ſhould ſtand.

By the Original Colour of an Object, is meant that Colour which it exhibits to the Eye when directly expoſed to it in a full open uniform Light, and at ſuch a moderate ſmall Diſtance as to be clearly and diſtinctly ſeen.

This Colour receives an alteration from many Cauſes, the principal of which are theſe.

1. From the Objects being removed to a greater Diſtance from the Eye, whereby the Rays of Light which it reflects, are leſs vivid, and the Colour becomes more diluted, and tinged in ſome Meaſure with the faint bluiſh Caſt, or with the Dimneſs or Hazineſs of the Body of Air through which the Rays paſs.

[a] Sect. 1. B. I.

2. From the greater or leſs Degree of Light with which the Object is enlightned; the ſame Original Colour having a different Appearance in the Shade, from what it has in the Light, although at an equal Diſtance from the Eye, and ſo in Proportion as the Light or Shade is ſtronger.

3. From the Colour of the Light itſelf which falls upon it; whether it be by the Reflection of Coloured Light from any neighbouring Object, or by its Paſſage through a Coloured Medium; which will exhibit a Colour compounded of the Original Colour of the Object, and the other accidental Colours which the Light brings with it.

4. From the Poſition of the Surface of the Object, or of its ſeveral Parts with reſpect to the Eye; ſuch Parts of it as are directly expoſed to the Eye, appearing more lively and diſtinct than thoſe which are ſeen ſlanting.

5. From the Cloſeneſs or Openneſs of the Place where the Object is ſituated; the Light being much more variouſly directed and reflected within a Room, than abroad in the open Air; every Aperture in a Room giving an inlet to a different Stream of Light with its own peculiar Direction, whereby Bodies in ſuch a Situation will be very differently affected with reſpect to their Light, Shade, and Colours, from what they would be in an open Place.

6. Some Original Colours naturally reflect Light in a greater Proportion than others, though equally expoſed to the ſame Degrees of it; whereby their Degradation at ſeveral Diſtances will be different from that of other Colours which reflect leſs Light.

From theſe ſeveral Cauſes it ariſes that the Colours of Objects are ſeldom ſeen pure and unmixed, but generally arrive at the Eye broken and ſoftned by each other; and therefore in Painting, where the natural Appearances of Objects are to be deſcribed, all hard or ſharp Colouring ought to be avoided.

A Painter, therefore, who would ſucceed in Aereal Perſpective, ought carefully to ſtudy the Effects which Diſtance, or different Degrees or Colours of Light, have on each particular Original Colour, to know how its Hew or Strength is changed in the ſeveral Circumſtances above mentioned, and to repreſent it accordingly; ſo that in a Picture of various coloured Objects, he may be able to give each Original Colour its own proper Diminution or Degradation according to its Place.

Now, as all Objects in a Picture take their Meaſures in Proportion to thoſe *placed* in the Front; ſo in Aereal Perſpective, the Strength of Light, and the Brightneſs of the Colours of Objects cloſe to the Picture, muſt ſerve as a Meaſure, with reſpect to which, all the ſame Colours at ſeveral Diſtances, muſt have a proportional Degradation in like Circumſtances. But as in Muſick, it is not neceſſary to the Harmony, that the Inſtruments ſhould be tuned to the Concert Pitch, but they may be ſet above or below it, ſo long as they are in Tune to each other; ſo in Painting, it is not requiſite that the Meaſures on the Interſecting Line of the Picture, or the Brightneſs of the Light there, ſhould be equal to the Life, but they may be taken greater or leſs, ſo long as every Thing elſe in the Picture, bears a true Proportion to that which is choſen as the firſt Standard.

Hence, almoſt any Degree of Light may be taken for the greateſt Light in a Picture, when the leſſer Degrees of Light are expreſſed with darker or weaker Colours; for any Degree of Light may either repreſent a Light in reſpect of a darker, or it may ſerve as a Shade to a lighter; and it matters not in point of Keeping, how light or how dark a Picture is in general, ſo that its ſeveral Parts have proportional Degrees of Light and Shade given them.

In order, therefore, to the giving any Colour its due Diminution in Proportion to its Diſtance, it ought to be known what the Appearance of that Colour would be, were it cloſe to the Picture, regard being had to that Degree of Light which is choſen as the principal Light of the Picture; as in order to the giving any Object its due apparent Size, its true Size muſt be reduced to the ſame Scale with the Meaſures on the Interſecting Line.

For if any Colour ſhould be made too bright for another, or for the general Colours imployed in the reſt of the Picture, it will appear too glaring, and ſeem to ſtart out of its Place, and throw a Flatneſs and Damp on the reſt of the Work, or, as the Painters expreſs it, the Brightneſs of that Colour will kill the reſt.

No Painting can expreſs the dazzling Brightneſs of the Sun, or even its reflected Light coming from poliſhed Metals, with that ſparkling Vivacity as it appears in the *Camera Obſcura*, in the Images of poliſhed Surfaces on which the Sun ſhines; or if it could in ſome Sort be imitated in a Picture, by the Aſſiſtance of gilding, it would not have a good Effect with regard to the other Colours, which it would too much outſhine; and thereby hurt the Keeping: And this is one Defect which the Repreſentation of Objects in the *Camera Obſcura* is liable to; for by reaſon of the Refraction of the Rays by the Glaſs, thoſe Objects which naturally reflect leſs Light, loſe a greater Proportion of it, than thoſe which reflect Light more plentifully; whereby the due Keeping in the whole, is not ſo exactly preſerved as in Direct Viſion, the Lights and Shades appearing generally too ſtrong for each other.

And here, it may not be improper to add ſome farther Obſervations by way of Compariſon between a painted Picture, and the Repreſentations of Objects in a plain Looking-Glaſs, and in the *Camera Obſcura.*

A Picture painted in the utmoſt degree of Perfection, ſhould repreſent the Objects to the Eye in its true Poſition, in the ſame manner as they would appear, if the Eye looked at the Original Objects through the Picture, as through a tranſparent Plane.

This is exactly performed by a plain Looking-Glaſs, wherein the Objects appear in their proper Colours and Dimenſions, as if they really were behind the Glaſs, and were ſeen through it, while the Surface of the Glaſs itſelf is not perceived.

This is alſo done in the *Camera Obſcura*, but with ſome diſadvantage to the Appearance; for beſides what has been already mentioned with regard to the defect of Keeping, the Objects are here repreſented in a ſmaller Scale than the Life, and in an inverted Poſition, unleſs ſeveral Glaſſes be uſed, which darken the Image; and in the next place, the whole Proſpect cannot be clearly ſeen together, for if the nearer Objects be made diſtinct, the more remote will be confuſed, and ſo *vice verſa*, eſpecially when the Diſtances are ſmall, there not being ſo great a variation in the Focal Lengths at great Diſtances; whereas in a Looking-Glaſs there is no *Focus*, and all Objects, at whatever unequal Diſtances they be from it, appear alike diſtinct and plain, ſave only the natural Faintneſs which attends diſtant Bodies ſeen by the naked Eye, which is the ſame in the Looking-Glaſs.

Another Difference is, that in the *Camera Obſcura*, the Paper or Cloth which receives the Image, is in ſome meaſure perceived through it, and the Image ſeems as it were painted on it, and in this reſpect reſembles Painting more than the Image in a Looking-Glaſs does; which laſt is the perfect Reſemblance of the Objects themſelves, as directly ſeen by the Eye, their Shape on the Surface of the Glaſs, or, if the Expreſſion may be uſed, their Stereographical Shape, not being perceived or attended to, which in the *Camera Obſcura* ſhews itſelf diſtinctly.

Beſides, the Image in the diſtinct Baſe of the *Camera Obſcura*, depends on the Situation of the Glaſs with reſpect to the Objects, and has no relation to the Place of the Spectator's Eye, but continues the ſame from whatever Point it is looked at; and like a Picture, each Point of the Image is ſo fixed to the ſame Point of the Paper, whilſt this laſt continues unmoved, that if a proper Situation of the Eye be not taken (and which ſhould be as near as poſſible to the Glaſs itſelf) the Picture repreſented may appear deformed; whereas in a Looking-Glaſs, the Place of the Reflection depends upon the Place of the Eye, and varies always with it, ſo that wherever the Eye is, the Reflected Images conform themſelves to that Station, and appear juſt and true, notwithſtanding any greater or leſs Space of the Surface of the Glaſs which they cover, or ſtand againſt.

Another Defect in the *Camera Obſcura* is, that it can repreſent with clearneſs, only a certain compaſs of Objects according to the Convexity of the Glaſs, it being neceſſary to limit its Aperture in proportion to that Convexity; otherwiſe, the Objects repreſented towards the Edges of the Appearance, will be dark and confuſed: but in a Looking-Glaſs, there are no Limits for the Deſcription, provided the Objects be on the Reflecting Side of the Plane of the Glaſs, indefinitely extended; for the Objects will appear truly in the Glaſs, although the Eye ſhould be ſo near it as to touch its Plane; that is, a Looking-Glaſs is capable of repreſenting as much of Objects expoſed to its Reflecting Surface, as the Eye could be capable of ſeeing through a tranſparent Plane, were the Eye even in Contact with that Plane.

One Thing, indeed, there is in common to a Looking-Glaſs and the *Camera Obſcura*, which cannot be expreſſed in Painting; and that is the Repreſentation of Motion in the Objects viewed, which gives a Beauty and Life to the Deſcription, which it is impoſſible for the Art of Painting to arrive at: But in Recompence, Painting has the peculiar Advantage of being able to repreſent whatever Objects the Painter's Fancy ſuggeſts to him, who has full Scope to exerciſe his Invention and Judgment in forming artful Deſigns and Compoſitions, and bringing together the moſt agreeable Objects in the propereſt Attitudes, and introducing ſuch beautiful Scenes, either of Action or Proſpect, as are not to be met with together in common Nature; whilſt the *Camera Obſcura* and Looking-Glaſs can repreſent nothing but what is actually preſent, and immediately expoſed to them.

From what has been ſaid in this Compariſon, it appears that the Reflected Images of Objects in a plain Looking-Glaſs, are more natural and juſt than thoſe in the *Camera Obſcura*, and are repreſented in the utmoſt Perfection, in regard they cannot in a manner be diſtinguiſhed from the Reality; and were this Appearance as rarely ſeen as that of the *Camera Obſcura*, it would certainly have the Preference; but its frequency, afforded even by Nature itſelf, without Art, in the ſmooth Surface of ſtanding Water, makes it leſs admired, though not leſs beautiful.

We ſhall only add, that the Reflection of an Object in a Looking-Glaſs, at any Diſtance of the Eye, being the ſame with the *Stereographical* Appearance of that Object to an Eye placed at an equal Diſtance perpendicularly on the contrary Side of the Glaſs, ſuppoſing it tranſparent; it follows, that a painted Picture ſeen in a Looking-Glaſs, will not truly repreſent what it ought to do, unleſs the Diſtance of the Picture from the Glaſs, and the Diſtance of the Eye from the Glaſs, added together, be equal to the true Diſtance from whence the Picture ought to be ſeen, the Eye being at the ſame time placed in the true Radial of the Picture, according to which it was drawn; or in other Words, the Reflection of a Painted Picture ſeen in a Looking-Glaſs, has the ſame Effect, as if the Picture itſelf were viewed at a Diſtance equal to the Diſtance of the Picture from the Glaſs, added to that between the Glaſs and the Eye, the Planes of the Glaſs and Picture being ſuppoſed Parallel.

SECTION V.

Of the Poſition of the Picture, with reſpect to the Objects to be deſcribed.

ALTHOUGH in the foregoing Work, *Stereography* has been treated of in general, with reſpect to all its Kinds, whether Perſpective, Projective, or Tranſprojective, in order to render the Science Univerſal, and ſuited to all poſſible Situations of the Eye, the Picture, and the Object; yet, in what is here to be ſaid touching the Poſition of the Picture, and, in the following Sections, touching the Diſtance and Height of the Eye, and the Size of the Picture, we ſhall confine ourſelves to that part of *Stereography* which is ſtrictly Perſpective, it being that in which Paintings and Drawings are chiefly concerned.

The deſign of Painting being to repreſent Objects in the Picture, as near as may be, in the ſame manner as the Real Objects would appear to the Eye were they actually preſent, it is evident, the nearer the Artiſt can bring his Imitation to Nature, the more agreeable will his Performance be; and therefore, although he is not confined in the Choice of his Subject, or the particular Situation of the Objects he intends to repreſent, but has all imaginable Latitude for his Deſign, yet he muſt in that, keep within the Bounds of Nature and Probability, or at leaſt of Poſſibility; and what Objects he deſcribes, ought to be ſuch as may reaſonably be imagined to be in the Poſitions where they are made to appear: He will avoid building Caſtles in the Air, or making Cattle grazing on the Sea, and ſuch other Abſurdities, which are the Reſult of a diſturbed Imagination. And as Viſion itſelf is ſubject to many Ambiguities or Uncertainties, with reſpect to the true Shape and Figure of Objects ſeen in certain Poſitions, and as no Art can poſſibly imitate Nature ſo exactly, but that ſeveral other additional Ambiguities may ariſe from the Defect of the Imitation, it is requiſite the Painter ſhould chuſe ſuch a Poſition for his Picture, and the Objects he intends to repreſent, as that they may be leaſt liable to that Inconveniency.

The Picture then may have any Poſition given it with reſpect to the Objects repreſented, but theſe ought always to appear in a Situation natural to them with reſpect to the true Horizon; and conſequently whatever Relation the Picture may have to the Objects, it ought to be ſo placed with reſpect to the Spectator's Eye, that the ſame Relation may be preſerved; that ſuch Objects as, in their natural Situation, are uſully viſible by Rays parallel to the Horizon, may be ſeen in the Picture by the like Rays, and

thoſe

thoſe which uſually require an exalted or depreſſed Turn of the Eye to be obſerved, may demand the ſame in the Picture.

For the Ground, or Plane of the Horizon, being the natural or apparent Seat of all viſible Objects, even of the Celeſtial, which are judged high or low in Proportion as they appear more or leſs elevated above it, it is to that Plane to which the Situation of all Objects may and ought to be referred, and with reſpect to which they ought to have ſuch a Poſition given them as is agreeable to Nature.

The different Situations of the Picture are therefore moſt properly to be diſtinguiſhed by its Poſition in reſpect to the Ground or Plane of the Horizon, to which it may be either Perpendicular, Parallel, or Inclining.

The Perpendicular Situation of the Picture is beſt fitted for the Deſcription of the Ground itſelf, with the ſeveral Objects which ſtand upon it; and as this Situation is parallel to the moſt uſual Poſture of the Body, and in which the Eye is moſt accuſtomed to behold Objects, their Deſcription in this Manner appears the moſt natural and agreeable.

Here, the Ichnography of the Deſign on the Ground, is deſcribed as on a Plane perpendicular to the Picture, the Vaniſhing Line of which Plane paſſes through the Center of the Picture[a] parallel to the Horizon, which it repreſents, and is therefore in this Caſe called the Horizontal Line; and the Elevations of the upright Faces of Objects are in Planes perpendicular to the Ground, which Planes may be either parallel, perpendicular, or inclining to the Picture, but their Vaniſhing Lines are always perpendicular to the Horizontal Line[b], and all Lines which meaſure the perpendicular Heights of Objects above the Ground, are parallel to the Picture.

This is on a Suppoſition that the Ground deſcribed in the Picture is truly Horizontal; but if it be a riſing or ſinking Ground, the Picture continuing perpendicular to the Horizon, the Vaniſhing Line of the Ground will not then coincide with the Horizontal Line, but either riſe above or fall below it, according as the Ground is elevated or depreſſed. Nevertheleſs, the Lines which meaſure the perpendicular Heights of Objects, will ſtill be parallel to the Picture, as being perpendicular to the true Horizon, and will, with reſpect to the Ground, repreſent the Oblique Supports of the ſeveral Points above that Inclining Plane.

This Situation of the Picture is proper for Landſkapes, Views, Buildings, Hiſtory Pieces, and generally for all Paintings where the Spectator is ſuppoſed to ſtand on the Ground, and to direct his Eye in a Line parallel to it.

The Parallel Situation of the Picture is of two Sorts; either when the Eye is between the Ground and the Picture, or when the Picture is between the Eye and the Ground.

In the firſt Caſe, the Eye's Axe is ſuppoſed to be turned perpendicularly upwards, and conſequently the Ground being behind the Eye with reſpect to the Picture, no part of the Ground Plane can poſſibly appear in it; whence, no Terreſtial Objects ought there to be deſcribed, but what may be ſuppoſed to riſe above the Plane of the Picture, ſuch as the higher Parts of Mountains or Buildings, or elſe ſuch Objects as may be imagined to be in the Air; but all their upper Faces muſt neceſſarily be hid from the Eye.

Of this Kind are Paintings on flat Ceilings, to which thoſe on Cupola's or Vaulted Roofs may be reduced, as already mentioned[c], they all agreeing in the kind of Objects which are proper to be deſcribed on them; for want of a due Obſervance of which, ſo many Performances of that Sort appear unnatural and diſagreeable; nothing being more abſurd than to make the Sea, or Ground, or part of the Floor of a Building, or the upper Faces of Steps appear in a Picture in this Situation, although nothing is more commonly done by injudicious Painters.

In the other Caſe of the Parallel Situation of the Picture, the Eye is ſuppoſed to be at ſome Height in the Air above the Picture, and its Axe turned perpendicularly downwards; and here, no Part of the Sky can appear, nor any thing but what can be imagined to lye upon the Ground, or between that and the Picture, ſuch as the Pavement of a Church or other Building, the Plan of a Garden, or ſuch like Objects, and ſuch Parts of Buildings or other Things as ſtand on the Ground, and do not reach up to the Picture: ſo that this Situation of the Picture, with regard to the Objects proper to be deſcribed, is the moſt confined of any, and is little uſed, except for Curioſity, as on the Floor or Pavement of a Church or Dome, which, by an artful Diſpoſition of different coloured Marble or Stone, may be made to repreſent Objects proper for that Situation, or even the reflected Image of the Building itſelf, as appearing in a Looking-

Glaſs

[a] Cor. 1. Theor. 9. B. I.

[b] Cor. 3. Theor. 16 B. I.

[c] Sect. 3.

Glaſs or ſtanding Water to an Eye viewing it from ſome Gallery, or other convenient Station at the Top of the Building.

In both Caſes of the Parallel Situation of the Picture, the Ichnography of the Objects on the Ground is deſcribed as on a Plane parallel to the Picture, and is therefore ſimilar to its Original; and the Planes of the Elevations, and the Lines which meaſure the perpendicular Heights of the Objects above the Ground, are perpendicular to the Picture; the Vaniſhing Lines of which Planes therefore paſs through the Center of the Picture, which Center is the Vaniſhing Point of all the Lines of Height. But as in either Caſe, the Ground cannot cut the Picture, and ſo is not proper to be uſed as the principal Original Plane, there being in thoſe Caſes no Horizontal Line, any of the upright Sides of the Buildings intended to be repreſented, or any other ſubſtituted Plane perpendicular to the Picture, may be uſed for that Purpoſe, to which the deſigned Objects may be referred.

Between the Perpendicular and Parallel Situations of the Picture, there may be an infinite Variety of Inclinations given to it with reſpect to the Plane of the Horizon, which will have correſponding Effects on the Place of the Horizontal Line, and the Vaniſhing Point of Perpendiculars to that Plane, all which have been already ſufficiently explained[a]; and the Nature of the Objects proper for ſuch a Picture, muſt be governed by what has been ſaid of the Perpendicular and Parallel Situations, according as the Inclination of the propoſed Picture approaches nearer to the one or the other of thoſe Poſitions: But theſe kind of Oblique Situations are ſeldom taken, unleſs by Neceſſity, when obliged to it by the Poſition of the Wall on which the Picture is to be painted or placed.

But whatever Situation is given to the Picture, if it be ſuch wherein the Horizontal Line can appear, the Picture ought in Strictneſs to be ſo placed, as that the Eye, when in the true Point of Sight, may be on a Level with that Line; for then all the Objects deſcribed in the Picture, will appear to ſtand in their true and natural Poſitions with reſpect to the real Horizon.

This may, however, in ſome meaſure, be diſpenſed with; for if a Picture be drawn on a Suppoſition of being perpendicular to the Horizon, but it ſhould be neceſſary to place it ſo high, that the Eye could not reach up to a Level with its Horizontal Line; then, by inclining the Picture forwards, ſo that a Perpendicular to it from the Eye may meet its Center, the true Appearance of the Picture may be ſaved; for although the Ground deſcribed in the Picture will not then appear parallel to the true Horizon, but riſing above it, yet the Spectator viewing the Objects repreſented, and conſidering them according to their Relations to each other, without Regard to their Poſition with reſpect to other Objects without the Picture, or to the real Poſture of his own Body, his Imagination will readily overlook the want of the due Poſition, ſo long as all the Picture is conſiſtent with itſelf; and the Picture will then, in ſome ſort, reſemble the Appearance of the like Objects reflected by a Looking-Glaſs which inclines to the Horizon, whereby the Reflected Ground Plane is elevated or depreſſed with reſpect to the real Ground, but the Reflections of all other Objects retain the ſame relation to the Reflected Ground, as the real Objects have to the Ground itſelf; it is but for the Spectator to imagine he ſtands perpendicular to this Reflected Ground Plane, and then the Objects will have a natural Situation with reſpect to the Ground on which they appear to ſtand.

But this Liberty is not to be uſed too freely, and only for Detached Pictures, which are to be placed any where accidentally, as convenient room can be found; but for ſuch as are painted expreſſly for a certain fixed Situation, as on the Wall or Ceiling of a Building, it is not allowable, in regard that theſe Pictures have generally a more immediate relation to the reſt of the Building itſelf.

For if a Picture drawn for a Poſition perpendicular to the Horizon, were to be placed parallel to it, as on a Ceiling; the Spectator, to bring the Objects to their natural State, muſt fancy himſelf ſtanding erect on a Ground, which appears perpendicular to the true Horizon; which will require a Strength of Imagination, not readily to be met with, although there are not wanting Examples of Paintings on Ceilings, where ſuch an extraordinary Share of Imagination is requiſite to make them ſeem natural.

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[a] Sect. I.

SECTION VI.

Of the Diſtance of the Eye from the Picture.

IT being evident from the Nature of *Stereography*, that a Picture cannot appear ſtrictly true, unleſs the Eye be placed exactly in the Point of Sight for which it was drawn: It follows, that a Picture ought always to be placed in ſuch a Poſition, that it may be viewed from that Point.

Hence, in Paintings on Ceilings, the Diſtance of the Eye muſt be determined by the Height of the Room, deducting the Height of the Spectator's Eye from the Floor; and ſo in all other Poſitions of the Picture, where the Place is confined, ſuch a Diſtance muſt be taken as is within reach: But where the Eye can be placed more at large, there is then room for Choice of ſuch a Diſtance as may contribute towards making the Objects appear to the beſt Advantage.

The Perpendicular Situation of the Picture, being that which generally allows the greateſt liberty of Choice for the Diſtance of the Eye; we ſhall conſider the Effects of different Diſtances taken for a Picture in that Situation, all which may eaſily be applied to any other Poſition of the Picture.

The various Diſtances of the Eye, abſtracted from the Conſideration of its Height, (which we ſhall here ſuppoſe taken at Pleaſure, and to continue always the ſame) have an Effect, both on the Ichnography and Elevation of Objects.

As to the Ichnography, which repreſents the Field or Space which the intended Objects occupy on the Ground, it is affected by the Diſtance of the Eye ſeveral ways: with regard to the Quantity of the whole Space it occupies, with reſpect to the Proportions of that Space taken up by its different Parts, as they lye nearer to or farther diſtant from the Picture, and likewiſe as to the apparent Breadth of thoſe Parts.

It has been ſhewn, that if a Line not parallel to the Picture, be divided into any Number of equal Parts, the Complements of the Images of thoſe Parts will be in a continual Harmonical Proportion[a]; and that the farther thoſe equal Parts lye from their Directing Point, or the farther the Eye is removed from the Picture, the Images of thoſe Parts will become more nearly equal[b]; that the Diſtance of the Image of any Point in an Original Plane from the Vaniſhing Line, will be increaſed or diminiſhed, as the Diſtance of the Eye is taken greater or leſs, the Eyes Height continuing the ſame[c]; and that if through the Diviſions of a Line in an Original Plane, there be drawn Lines parallel to the Picture, the Diſtances between the Images of theſe Parallels will be governed by the ſame Rules[d].

Now, in chuſing the Diſtance of the Eye, two Things ought principally to be guarded againſt, with reſpect to the Ichnography: *Firſt*, That the apparent Decreaſe of equal Parts of the Ichnography, as they grow more diſtant, may not be too quick; and, *Secondly*, That ſuch diſtant Parts of the Ichnography as are intended to be plainly deſcribed, may not run too near the Horizontal Line, which would occaſion their Images to be ſo ſmall and crowded together, that they could not be diſtinctly expreſſed; both which Defects may be remedied by a proper Choice of the Diſtance of the Eye.

Thus, if the Ichnography of the Deſign were contained within the Parallelogram Fig. 222. ABCD, ſubdivided into ſmaller, as in the Figure; the Line KP being choſen for the Line of Station, which determines the Poſition of the Ichnography with reſpect to the Picture: Having then taken any Height P*o* for the Eye, at pleaſure[1], draw out the Vertical Plane IK*o*P, the Line KP repreſenting the ſame Line, and being divided in the ſame manner, as it is in the Ichnography by the croſs Diviſions.

Then, if any Diſtance I*o* be taken for the Eye, and the Images *l*, *m*, *n*, *q*, *r*, *s*, of the Diviſions L, M, N, Q, R, S, and conſequently the Diſtances of the Images of the croſs Diviſions of the Ichnography which paſs through thoſe Points, ſhould appear to be too unequal, or to fall off too quickly, and the Image of the moſt diſtant S, ſhould be carried too far up towards the Vaniſhing Point *o*; Theſe Defects may be remedied by removing the Eye from I to a more diſtant Point *J* in the Radial I*o*.

If the Diſtance of the Eye be taken equal to the firſt Diviſion PL from the Picture, the Heights of the Images of L, M, N, Q, R, S, above the Interſecting Line, will be $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{6}{7}$, of P*o* the Depth of the Original Plane, their reſpective Diſtances from the Vaniſhing Line being $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$, of that Depth; and the Images of the Parts PL, LM, MN, &c. will be $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{20}$, $\frac{1}{30}$, $\frac{1}{42}$, and $\frac{1}{56}$, of the ſame Depth[e].

[a] Theor. 29. B. I.

[b] Cor. 2. and 3. Theor. 29. B. I.

[c] Cor. 3. Theor. 25. B. I.

[d] Gen. Cor. Theor. 36. B. I.

[1] Fig. 223.

[e] Cor. 4. Theor. 29. B. I.

If by this means, the Parts near the Picture ſhould occupy a greater Space in reſpect to the ſucceeding, than is agreeable, or that the Objects between R and S ſhould appear too ſmall and crowded, the Diſtance of the Eye may be taken equal to the two firſt Parts PL and LM, whereby the Images of the Diviſions M, Q, and S, will be brought down to the ſame Heights where thoſe of L, M, and N, ſtood before; and thus ſuch a Diſtance may be found, as may bring any Point of the Ichnography to appear at any propoſed Height in the Picture, within the Limits of the Depth of the Original Plane.

And if it were required to find ſuch a Diſtance of the Eye, that any particular Part of the Ichnography, as for Example, that which lies between L and Q, may occupy the largeſt poſſible Space in Depth; the Diſtance I*o* muſt be taken a mean proportional between PQ and PL, from which Diſtance the Image of the Space LQ will occupy a larger Field in Depth, than it would do at any other Diſtance of the Eye in the Radial I*o*, either nearer to or farther from the Picture[a].

Hence it may be obſerved, that the Diſtance of the Eye is that which principally governs the Diſtance of the fartheſt Ground that can be deſcribed with tolerable Diſtinctneſs; for if the Ground beyond the Picture be divided into Spaces equal to the Diſtance of the Eye, the Seats of all Objects which lye within the Ninth Space from the Picture, can occupy no more than one ninetieth Part of the Depth of the Original Plane, and the Image of the Extremity of that Space being diſtant but one tenth Part of that Depth from the Vaniſhing Line, that one tenth is the whole room left for the Deſcription of all poſſible Spaces beyond the Ninth.

Thus, if the Height of the Eye were 5 Feet, and its Diſtance 20; the Seat of all Objects, whoſe Diſtance is between 160 and 180 Feet from the Picture, can occupy no more Space than one ninetieth Part of 5 Feet, or two thirds of an Inch; and this reaching within 6 Inches of the Vaniſhing Line, the Seats of all Objects on the Ground from 180 Feet beyond the Picture to any aſſignable Diſtance, muſt be confined within that ſix Inches; ſo that even at ſo great a Diſtance of the Eye as 20 Feet, the Seats of Objects 20 Feet in Depth, at the Diſtance of 60 or 70 Yards, can be repreſented but imperfectly, and all beyond that will be faint and confuſed; and if the Diſtance of the Eye be leſſened, the Space which can be diſtinctly deſcribed will be proportionally decreaſed.

Another Effect of the Diſtance of the Eye upon the Ichnography is, that as by enlarging that Diſtance, the Images of the ſeveral croſs Diviſions of the Ichnography are brought lower towards the Interſecting Line, ſo their apparent Meaſures are proportionally increaſed[b], and conſequently the Lines which meaſure the Breadth of Objects parallel to the Picture, appear larger, at the ſame Time that thoſe which meaſure their Depths become leſs; and as the apparent Breadths of Objects are thus increaſed, ſo alſo are their apparent Heights or Elevations, thoſe Heights being ſuppoſed parallel to the Picture; ſo that upon the whole, by enlarging the Diſtance of the Eye (the Picture and original Objects retaining their Places) their apparent Breadths and Heights, or their Dimenſions which are parallel to the Picture are increaſed, but their Depths are leſſened.

From theſe Conſiderations, it will be eaſy to find ſuch a Diſtance of the Eye ſuitable to the intended Deſign, that the principal Objects may have the moſt advantageous Situation in the Picture, that the more diſtant which are intended to be plainly expreſſed, may not be advanced too near the Horizon, that the Decreaſe of the Depths of the ſeveral Objects may not be too ſudden, and that a due and agreeable Proportion between their apparent Heights, Breadths, and Depths may be preſerved.

What has been ſaid of the ſeveral Diſtances capable of being plainly repreſented in Painting, muſt be underſtood to be when the Scale by which the Picture is painted, is equal to the true Meaſures of the original Objects, that is, when the Objects adjoining to the Picture are repreſented as big as the Life; but as this Scale may be diminiſhed in any Proportion, a Picture may be thereby made capable of repreſenting much greater Diſtances than thoſe before mentioned, and yet the Diſtance which the Spectator is obliged to ſtand at, to ſee the Picture truly, may be greatly leſſened.

Fig. 224. Thus, let LMGH repreſent the Ground Plane, on which the Ichnography of the intended Objects is ſuppoſed to be deſcribed in its true Dimenſions, and let GH be the Interſection of the Ground with a ſuppoſed Picture EFGH, and conſequently the neareſt Part of the Ground propoſed to be deſcribed in the Picture: If then the Diſtance KP of the Eye, neceſſary to make any required Part of the Objects appear with ſufficient Diſtinctneſs, be too large for the Place where the Picture is intended to be ſet, the Scale may be diminiſhed in this Manner.

Draw any Line gh in the Original Plane parallel to GH, and between it and K the Point

[a] Theor. 35. B. I.

[b] Cor. 5. Theor. 23. B. I.

Plate 128 Book 7.
Sect. 6.7.8.

Fig. 224.

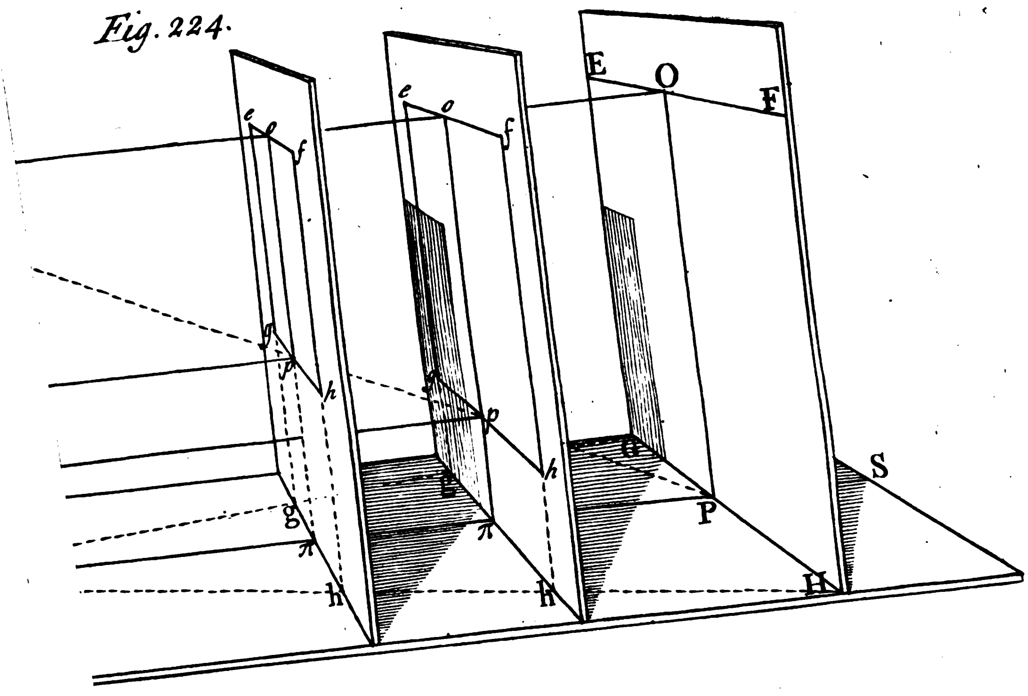
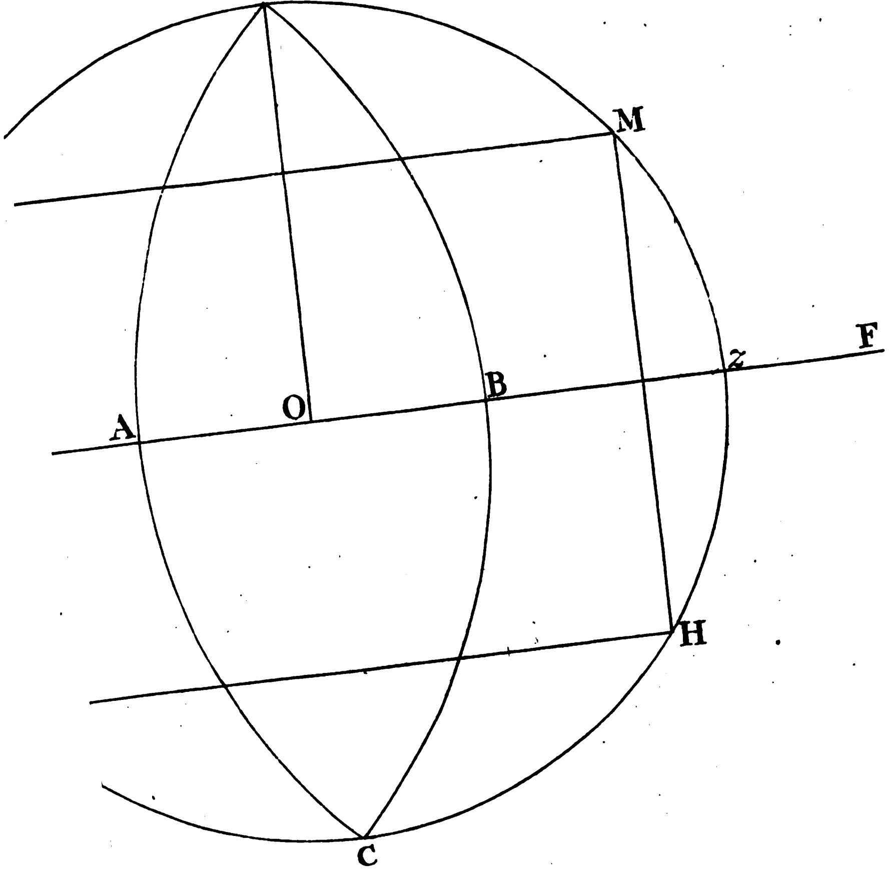


Fig. 225.



J. Mynde sc.

Point of Station, and on gh erect a Plane parallel to the Plane EFGH, and find the Image *efgb* of the Picture EFGH on that Plane; then *efgb* being taken for the Picture, and I*o* and I*k* for the Diſtance and Height of the Eye, and the true Meaſures of the intended Objects being reduced to the Proportion of *gb* to GH, all Objects de- ſcribed in the Picture *efgb* with theſe proportional Meaſures, and with the Diſtance I*o* and Height I*k* of the Eye, will be ſimilar to a Picture of the ſame Objects deſcribed on the Plane EFGH with their true Meaſures, and with the Diſtance IO and Height IK of the Eye, only proportionally leſſened.

For it is evident that all Lines in the Picture *efgb* will be to the Correſponding Lines in the Picture EFGH as I*o* to IO, or as I*k* to IK, or *op* to *oπ*; and that the Section of the Optick Pyramid whoſe Baſe is the Picture EFGH, by the Plane *efgb*, will be exactly ſimilar to it [a]: And thus the Scale, and conſequently the Diſtance of the Eye, may be reduced to any convenient Meaſure to ſuit the Place where the Picture is to be hung up.

In this Caſe, the Space between gh and GH is intirely hid, no Part of it being to have any Repreſentation in the Picture; and as GH is the true Interſecting Line of the Ground, on which the true Meaſures of the Objects ought to be taken were the Picture to ſtand there, ſo *gb* the Repreſentation of GH is to be conſidered as the Interſecting Line of the Picture *efgb*, on which the proportional Meaſures according to the dimi- niſhed Scale muſt be uſed.

It is viſible alſo, that if the Picture *efgb* were continued down to gh its Interſecti- on with the Original Plane, the Image of the Space between gh and GH will fall between gh and *gb*, and that the Objects which lye there, may be deſcribed either by ſetting off their true Meaſures on gh as the Interſecting Line, or their proportional Meaſures on *gb* according to the diminiſhed Scale; and that whether the one or the other of theſe Meaſures be uſed on thoſe reſpective Lines, ſuch Part of the Deſcripti- on as falls within *efgb* will be the ſame.

After this manner, a Picture may be made to repreſent a very large extenſive Pro- ſpect with a moderate Diſtance of the Eye, the Picture *efgb* being conſidered as a Window in a ſecond or third Floor through which the Spectator views the Objects; for in Proportion to the Height of *gb* above the Ground, and to the Diſtance I*o* ta- ken for the Eye, the viſible Interſecting Line GH of the Ground, from whence the Deſcription begins, may be removed to any required Diſtance, which will have a pro- portional Effect on the Space capable of being plainly repreſented.

And here, as the Picture *efgb* is a kind of Miniature of that on EFGH, the near- eſt Objects repreſented in it muſt be leſs than the Life, they being ſuppoſed equal to the Life at GH. But if in the Picture *efgb*, Objects were repreſented according to their natural Sizes, that Picture projected on EFGH would be bigger than the Life; which kind of projected Images bigger than the Life may be uſed, when the Diſtance of the Picture is neceſſarily very great, as on the Roofs of very high Churches, and it is intended that the Objects ſhould appear nearer to the Eye than where they are painted; which they will naturally do when painted in this Manner, provided they are done with ſufficient Strength of Colour, which ought to be augmented above the Life, as well as the Size, to produce the deſired Effect.

Hence Objects deſcribed by a Scale leſs than the Life, will appear or be judged more diſtant than their Picture, thoſe deſcribed equal to the Life will appear equally diſtant, and thoſe deſcribed bigger than the Life will appear nearer than their Picture, ſo long as the requiſite Degrees of Strength and Colour are obſerved.

Nevertheleſs, in Miniature Paintings, it is not neceſſary to deſcribe the Figures with that Faintneſs and Weakneſs of Colour with which the Originals would appear, were they really ſo far diſtant as to be reduced to that Size by the ſmallneſs of the Angle under which they were ſeen; but a greater Diſtinction of Parts and Vivacity of Colour is allowable to be uſed, provided only that a due Diminution be obſerved amongſt the ſeveral Objects repreſented in the Picture, with reſpect to each other; for ſtill the Picture thus drawn, may be conſidered to be ſuch a Repreſentation of the Objects, as would be produced by looking on them through a Concave Glaſs, which although it diminiſhes their Sizes, yet does not take off their Diſtinction of Parts or Strength of Colour in ſo great a Proportion.

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[a] Cor. 3. Theor. 23. B. I.

SECTION VII.

Of the Height of the Eye.

THE Height of the Eye is that which governs the Depth of the Original Plane, to which it is always equal[a]; and conſequently gives Bounds to the Field or Space within which the whole Ichnography of all poſſible Objects on the Original Plane muſt be confined.

It has been ſhewn that the Image of a Line in a Plane parallel to the Picture, will be of the ſame Length where-ever the Eye be placed in the Directing Plane[b]; wherefore the raiſing or lowering the Height of the Eye, makes no Difference in the apparent Heights and Breadths of Objects, or ſuch of their Dimenſions as are parallel to the Picture, which continue of the ſame Length at all different Heights of the Eye, while its Diſtance from the Picture is not varied.

It has been alſo ſhewn that the Images of any determinate Parts of an Original Line which inclines to the Picture, will have the ſame Proportion to each other at all different Stations of the Eye in the Directing Plane[c]; and therefore the Alteration of its Height without altering its Diſtance, hath no Influence on the Quickneſs or Slowneſs of the apparent Decreaſe of the equal Parts of Lines which Meaſure the Depths or Diſtances of Objects; theſe continuing ſtill to have the ſame Proportion to each other, whatever Height of the Eye be taken, and are only decreaſed or diminiſhed proportionally to that Height.

It has likewiſe been ſhewn, that if any determinate Part of the indefinite Image of a Line inclining to the Picture, be taken, and the Proportional Meaſure of that Part on a Parallel to the Interſecting Line drawn through its neareſt Extremity, be found; then, if the Complement of the propoſed Image be equal to, or bigger, or leſs than the Radial of that Line, the aſſumed Part of that Image will be equal to, or bigger, or leſs than its Proportional Meaſure[d].

Fig. 223. Hence, if the Height of the Eye IK ſhould be taken ſo great in Proportion to its Diſtance I*o*, that the Image *lm* of any Part LM of an inclining Line in the Original Plane, ſhould fall at a greater Diſtance from its Vaniſhing Point *o*, than the Length of the Radial I*o* of that Line; then the Image *lm* will be larger than the Proportional Meaſure of its Original LM, taken on a Line parallel to the Picture paſſing through its neareſt Extremity L.

Now, as it muſt ſeem unnatural, that the Image of a Line inclining to the Picture, ſhould appear equal to, or bigger than the Image of a Line of the ſame Length parallel to the Picture, and directly expoſed to the Eye, at a Diſtance no greater than the neareſt Extremity of the inclining Line; ſuch an Appearance muſt be deformed and diſagreeable, and ought therefore to be avoided.

This may be done, by taking the Height of the Eye ſo, that the indefinite Image of any Line in the Original Plane may not be longer than its Radial; for then, whatever part of the Original Line comes to be deſcribed within theſe Bounds, its Image will always be leſs than its Proportional Meaſure.

By this it appears, that the Height of the Eye has an immediate Dependance upon its Diſtance, which it ought by no means to exceed, nor indeed to be ſo great; for if it ſhould be equal to the Eye's Diſtance, the whole Perſpective P*o* of the Line of Station will be equal to its Radial I*o*, and ſo the Image of no part of that Line can, in ſuch Caſe, be greater than its Proportional Meaſure; but the whole Perſpectives of all other Lines in the Original Plane parallel to the Line of Station, will be longer than I*o*, and conſequently liable to the Inconvenience abovementioned.

But although this ſhews the Limits which the Height of the Eye ought not to exceed, yet that Height may be taken leſs in any Proportion, according to the Nature of the Deſign; for as it is the Height of the Eye which governs the Quantity of the Field or Space on which the intended Objects are to ſtand, that Height may be taken greater or ſmaller (within the abovementioned Limits) according as the Artiſt would give more or leſs room for the Depths of his Objects, to obſerve an agreeable Proportion between their Ichnography and Elevations.

Although it was ſaid[e], that in ſtrictneſs, the Eye ought always to be placed on a Level with the Horizontal Line of the Picture, yet the Height of the Eye is not confined to that of a Man ſtanding on the Original Plane, unleſs that Plane be what the

Spectator

[a] Cor. 3. Def. 18. B. I.

[b] Cor. 4. Theor. 23. B. I.

[c] Cor. Theor. 27. B. I.

[d] Cor. 3. Prob. 9. B. II.

[e] Sect. 5.

Spectator actually ſtands upon; for the Eye may be ſuppoſed on an Eminence, at a conſiderable Height above the Original Plane deſcribed; as when low Grounds, and Objects thereon, are to be repreſented as ſeen by an Eye ſtanding on a Hill, or the upper Part of ſome high Building.

Fig. 224. Thus, if LMGH be the Original Plane on which the Objects are ſuppoſed to ſtand, and GH be the Boundary of the neareſt Ground that is to be deſcribed; the Height of the Eye IK is not confined to the Height of a Man's Eye ſtanding at K, but it may be greater in any Proportion, provided the Picture can be ſo placed, that the Spectator's Eye may be at I, on a Level with the Horizontal Line EF, although his Feet may not reach lower than *k*.

SECTION VIII.

Of the Size of the Picture.

THE Rule already propoſed for limiting the greateſt Height proper to be given to the Eye, equally ſerves for giving Bounds to the Size of the Picture; for as there, the principal Inconvenience to be avoided, is the Exceſs of the Image of any Part of a Line in the Original Plane or Ground, above its Proportional Meaſure, the ſame Reaſoning muſt equally hold with reſpect to all other Planes and Lines, which it may be neceſſary to deſcribe in the Picture.

Fig. 225. If then from the Center O of the Picture, a Circle be deſcribed with a *Radius* OI equal to the Diſtance of the Eye, and that Circle, or any Square or Parallelogram LMGH, or other Rectilinear Figure inſcribed in it, be made the Bounds of the Picture; the Image of no Line whatever which is perpendicular to the Picture, can, within theſe Bounds, be extended farther than the Length of its Radial; and therefore when the Principal Lines of Depth in the Deſign are perpendicular to the Picture, the Size of the Picture may be limited in that manner.

But if the Principal Lines of Depth incline to the Picture, as when Buildings are to be repreſented in an Oblique Poſition; then the Vaniſhing Points of the Ichnography of their inclining Faces, muſt be taken as Centers, and the reſpective Diſtances of thoſe Vaniſhing Points as *Radii*, by which, Circles being deſcribed, they will, by their mutual Interſections, mark out the Space beyond which no part of the Images of thoſe Faces ought to extend.

Thus, if *y* and *z* were the Vaniſhing Points of the Ichnography of the Faces of a Building; then *y* and *z* being taken as Centers, and their Diſtances *y*I and *z*I as *Radii*, the Arches IBC, IAC, will include the proper Space within which the Image of the propoſed Building ought to be confined.

But theſe Rules more particularly regard the Deſcription of Pieces of Architecture, or the Bodies of Men and Animals, or ſuch other Objects as have certain, known, and determinate Shapes, that their Images may not be thrown too far diſtant from the Center of the Picture; but for Objects of uncertain variable indeterminate Shapes, ſuch as Clouds, Hills, Mountains, or the like, a greater Latitude is allowable.

For, although it is certain, that a Picture drawn according to the true Rules of *Stereography*, if ſeen from the true Point of Sight, will juſtly repreſent the Objects intended, where-ever that Point be taken; yet, if the Diſtance of that Point be too ſmall for the Size of the Picture, the Images of Objects towards the Sides of the Picture, will be drawn out to great Lengths, and occupy more Space on the Surface of the Picture, than the Objects themſelves would do if ſeen directly; and when the Picture thus drawn, comes to be looked at from a different Situation, the Images of thoſe Objects will appear deformed or diſtorted, and diſagreeable to the Eye.

It is therefore neceſſary, ſo to ſuit the Size of the Picture to the Diſtance of the Eye, that nothing in it may appear monſtruous or unnatural, where-ever the Eye be placed to view it; for although a Picture can in ſtrictneſs be truly ſeen, only from the true Point of Sight, yet when the Diſtance of the Eye is pretty large with reſpect to the Size of the Picture, ſo that the greateſt Dimenſion of the Picture may be ſeen under a Right Angle or leſs, any little Deviation of the Eye from its true Place, will not have ſo ſenſible an Effect on the Appearance of the Picture, as when the Diſtance is ſmaller, or the Picture of a greater Extent.

And as Pictures are generally, if not always placed in ſuch Poſitions, that they may be viewed from ſeveral different Situations; they ought to be ſo drawn, that in any of thoſe Situations, fronting them, they may appear as little diſagreeable to the Eye as may be; and if nothing in the Picture in theſe Views, appear remarkably deformed, the Eye will overlook little Variations from the ſtrict Appearance the Objects ought to have, and the Imagination will be ready to ſupply the Defect.

SECTION IX.

Of the Conſequences of viewing a Picture from any other Point than the true Point of Sight.

WHEN the Eye is placed in the true Point of Sight to view a Picture, the Imagination doth not ſtop at the Lines and Figures actually drawn in the Picture, but is carried on beyond it, to the Original Objects which are ſuppoſed to produce thoſe Images, the Picture itſelf being only conſidered as a tranſparent Plane through which the Objects are ſeen.

Hence, where-ever the Eye is placed to look on a Picture, the Originals of the Objects there repreſented, will be conceived to be ſuch, as would produce the Images as deſcribed in the Picture, were the Point of Sight the ſame with the preſent Situation of the Eye; and conſequently, if the Eye be not in the true Point of Sight, the Images in the Picture not being capable of varying, the Originals will be judged different from what they really are, or were intended.

Every different Poſition of the Eye out of the true Point of Sight, muſt therefore have a correſponding Effect on the apparent Places of the ſeveral Preparatory Lines and Points in the Picture, and conſequently on the Judgment to be formed of the Objects repreſented.

The firſt Thing then to be laid down is, that where-ever the Eye is placed, a Perpendicular from thence to the Picture will therein mark that Point which will be judged to be its Center, and the real Diſtance of the Eye from that Point will be taken as the true Diſtance of the Picture.

This being premiſed, we ſhall conſider in what reſpects the Appearances of the Objects in the Picture, are affected by the Alteration of the Place of the Eye.

CASE I.

If the Eye be placed in any Point of the Eyes Parallel relating to an Original Plane, the Vaniſhing Line of that Plane and the Height of the Eye will remain unaltered; but the Center of the Vaniſhing Line will appear to be moved ſo far on the ſame Side of the true Center, as the Place of the Eye is diſtant from the Point of Sight.

In this Caſe, thoſe Dimenſions of Objects which are parallel to the Picture, will not appear varied either in Size or Diſtance[a]; the Original Plane, if it be the level Ground, will ſtill ſeem Horizontal, and the Elevations of Objects will appear perpendicular to that Plane; and all Lines in the Ichnography parallel to the Picture, will be judged at the ſame Diſtances from each other, and from the Picture, as they would be from the true Point of Sight. The only Variation from the Truth, will be in the Appearance of the Angles, which the inclining Lines in the Ichnography, and the Planes raiſed on them, make with each other and with the Picture, and in the apparent Meaſures of the ſeveral Parts of ſuch inclining Lines.

[a] Cor. 4. Theor. 23. B. I.

For as the Images of all Lines in the Ichnography which are perpendicular to the Interſecting Line, have the true Center of the Vaniſhing Line for their Vaniſhing Point; when the apparent Place of this Center is changed, the true Center becomes the Vaniſhing Point of Lines in the Original Plane inclining to the Line of Station in an Angle, whoſe Tangent is equal to the Diſtance between the Real and Apparent Centers, putting the Diſtance of the Eye as *Radius*; and a correſponding Alteration will be made in the Appearance of all other Lines in the Original Plane, whoſe Images tend to any other Vaniſhing Points, and conſequently in the apparent inclination of any Elevated Planes, whoſe Vaniſhing Lines paſs through thoſe Points.

Fig. 226. Thus, let EFGH be the Picture, O its Center, IO its Diſtance, and NR the Parallel

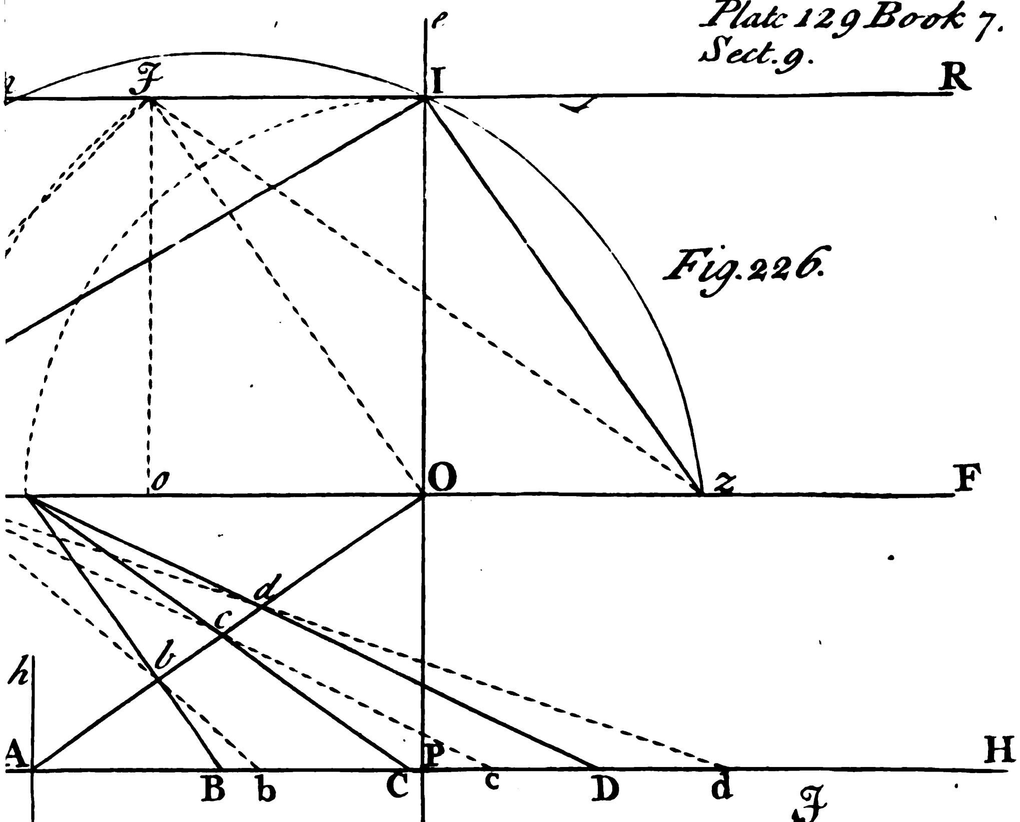

Fig. 226.

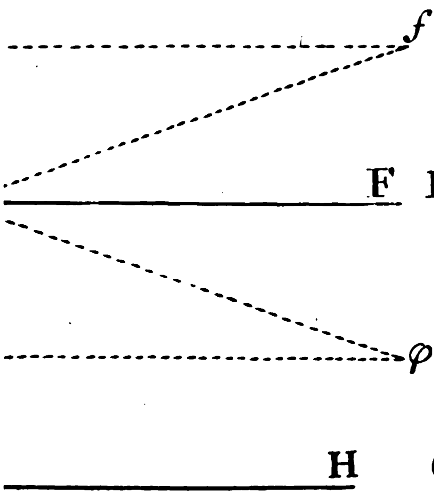

Fig. 227.

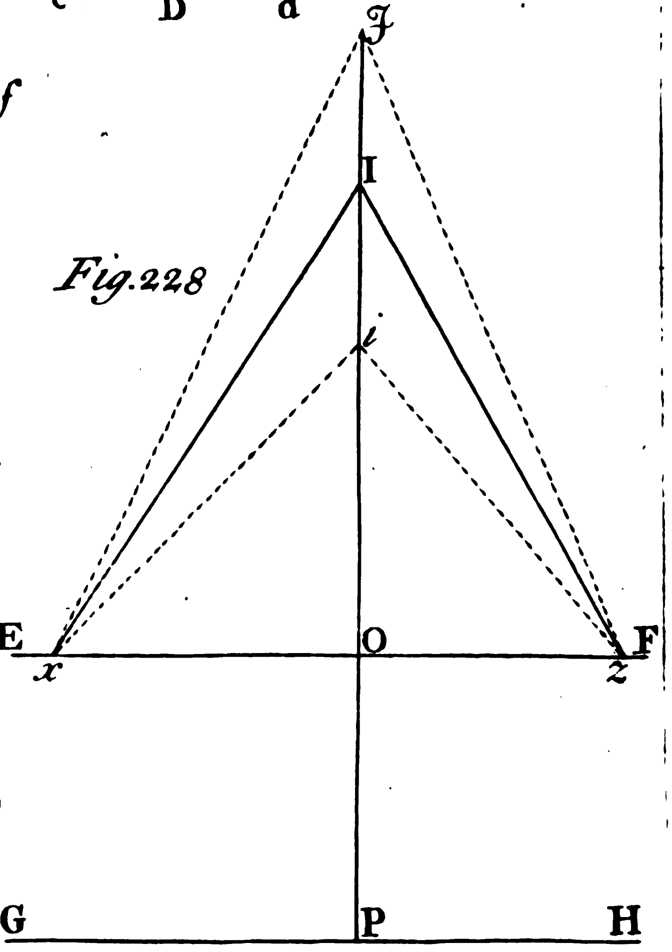

Fig. 228

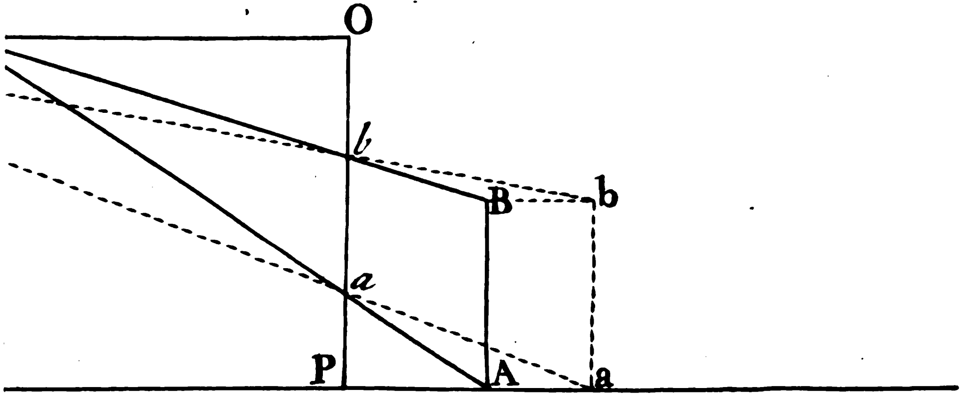

Fig. 229.

J. Mynde sc.

rallel of the Eye relating to the Original Plane, whoſe Vaniſhing Line is EF.

If the Eye be moved from I to J in the Line NR, o becomes the Apparent Center of the Vaniſhing Line EF; and conſequently the Image AO of a Line in the Original Plane perpendicular to its Interſecting Line, appears as if it inclined to the Line of Station in an Angle oJO, of which Oo is the Tangent, putting Jo or IO as *Radius*; and if IO or ef were the Vaniſhing Line of a Plane paſſing through AO perpendicular to the Picture and to the Original Plane, it will, from the Station J, appear as a Plane, perpendicular indeed to the Original Plane, but inclining to the Picture in the Angle JOo; and hence it is, that the Plane $efgh$, which at the Station I would appear perpendicular to the Picture, will, as the Eye is moved in the Line NR, appear to incline more and more towards the Picture, and always the contrary Way to which the Eye is moved; that is, as the Eye is moved towards N, the Plane $efgh$ appears to incline more and more towards R, and if the Eye were moved towards R, that Plane would appear to incline towards N.

Now if any two Vaniſhing Points x and z be taken in the Line EF, ſubtending with the true Point of Sight I, any Angle xIz, theſe two Points from the Station J, will appear to ſubtend an Angle xJz, and conſequently all Lines in the Original Plane, which tend to x and z, will from J, appear to incline to each other in the Angle xJz; which Angle will be either bigger, equal, or leſs than the true Angle xIz, according as x and z happen to fall in EF, with reſpect to the Real and Apparent Centers O and o.

For if on xz a Portion of a Circle xnIz, containing the Angle xIz, be deſcribed in the Vaniſhing Plane, paſſing through I; it is evident, that if the Eye be placed any where in that Circumference, the Angle ſubtended by x and z will appear the ſame: If then the Eye be placed at n, where the Circle cuts the Line NR, the Angle xnz will be true; if the Eye be within the Circle, the apparent Angle will be larger, and if it be without the Circle, the apparent Angle will be leſs than the true Angle xIz.

And as the Line AO from the Station J, appears to incline to the Picture, ſo JO becomes its apparent Radial, and therefore if AO be anywiſe divided in the Points b, c, d, the apparent Meaſures of the Parts Ab, bc, cd, of that Line, will be increaſed or diminiſhed in the ſame Proportion as the apparent Radial JO bears to the true Radial IO of that Line[a]; ſo that inſtead of repreſenting their true Meaſures AB, BC, CD, they will from the Station J, be judged equal to Ab, bc and cd. And as according to the Poſition of any Vaniſhing Point in EF with reſpect to the apparent Center o, the apparent Radial of that Point may be either bigger, equal, or leſs than its true Radial, (except only the Point O, whoſe apparent Radial can never be leſs than IO) ſo the Images of the Parts of any inclining Line in the Original Plane, may accordingly repreſent Parts bigger, equal, or leſs than their true Originals. Nevertheleſs, ſo long as the Eye continues in the Line NR, the perpendicular Diſtances of the Points b, c, d, from the Picture, will appear true[b].

Likewiſe, with regard to the Elevated Plane $efgh$; although its Center is not changed while the Eye is moved in the Line NR, yet its apparent Radial becomes equal to JO; and as That is bigger than its true Radial, it will have a correſponding Effect on the apparent Radials of all other Vaniſhing Points in ef, and on the apparent Angles they ſubtend with each other, and alſo on the apparent Diſtances of the Originals of any Points in that Elevated Plane from its Interſecting Line, though not on their apparent perpendicular Diſtances from the Picture.

The ſame is to be underſtood of any other Elevated Plane, whoſe Vaniſhing Line is parallel to ef, and paſſes through any other Vaniſhing Point in the Line EF.

CASE. 2.

If the Eye be placed in any Point of the Eye's Director relating to the Original Plane; thoſe Dimenſions of Objects, which are parallel to the Picture, will continue unvaried in their Appearance, both as to Size and Diſtance[c], as in the preceeding Caſe; but the apparent Place of the Horizon will be altered, and ſeem higher or lower than the true Horizontal Line, as the Place of the Eye is taken higher or lower than the true Point of Sight, and the Original Plane will appear depreſſed below, or elevated above the Horizon accordingly; and therefore the perpendicular Supports of all Points on the Original Plane, will appear only as their Oblique Supports[d]: and as by the Alteration of the apparent Center of the Picture, the apparent Radial of the Original Plane is enlarged, the Originals of the Parts of all inclining Lines in the Original Plane, will be judged proportionably larger than they really are; ſo that the Objects will ſeem to occupy more Space in Depth on this inclining Ground, than they really do, though their

perpendicular

[a] Cor. 4. Prob. 8. B. II.

[b] Cor. 1. Theor. 25. B. I.

[c] Cor. 4. Theor. 23. B. I.

[d] Def. 3. and 5. B. IV.

perpendicular Diſtances from the Picture are not affected: But as the Eye is ſuppoſed to continue in the ſame Director, the Vertical Line of the Original Plane will ſuffer no Change; and conſequently all Planes of Elevation will appear to incline to the Picture in the ſame Angles as they would do from the true Point of Sight, but they will not appear perpendicular to the Original Plane, but only as Planes of the Oblique Seats of Lines on that Plane[a]; and thoſe Lines in the Planes of Elevation which ſhould appear parallel to the Horizon, ſuch as the Ranges of Windows, or the Cornices, or other Members of Architecture in Buildings, will, by the apparent Change of the Centers of thoſe Planes, ſeem elevated above, or depreſſed below the Horizon, according as the Eye is placed lower or higher than the Point of Sight: likewiſe, the apparent Radials of the Elevated Planes being altered by the apparent Change of their Centers, that will have a correſponding Effect on the apparent Angles of Inclination of all inclining Lines in thoſe Planes to each other, and on the apparent Sizes of the Parts of thoſe Lines; nevertheleſs, while the Eye continues in the ſame Director, which, with regard to the Elevated Planes, is the Parallel of the Eye, the Perpendicular Diſtances of all Points in thoſe Planes from the Picture, will continue to appear true.

Fig. 227. All this is ſufficiently evident by what has been already ſaid, and by the Figure; where EF is the true Horizontal Line, O the Center of the Picture, and IO its Diſtance; *ef* repreſents the apparent Horizon, and *o* the Center of the Picture, when the Place of the Eye is taken at *J* above the Point of Sight I, and *εφ* the apparent Place of the Horizon, and *w* the Center, when the Eye is at *i* below that Point; the Diſtances IO, *Jo*, and *iw* being ſuppoſed equal.

CASE 3.

If the Eye be placed any where in the Radial of the Original Plane, farther from, or nearer to the Picture, than the true Point of Sight; the Center of the Picture and the Horizontal and Vertical Lines will not be altered, and all Planes of Elevation will continue to appear perpendicular to the Original Plane; but in regard that by the Motion of the Eye in the Radial of the Original Plane, its Diſtance from the Picture is varied, the apparent Radial of the Original Plane, and conſequently thoſe of all the Vaniſhing Points of Lines in that Plane, will become greater or leſs than the true Radials, as the Eye is removed farther from, or brought nearer to the Picture, than the true Point of Sight; which will have a correſponding Effect on the apparent Sizes of the Parts of all inclining Lines in the Original Plane, and conſequently on the apparent Diſtances of thoſe Parts from the Picture; and likewiſe on the apparent Angles ſubtended by the Vaniſhing Points of Lines in that Plane, which will appear greater or leſs than the true, as the Diſtance of the Eye is leſſened or increaſed; which will in like manner affect the apparent Inclinations of all Elevated Planes, and of the Lines in them, towards each other and to the Picture; except only as to the Right Angle contained between the Parallels and Perpendiculars to the Interſecting Lines of the ſeveral Planes, all which will continue to appear Perpendicular to each other, at all Stations of the Eye in the Radial of the Original Plane; ſeeing that the Center of the Picture not being thereby varied, the Centers of all Vaniſhing Lines in the Picture will remain unchanged[b].

Fig. 228. Thus, if EFGH be the Picture, O its Center, and IO its Diſtance; if that Diſtance be increaſed as to *J*, the apparent Radial will be *J*O, and the apparent Angle *xJz* ſubtended by the Vaniſhing Points *x* and *z*, will be leſs than *x*I*z* the true Angle; as it will become greater, if the Diſtance of the Eye be leſſened by moving it to *i*: the reſt is ſufficiently evident from what went before.

But although the apparent Diſtances of all Points in the Original Plane from the Picture, and conſequently of all Parallels to the Picture drawn through thoſe Points, will be greater or leſs as the Diſtance of the Eye is increaſed or diminiſhed; nevertheleſs, the Sizes of the Parts of thoſe Parallels, and of all other Lines which meaſure the Dimenſions of Objects parallel to the Picture, will be judged the ſame, at whatever Diſtance the Eye is placed.

Fig. 229. Thus, if IOKP repreſent the Vertical Plane, O the Center of the Picture, IO its Diſtance, and KP the Line of Station; let AB be an Original Line in that Plane perpendicular to the Original Plane and parallel to the Picture, the Image of which, from the true Point of Sight I, is *ab*: If then the Eye be moved from I to *J* in the Radial IO, whereby its Diſtance will be increaſed; it muſt be ſhewn that the apparent Original ab of the Image *ab* from the Station *J*, although it appear more diſtant from the Picture than its true Original AB, yet it will be judged of the ſame Size with AB, that is, that AB and ab will be equal.

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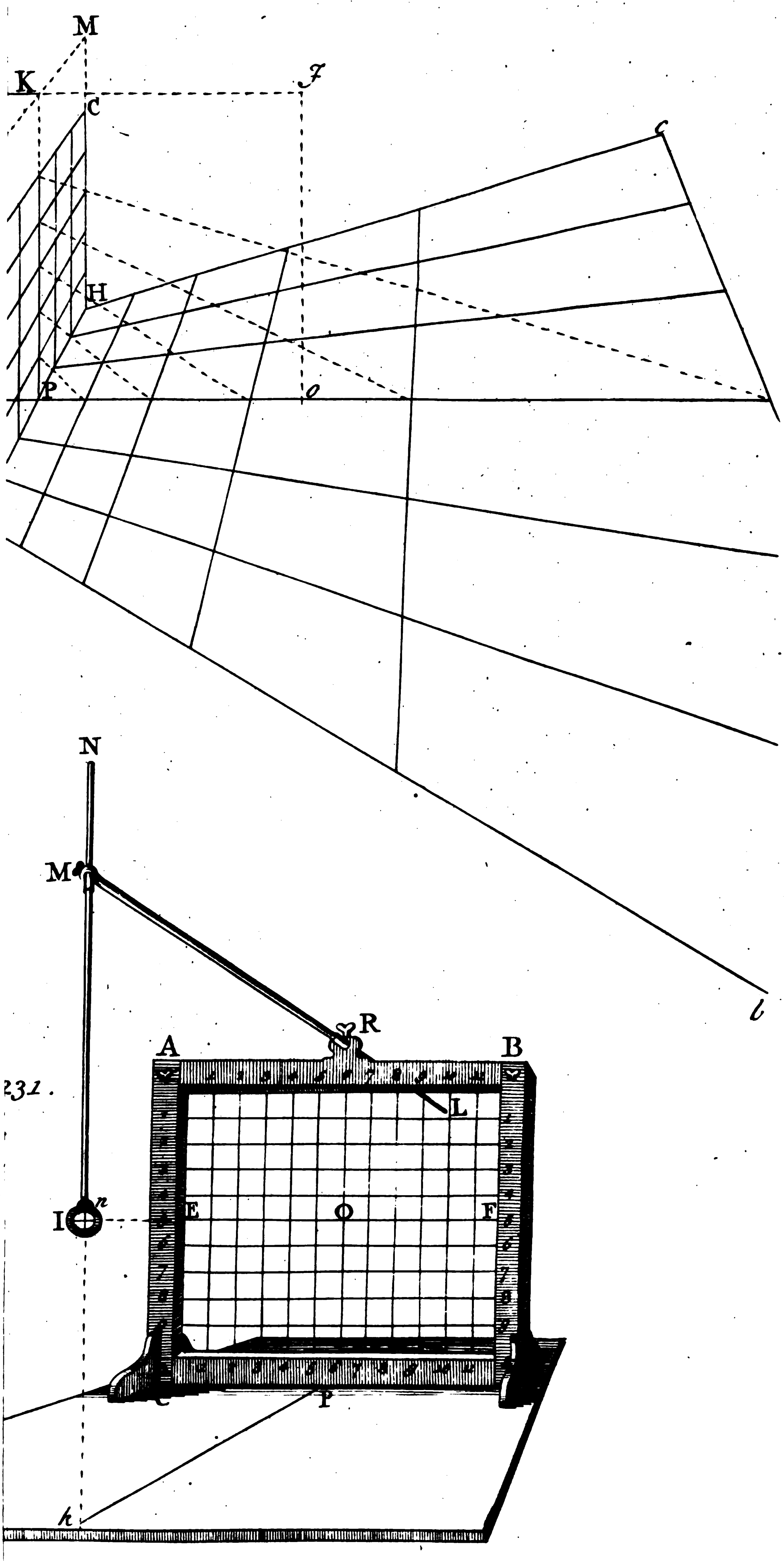
[a] Def. 6. B. IV.

[b] Cor. 2. Def. 15. B. I.

Plate 130 Book 7.
Sect. 10. 11



231.

J. Mynde f.

In the Similar Triangles $J\mathrm{I}a$, $a\mathrm{Aa}$, $Ja : a\mathrm{a} :: \mathrm{I}a : a\mathrm{A}$
And by Compoſition $Ja + a\mathrm{a} = J\mathrm{a} : Ja :: \mathrm{I}a + a\mathrm{A} = \mathrm{IA} : \mathrm{I}a$
But in the Similar Triangles Jab, $J\mathrm{ab}$, $J\mathrm{a} : Ja :: \mathrm{ab} : ab$
And in the Similar Triangles $\mathrm{I}ab$, IAB, $\mathrm{IA} : \mathrm{I}a :: \mathrm{AB} : ab$
Therefore by Parity of Reaſon $\mathrm{ab} : ab :: \mathrm{AB} : ab$
Conſequently $\mathrm{ab} = \mathrm{AB}$.

If *J* be conſidered as the true Point of Sight, and I as the Place of the Eye; the ſame Demonſtration will ſerve to ſhew, that if the Diſtance of the Eye be leſſened, the Original of *ab* will be judged nearer the Picture than it really is, but ſtill of the ſame Size.

Now, as AB is to *ab*, ſo are all other Lines in a Plane parallel to the Picture paſſing through AB, to their Images from the ſame Station I[a]; and in like manner, as ab is to *ab*, ſo are all other Lines in a Plane parallel to the Picture paſſing through ab, to their Images from the Station *J*: Therefore as AB and ab are equal, the Originals of all Lines in the Picture, which meaſure the Dimenſions of Objects parallel to it, will be judged equally long, at whatever Diſtance the Eye be placed in the Line IO.

CASE 4.

What has been ſaid of the ſeveral Poſitions of the Eye, either in the Eye's Parallel, or in the Eye's Director, or in the Radial relating to the Original Plane; is eaſily applicable to any other Poſition of the Eye whatever out of the Point of Sight: for if the Eye be placed any where in the Directing Plane, the Conſequences of that Poſition will be governed by what has been ſhewn under the two firſt Caſes; and if the Eye be placed any where out of the Directing Plane, either nearer to or farther from the Picture, the Conſequences are to be gathered from all the three.

Upon the whole, it may be obſerved, that the placing the Eye at a greater or leſs Diſtance from the Picture, ſo long as it remains on a Level with the Horizontal Line, has not ſo diſagreeable an Effect on the Look of the Picture, as when the Eye is placed higher or lower than that Line; but in any Situation of the Eye out of the true Point of Sight, within a reaſonable Compaſs, the Conſequences are not ſo conſiderable when the Picture is painted on a plain Surface, but that the Fancy will be ready to give ſome Aſſiſtance towards correcting what is not ſtrictly right, ſo that ſome Latitude may be allowed for looking on ſuch Pictures, without any groſs Inconveniency: But when Pictures are drawn on uneven Grounds, ſuch as Vaults, or Arched Roofs, or Walls, or otherwiſe irregular Surfaces, there, it becomes neceſſary to have the ſtricteſt Regard to the true Point of Sight; for any the leaſt Variation from it, occaſions the Figures to be disjointed, broken, or diſtorted, and conſequently deſtroys the Unity of the Repreſentation, which can alone be preſerved by the Eyes being placed in the true Point of Sight, and thence, as through a ſmall Hole or Sight, viewing the intire Deſign.

SECTION X.

Of Anamorphoſes, *or Deformations.*

AN *Anamorphoſis*, or Deformation, is a ſort of Painting or Drawing, wherein the Images of Objects are projected in ſuch manner as to appear monſtrous and miſhapen, when the Picture is directly expoſed to the Eye; but when viewed ſlantingly from one certain determinate Point, it repreſents the intended Objects in their due Proportions.

To this End, a regular Picture of the propoſed Objects being drawn on a Plane, for a Poſition perpendicular to the Horizon; this Picture, which is uſually called the *Prototype*, is conſidered as an Original Object; which being placed erect on the Plane on which the *Anamorphoſis* is to be deſcribed, the ſeveral Lines of the *Prototype* are projected on that Plane, from the ſame Point which was choſen for the Place of the Eye in drawing it; and the Projection thus formed, is the *Anamorphoſis* required; which being viewed from that Point, will exactly repreſent the *Prototype*, and conſequently the Original Deſign propoſed.

Thus, let LMGH repreſent the *Prototype*, or Picture of the intended Objects, as Fig. 230.
they are deſigned to appear to the Eye at I, and GH*cb* the Plane on which the *Ana-*

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[a] Cor. 1. Theor. 23. B. I.

morphoſis is to be deſcribed, ſuppoſed to be parallel to the Horizon; then the *Prototype* LMGH being regularly Reticulated by Lines parallel to its Sides, and that Reticulation being projected on the Plane GH*cb* from the Point I, as is repreſented in the Figure; the Objects in each Cell of the Reticulation of the *Prototype*, muſt be transferred to the correſponding Cells of the projected Reticulation, and thereby the Projection of the intire *Prototype* will be obtained; which Projection or *Anamorphoſis*, when viewed directly, will ſeem very deformed, but when placed Horizontally, and looked at from the Point I, will then appear a juſt Deſcription of the intended Objects in their due Proportions.

For although the Axe of the Eye is generally ſuppoſed to be directed to the Center of the Picture, in which Poſition of the Eye alone, the Picture is, in ſome ſort, ſimilar to its Repreſentation on the *Retina*; yet, whatever Turn the Eye makes to ſurvey the ſeveral Parts of the Picture, ſo long as its Center continues in the true Point of Sight, the Picture will appear juſt; and ſuch of the Images as, by Reaſon of their Diſtance from the Center of the Picture, are greatly protracted, being ſeen very ſlantingly, are thereby fore-ſhortened and reduced to their natural Appearance: For when the Poſition of the Eye with reſpect to an Original Object, is once fixed, the Viſual Pyramid, and conſequently the Image of that Object perceived by the Eye, can undergo no Change; and a Picture of the Object being only a Deſcription of the Section of its Viſual Pyramid by the Plane of that Picture, although different Sections by Planes variouſly inclined to this Pyramid, will produce Pictures different from each other, both in Size and Shape, yet all of them muſt ſtill exhibit the ſame Appearance to the Eye in its true Place, as the Object itſelf would do were the Picture removed.

From what has been ſaid, it appears that the Method of drawing an *Anamorphoſis* is exactly conſonant to the Rules of *Stereography*; for if the *Prototype* LMGH be conſidered as an Original Plane in which the Objects lye that are to be deſcribed, and EFGH*bc* as the Picture on which the Deſcription is to be made, then IK will repreſent the Height of the Eye, and IO its Diſtance, K the Point of Station, and O the Center of the Picture; and the *Anamorphoſis* will be the Projection of the Part BCGH of the Original Plane, which lyes between its Directing and Interſecting Lines LM and GH; which Projection therefore falls wholly beyond the Interſecting Line GH of the Picture, ſeeing no Part of it can fall between its Vaniſhing and Interſecting Lines EF and GH. And here, it is evident that the Objects deſcribed in the *Prototype* muſt not reach up to the Line LM, ſeeing the Projection of that Line on the Plane GH*bc* is infinitely diſtant, and cannot therefore be repreſented.

In another View, the Eye remaining at I as before, the Plane of the Projection GH*bc* may be conſidered as an Original Plane, and LMGH as a Picture placed upon it, whereby IO becomes the Height of the Eye, and IK its Diſtance; and the *Prototype* then repreſents the Image of the *Anamorphoſis* as ſeen from I: wherefore if the Eye keeps its Station, and the *Prototype* LMGH be removed, the *Anamorphoſis* being viewed from thence, will exhibit the ſame Repreſentation as the *Prototype* did.

The only Thing which makes Pictures of this kind pleaſing, is the proper Choice of the Objects to be repreſented; which ought chiefly to be ſuch as are to appear erect on the Plane of the Projection, as Buildings, Trees, or Figures of Men and Animals, &c. for the Images of ſuch Objects being drawn on the Plane LMGH, their Projections on the Plane GH*bc* will be thrown out into ſtrange uncouth Figures, having no intelligible Shape, till, when looked at from I, they ſuddenly ſeem to ſtart out of the Plane in which they lye, and recover their proper erect Poſture, which gives an agreeable Surprize to the Beholder: Whereas, if ſuch Objects were choſen as were deſigned to appear as lying in the Plane GH*bc*, conſidered as the Ground, ſuch as a flat Proſpect of a Garden or Parterre, diverſified with Walks and Alleys, which might appear beautiful enough in the *Prototype* LMGH; the *Anamorphoſis* or Projection, in that Caſe, would be only the Geometrical Deſcription or Plan of the Objects propoſed, which would have nothing remarkable in it to raiſe the Attention, or to produce any agreeable Effect.

The *Anamorphoſes* of the kind here deſcribed, may alſo be Rectified, or made to appear juſt, by the help of a Reflecting Plane.

Thus, if the *Prototype* LMGH were ſuppoſed to be a Looking-Glaſs, ſtanding on the Picture GH*bc*, with its reflecting Face towards the Projection; then, if the Eye be placed at *J* in the Line IK, produced to an equal Diſtance on the contrary Side of that Plane from I, the Reflected Projection will have the ſame Appearance from *J*, as the Projection itſelf when directly ſeen from I[a].

Many

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[a] Art. 16. Sect. 3. B. V.

Many other Sorts of *Anamorphoſes*, or Deformations, may alſo be drawn, ſo as to be Rectified by Reflection from poliſhed Surfaces of various Shapes, either Curvilinear, or Angular; as Cylinders, Globes, Cones, Priſms, Diamond-cut Surfaces, and the like: But all theſe being much more of Curioſity than Uſe, and the Method of drawing ſuch Projections depending principally on the Rules of Catoptrics, which are out of the Limits of our preſent Subject, we ſhall not here ſpend Time in examining them more particularly; the rather for that all ſuch *Anamorphoſes*, give, at beſt, but very imperfect Repreſentations of the Objects propoſed.

For the Agreeableneſs of Paintings of this Sort, ariſes principally from the Imaginations being able readily to refer the *Anamorphoſis* to the Surface on which the *Prototype* is ſuppoſed to be drawn, whereby the diſtorted Figures in the Picture put on an intelligible Form; But, as in a Reflecting Surface, the Reflected Image of an Object doth not appear as in that Surface, but at ſome Diſtance behind it, the Image perceived in the Reflection, will not naturally repreſent the Objects in an erect Poſture, as was intended, but rather a Reduplication of the ſame *Anamorphoſis* inverted, the one as difficult to be underſtood as the other. This is evidently the Caſe, when the reflecting Surface is a Plane, and it muſt be ſtill worſe when the Surface is Curvilinear, either Spherical, Cylindrical, Conical, or of any other Curvature.

SECTION XI.

Of the Perſpective Frame.

HAving now treated of *Stereography* in all its uſeful Branches, it may not be unacceptable, by way of Concluſion of this Work, to give the Deſcription of a very plain and ſimple Inſtrument, called a *Perſpective Frame*; which, although no new Invention, yet, with the ſmall Improvements here added, affords a very eaſy and exact Mechanical Way, whereby a Perſon moderately ſkilled in Drawing, may at Sight deſcribe the Proſpect or View of any Landſkape, Buildings, Gardens, or other Objects that preſent themſelves, without the Trouble of meaſuring any of them, or their Diſtances from the Picture.

Fig. 231.

The Body of the Inſtrument is a Rectangular Parallelogram ABCD, compoſed of four Rulers of Box Wood, near half an Inch thick, and two Inches and a half broad, with Braſs Mortiſes and Tenants at the Ends to be fitted into one another, and fixed by a Screw at each Angle, when uſed.

The inner Edge of each Ruler is lined with a thin Braſs Plate, the Edge of which projects about a quarter of an Inch beyond the Wood, and runs parallel to it, which Plates, when the Rulers are fitted together, form an Inner Parallelogram.

The upper and lower Limbs of this Parallelogram are 30 Inches clear, and the Sides 26 Inches, all divided into Inches, and half Inches; which Diviſions are carried on upon the wooden Part of the Inſtrument, the Inner Edges of which are worked down ſlanting to the Braſs, like a common Ruler; and each whole Inch Diviſion is numbered by a Figure on the Wood, from 1 to 30 at Top and Bottom, *viz.* from A to B, and from C to D, that is, from the Left to the Right; and on the Sides, from 1 to 26 downwards, from A to C, and from B to D.

Through each Diviſion of the Braſs inner Edges, a ſmall Hole is drilled, of a fit Size to let through a Thread of well twiſted Silk (ſuch as Peruke-makers uſe) and made ſmooth enough not to cut or fret the Threads; and when the Inſtrument is put together, it is Reticulated, by drawing the Threads through the Holes from every Inch Diviſion on one Side, to its correſponding Diviſion on the other, and ſo from Top to Bottom, till the whole Aperture of the Inſtrument is ſubdivided by the Threads into ſquare Inches. This Reticulation muſt be made on that Side of the Inſtrument which is turned towards the Objects, but the Diviſions and Figures are ſet on the Side next the Eye.

Or, inſtead of drilling Holes in the Diviſions, there may be fixed in each, a ſmall Pin with a round Head to wind the Silk over for making the Reticulation; but then, to prevent their being liable to be broken off, it will be convenient to let in the Braſs Plates a little way into the Subſtance of the Wood, that the Heads of the Pins may not project beyond the back Surfaces of the Rulers.

In the middle of the uppermoſt Limb AB, there is fixed a Braſs Socket R, about three Inches in Length, to receive a round Rod ML, about thirty Inches long, which is made to ſlide pretty tight in the Socket R, perpendicularly to the Plane of the Inſtrument, and in the Socket there is a ſmall Screw to fix the Rod to any propoſed Length.

At one End of this Rod there is another little Braſs Socket M, for a ſmaller Rod NI about two Feet long, to paſs through, perpendicular to the other, to be fixed likewiſe by a Screw at M to the Length deſired; the lower End of which laſt Rod carries a flat Braſs Plate I, either with a Circular Aperture in it, about an Inch Diameter, with croſs Hairs, or elſe a ſmall Hole to ſerve for a Sight; which Braſs Plate is moveable ſideways by a Joint at *n*.

Here, the Inſtrument ABCD repreſents the Picture, the Rod ML ſerves to meaſure the Diſtance of the Eye from M to R, and the Center of the Plate at I repreſents the Point of Sight. And as the Diſtance of the Eye may be increaſed or leſſened by ſliding the Rod ML in the Socket R, ſo may the Height of the Eye be made greater or leſs by ſliding the Rod NI in the Socket M; and the Point of Sight may be alſo brought perpendicularly againſt any Point of the Inſtrument, where the Center of the Picture is deſired to be, by turning the Rod ML in the Socket R, the Eye Plate at I being kept parallel to the Plane of the Inſtrument, and at the ſame Time, by the help of its Joint *n*, ſo turned, that the croſs Hairs may retain their due Situation with reſpect to the Horizon, the one parallel and the other perpendicular to it, to repreſent the Parallel and Director of the Eye, with reſpect to the Original Plane or Ground; and it may be likewiſe convenient to place the Point of Sight ſo, that from thence the true Horizon may appear againſt ſome one of the croſs Diviſions of the Inſtrument, as EF, which will then repreſent the Horizontal Line.

The Inſtrument, when taken to Pieces, is eaſily portable with the neceſſary Apparatus, in a ſmall wooden Box, of the Length of the longeſt Ruler, to any Place or Station where the Proſpect is to be taken; and when put together and Reticulated, may be ſet perpendicular to the Horizon upon a Table, by the help of two Brackets fixed upon it, as at D and C, or in a Window fronting the View; and the Inſtrument being firmly fixed in its Place, the Artiſt may then chuſe ſuch a Diſtance and Height for his Eye, as may make the Objects appear the moſt agreeably in his intended Picture; all which he may ſettle, by looking through the Point of Sight, and obſerving how the Objects fall againſt the Reticulations of the Inſtrument, and altering the Height or Diſtance of the Eye, till he makes them come to the Poſition he approves of; for the better doing of which, the Rules already laid down for the Choice of the Diſtance and Height of the Eye, and the Size of the Picture[a], may be of Service.

Theſe Things being thus previouſly ſettled; nothing remains but to Reticulate the Paper or Cloth intended for the Picture, with croſs Lines in the ſame Manner as the Inſtrument, and figured alike in the Margents, and to tranſcribe into the ſeveral Cells of this Reticulation, the Images of ſuch Objects as appear through the correſponding Cells in the Inſtrument; taking Care in viewing the Objects, that the Eye be always placed as near as poſſible to the croſs Hairs or Point of Sight: for if that be not ſtrictly obſerved, the Objects will alter their apparent Places, with reſpect to the Reticulation of the Inſtrument, which will lead the Artiſt to place their Images wrong in the Picture.

If any Part of the Proſpect ſhould be fuller of Work than the reſt, the Reticulation in that Place of the Inſtrument where thoſe Objects appear, may be ſubdivided at Pleaſure, by the help of the intermediate Pins or Holes in the inner Edges of the Inſtrument; and the correſponding Part of the Reticulation in the Picture, being ſubdivided in like manner, the propoſed Objects may be thereby deſcribed with greater Exactneſs. And if the Picture were required to be drawn bigger or leſs than the Appearance of the real Objects through the Inſtrument, it may be done by making the Reticulation in the Picture proportionably larger or ſmaller.

The principal Thing to be regarded in the Conſtruction of the Inſtrument is, that the Rods which carry the Point of Sight be ſufficiently ſtrong and firm, ſo as not to ſhake or waver about; it being on the Steddineſs of the Point of Sight, that the Truth of the Work chiefly depends: And the Inſtrument, when once placed, ought not to be moved, till the apparent Places and Dimenſions of all the principal or remarkable Objects are deſcribed, or at leaſt a Sketch of their Out-lines be made in the Picture, to be afterwards finiſhed at a more convenient Time or Place.

F I N I S.

[a] Sect 6, 7, 8.

A
TABLE
OF THE
PRINCIPAL MATTERS
Contained in this WORK.

BOOK I.

SECTION I.
Of Plain Viſion.

THIS Section treats of the Nature of Light, and of Colours ariſing from the different Reflexibility and Refrangibility of the Rays of Light, and the Effect theſe have on the apparent Figures of Bodies; Of the Conſtruction of the Eye, and of the Area or Extent of Objects which it is capable of taking in at the ſame View, and of diſtinct and confuſed Sight; Of the Optic Angle, and whether Objects appear bigger or leſs in Proportion to the Angles under which they are ſeen; Of the apparent Changes of the Shape, Light, and Colour of Objects according to their different Diſtances or Obliquity; Of erect and refracted Viſion; Of the Difference between the Impreſſion made by an Object on the Organ of Sight, and the Judgment concerning the Object itſelf formed in the Mind in Conſequence of that Impreſſion, and of the natural Rules by which the Mind is guided in forming ſuch Judgment, with ſome Obſervations in what Caſes theſe become uncertain, ambiguous, or falſe: Of the Difference between looking at Objects with a ſingle Eye, or with both Eyes at once, with ſome other Particulars relating to this Subject, all which are briefly diſcourſed of by way of Introduction. The whole is contained in twenty five Articles. Page 1

SECTION II.
Of the Difference between the Art of Drawing and Stereography.

Defines the Art of Drawing *to be an acquired Habit of delineating the Appearances of Objects by Imitation or copying, without the Aſſiſtance of Mathematical Rules, in Contradiſtinction to* Stereography *or* Perſpective *which teaches to deſcribe thoſe Appearances by certain Methods grounded on Mathematical Reaſoning, and to be performed only by Rule and Compaſs; from which Diſtinction it is ſhewn what is the peculiar Province of each of theſe Arts, what is the proper Buſineſs of each to perform, and how far their mutual Aſſiſtance is requiſite to the compleating a finiſhed Piece, they being both neceſſary, but neither of them alone ſufficient for that Purpoſe.* 10

SECTION III.
Of the different Methods of deſcribing Objects by Mathematical Rules.

This Section gives an Account of the principal Ways of deſcribing Objects on a Plane by Mathematical Rules, which are two, Geometrical *and* Stereographical. 13

The Geometrical *Deſcription of an Object is when its Repreſentation or Image on the propoſed Plane is formed by the Interſections of that Plane with parallel ſtraight Lines, falling either perpendicularly or with the ſame Angle of Inclination on it, from the ſeveral Points of the Object. In this manner only two Dimenſions of the Object can be repreſented at a time, ſuch as Heighth and Depth without regard to Thickneſs, or Breadth and Length without reſpect to Depth, ſo that no Part of the Figure is deſcribed with regard to its being either nearer or farther from the Eye, but the Eye may be ſuppoſed to be infinitely diſtant from the Plane of the Section, its Diſtance being no wiſe concerned in the Deſcription.*

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Of this Sort are the ſeveral Projections uſually called a Plan, Ichnography, Elevation, Profil, &c. *all which are particularly defined.*

The Stereographical *Deſcription is when the Lines which by their Interſection with the Plane of the Section form the Image of the Object, are not parallel, but are all ſuppoſed to meet in ſome one Point, which Point is taken as the Place of the Eye; ſo that this kind of Deſcription regards the Appearance that Objects have when ſeen from one certain Point, and is therefore capable of repreſenting all the three Dimenſions at a Time, as Length, Breadth, and Thickneſs, or as it were the Solidity of Objects, whence it takes the Name of* Stereography.

This kind of Deſcription is of three Sorts which are denominated from the different Poſitions of the Object and the Plane of the Section with reſpect to the Eye.

When the Plane of the Section is between the Eye and the Object, it is called Perſpective; *And here the Rays proceeding from the Object to the Eye are ſuppoſed to be cut by the Plane of the Section, and by their Interſection with it to form the Image of the Object, and it is therefore called* Perſpective, *the Object being as it were ſeen through a Plane placed between the Eye and it, as if that Plane were tranſparent.*

When the Object is between the Eye and the Plane of the Section, it is then called Projection, *the Rays which proceed from the Object to the Eye being ſuppoſed to be continued on beyond the Object till they meet that Plane, whereby the Image of the Object is in a manner projected or thrown forward upon a Plane beyond it.*

Laſtly, If the Eye be ſuppoſed to be between the Object and the Plane of the Section, then the Eye muſt be conſidered only as a Point through which the projecting Rays paſs from the Object, and are continued on till they cut the Plane of the Section on the oppoſite Side. This kind of Deſcription is therefore called Tranſprojection, *the Image of every Point of the Object being in a manner projected through the Place of the Eye upon the Plane of the Section; which laſt kind of Projection, altho' only imaginary, is in many Caſes requiſite to be found.*

This Section alſo gives a ſhort Account of the common Projections of the Sphere, and how far they partake of the Geometrical or Stereographical kind, it likewiſe deſcribes Uneven Stereography, *which is when the Surface on which the Deſcription is made is not a Plane, but Concave, Convex, or otherwiſe uneven; alſo* Stereography by Reflection, *and a kind of Deſcription called* Military Perſpective. *And contains thirty one Articles.*

SECTION IV.

Of the ſeveral preparatory Planes, Lines, and Points uſed in Stereographical Deſcriptions, their Definitions and Relations to each other, and to the Objects intended to be repreſented.

Definition 1. The Plane of the Section, *or that Plane by which the Rays from the Object to the Eye are ſuppoſed to be cut, is called* the Plane of the Picture, *or ſimply* the Picture, *and is repreſented by the Plane* EFGH. *Fig.* 8. 18

Definition 2. *The Point where the Rays are ſuppoſed to meet is ſometimes called* the Point of Sight, *and being that where the Eye is ſuppoſed to be, is therefore alſo called* the Place of the Eye, *or ſimply* the Eye, *and is marked with the Letter* I.

Definition 3. *By* Original Object *is meant the real Object intended to be repreſented, placed in its true Situation with reſpect to the Eye and the Picture.*

Definition 4. *By* Original Plane *is meant the Plane in which any Original Object lies or is geometrically deſcribed, this is repreſented by the Plane* LMGH *in which* QB *is an* Original Line.

Definition 5. *The Stereographical Deſcription of any Original Object, whether* Perſpective, Projective, *or* Tranſprojective, *is called* the Image of that Object.

Definition 6. *A Line* IO *drawn from the Eye perpendicular to the Picture cuts it in* O, the Center of the Picture; *the Line* IO *is called* the Axe of the Eye, *and being the Meaſure of the Diſtance of the Eye from the Picture, is in that View called the* Diſtance of the Picture *or* the Diſtance of the Eye. 19

Definition 7. *The Plane* LMNR *paſſing through the Eye parallel to the Picture is called* the Directing Plane. *Two* Corollaries.

Definition 8. *The Interſection* G H *of the Original Plane with the Picture is called* the Interſecting Line *of that Plane, and its Interſection* LM *with the Directing Plane is called its* Directing Line.

Definition 9. *A Plane* NREF *paſſing through the Eye parallel to the Original Plane* LMGH *is called the* Vaniſhing Plane *of that Original Plane.*

Definition 10. *The Interſection* EF *of the Vaniſhing Plane with the Picture is called the* Vaniſhing Line *of the Original Plane, and its Interſection* NR *with the Directing Plane is called the* Parallel of the Eye. *Two* Corollaries.

Definition 11. *The Plane* IKoP *which paſſes through the Axe of the Eye* IO *perpendicular to the Original Plane* LMGH *is called* the Vertical Plane. Corollary.

Definition 12. *The Interſection* IK *of the Vertical Plane with the Directing Plane is called* the Director of the Eye, *and being the Meaſure of the Diſtance of the Eye from the Original Plane, it is alſo called* the Height of the Eye, *and the Point* K *where it cuts the Directing Line is called* the Foot of the Eye's Director, *or* the Point of Station. 20

Defi-

Of the general Relations of Objects to the preparatory Planes, Lines, and Points uſed in Stereography.

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THEOREM

SECTION V.

Of the Proportions of the Images of determinate Original Lines.

B O O K II.

Treats of the various Methods of deſcribing the Images of Points, Lines, and Figures lying in an Original Plane whoſe Vaniſhing and Interſecting Lines are given, the Center and Diſtance of the Picture being conſtantly ſuppoſed to be known.

SECTION I.

Of the Preparation of the Picture and Original Plane.

Of the Indefinite Images of Lines in the Original Plane.

SECTION II.

Of the Determinate Images of Points and Lines in the Original Plane.

SECTION III.

Of the Images of Figures in the Original Plane.

BOOK III.

SECTION I.

Of the ſeveral Curves produced by the Image of a Circle in different Poſitions.

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SECTION III.

Of the Tranſmutation of the Conic Sections into each other by the Rules of Stereography.

2. *The*

Of the Methods of deſcribing the Conic Sections.

BOOK IV.

Treats of the various Methods of deſcribing the Images of Points, Lines, and Figures which do not lie in a given Plane, the Situation of the propoſed Objects with regard to the Picture, or to ſome known Plane, being given.

SECTION I.

Of the Seats of Points and Lines on an Original Plane.

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SECTION II.

Of the Images of Points, Lines, and Plane Figures whoſe relations to the Picture or to any known Original Plane are given.

BOOK V.

SECTION I.

Of the Projections of Points, Lines, and Plane Figures on a given Plane from a given Point.

Note,

SECTION II.

Of the Reflection of Light from a Poliſhed Plain Surface.

SECTION III.

Of the Reflected Images of Points, Lines, and Plain Figures as they appear in Reflecting Planes.

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BOOK VI.

Of Solid Bodies.

SECTION I.

Of the five Regular Solids.

Of the Ichnography and Elevation of the five Regular Solids.

Of the Images of the five Regular Solids.

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Of the Image of the Cone and its Sections.

SECTION III.

Of the Image of the Cylinder and its Sections.

Scholium

SECTION IV.

Of the Image of the Sphere or Globe, and its Sections.

SECTION V.

Of the Annulus, and its Image.

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BOOK VII.

Of ſeveral Matters relating to the general Practice of Painting, whether intended for Public or Private Buildings, either on plain or uneven Gounds.

SECTION I.

Of fixed or immoveable Painting on flat Grounds.

SECTION II.

Of Scenography.

SECTION

SECTION IX.

Of the Conſequences of viewing a Picture from any other Point than the true Point of Sight.

SECTION X.

Of Anamorphoſes or Deformations.

SECTION XI.

Of the Perſpective Frame.

ERRATA in the TABLE.

Pag. 3. Line 1. for *callea* read *called*
4. Theo. 9. 1. *Director, and* *Director and*
12. Lem. 23. 2. *ontinual* *continual*
14. Prop. 13. 3. after *either a Circle* add *or El-*
16. in the Margent, for Page 150. read Page 147.